Boundary control cost for parabolic systems

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GOAL:

We will present new results on the **cost of the boundary null controllability** of parabolic systems at time $T > 0$. In particular, we will study sharp estimates of the control cost at time T when the eigenvalues of the generator of the C_0 semigroup **accumulate** and do not satisfy a **gap condition**.

Some keywords:

1. Control cost for **boundary null controllability problems**.
2. **Coupled** parabolic systems.
3. Eigenvalues not satisfying a **gap condition** (with possible **condensation**).

Contents

1. The control cost of the one-dimensional heat equation

1. Control cost for the heat equation

Let us fix $T > 0$. We consider the scalar parabolic problem:

$$(1) \qquad \boxed{\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}}$$

where $y_0 \in H^{-1}(0,\pi)$ is the initial datum, $v \in L^2(0,T)$ is a scalar control function and $y = y(x,t)$ is the state.

Recall

System (1) is null controllable at time $T > 0$ if $\forall y_0 \in H^{-1}(0,\pi)$, $\exists v \in L^2(0,T)$ such that the corresponding solution to (1) satisfies

$$\boxed{y(\cdot,T) = 0} \quad \text{in } (0,\pi).$$

$$
(1) \qquad \begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

Null controllability at time $T > 0$:

Theorem

System (1) *is null controllable at time T for any $T > 0$.*

In particular, the set

$$
\mathcal{Z}_T(y_0) := \{v \in L^2(0,T) : y(T) = 0\} \neq \emptyset.
$$

We can then define the **control cost** for system (1) at time T as

$$
\boxed{\mathcal{K}(T) = \sup_{\|y_0\|_{H^{-1}(0,\pi)}=1} \left(\inf_{v \in \mathcal{Z}_T(y_0)} \|v\|_{L^2(0,T)} \right)}, \qquad \forall T > 0,
$$

1. Control cost for the heat equation

$$(1) \quad \begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Theorem

There exist $\tau_0 > 0$ *and* $C_0, C_1 > 0$ *such that*

$$\exp\left(\frac{C_0}{T}\right) \leq \mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0,\tau_0).$$

1. Control cost for the heat equation

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1. Control cost for the heat equation

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Moment Method (Fattorini-Russell)

Elements:

- $\sigma(-\partial_{xx}) = \{k^2\}_{k\geq 1} := \{\lambda_k\}_{k\geq 1}$.
- $\{\Phi_k\}$ a orthogonal basis of $H_0^1(0,\pi)$, where $\Phi_k = \sqrt{2/\pi}\sin kx$ are eigenfunctions of the operator $\boxed{-\partial_{xx}}$.

Consequence

We can obtain an explicit solution of problem (1) as a Fourier series.

1. Control cost for the heat equation

Thus $\boxed{y(T) = 0}$ $\iff$ $\exists v \in L^2(0,T)$ such that

$$\boxed{\sqrt{\frac{2}{\pi}}k\int_0^T v(T-t)e^{-\lambda_k t}\,dt = -e^{-\lambda_k T}\langle y_0, \Phi_k\rangle, \quad \forall k \geq 1}$$

$\iff$ $\exists v \in L^2(0,T)$ such that

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = -\sqrt{\frac{\pi}{2}}\frac{1}{k}e^{-\lambda_k T}\langle y_0, \Phi_k\rangle \equiv e^{-\lambda_k T}c_k, \quad \forall k \geq 1}$$

1. Control cost for the heat equation

$$(1) \qquad \begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\pi) \times (0,T), \\ y(0,\cdot) = v, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Summarizing: Moment problem

System (1) is null controllable at time $T > 0$ if and only if for any $y_0 \in H^{-1}(0,\pi)$ there exists $v \in L^2(0,T)$ such that

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T} c_k, \quad \forall k \geq 1$$

where $c_k = c_k(y_0)$ is such that $\sum_{k\geq 1} c_k^2 \leq C\|y_0\|^2_{H^{-1}(0,\pi)}$ ($\lambda_k = k^2$).

1. Control cost for the heat equation

$$\boxed{\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T}c_k, \quad \forall k \geq 1.}$$

Definition

Let $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ be a complex sequence and $T > 0$. We say that the family of functions $\{q_k\}_{k\geq 1} \subset L^2(0,T;\mathbb{C})$ is a **biorthogonal family** to the sequence of complex exponentials $\left\{e^{-\lambda_k t}\right\}_{k\geq 1}$ in $L^2(0,T;\mathbb{C})$, if

$$\int_0^T e^{-\lambda_k t}\,\overline{q}_n(t)\,dt = \delta_{kn}, \quad \forall k,n \geq 1.$$

Formal solution ($\lambda_k = k^2$):

$$\boxed{v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t)}.$$

1. Control cost for the heat equation

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T} c_k, \quad \forall k \geq 1.$$

Formal solution ($\lambda_k = k^2$):

$$v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t).$$

Questions:

1. Existence?
2. Convergence in $L^2(0,T)$ of the previous series? Related to the behaviour of $\|q_k\|_{L^2(0,T)}$ as $k \to \infty$.

1. Control cost for the heat equation

Existence: In our case $\lambda_k = k^2$. This implies

$$\boxed{\sum_{k\geq 1} \frac{1}{\lambda_k} < \infty}$$

and we deduce that the set $\left\{e^{-\lambda_k t}\right\}_{k\geq 1} \subset L^2(0,T)$ is **minimal** (linearly independent) and:

Theorem (Existence)

Assume $\{\lambda_k\}_{k\geq 1} \subset (0,\infty)$ is an increasing sequence satisfying

$$\boxed{\sum_{k\geq 1} \frac{1}{\lambda_k} < \infty.}$$

Then, $\left\{e^{-\lambda_k t}\right\}_{k\geq 1} \subset L^2(0,T)$ admits a ***biorthogonal family*** *$\{q_k\}_{k\geq 1}$ in $L^2(0,T)$.*

1. Control cost for the heat equation

Explicit estimates of the biorthogonal family w.r.t. k and T:

Theorem (Fattorini-Russell)

Assume $\boxed{\lambda_k = A\,(k+\omega)^2 + O(1)}$, $A, \omega > 0$, $\boxed{\lambda_k \neq \lambda_n}$, $k \neq n$. *Then,* $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T)$ *admits a* ***biorthogonal family*** $\{q_k\}_{k\geq 1}$ *in* $L^2(0,T)$. *Moreover, there exist* $\tau_0 > 0$ *and* $C > 0$ *such that, if* $T \in (0, \tau_0)$, *one has*

$$\boxed{\|q_k\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k}} e^{C/T}, \quad \forall k \geq 1.}$$

1. Control cost for the heat equation

A formal solution to

$$\int_0^T v(T-t)e^{-\lambda_k t}\,dt = e^{-\lambda_k T}c_k, \ \forall k \geq 1$$ is $$v(T-t) = \sum_{k\geq 1} c_k e^{-\lambda_k T} q_k(t).$$

Then,

$$\|v\|_{L^2(0,T)} \leq Ce^{C/T}\sum_{k\geq 1}|c_k|e^{-\lambda_k T}e^{C\sqrt{\lambda_k}}$$

$$\leq Ce^{C/T}\|y_0\|_{H^{-1}(0,T)}\left(\sum_{k\geq 1} e^{-2\lambda_k T}e^{2C\sqrt{\lambda_k}}\right)^{1/2}$$

But, $2C\sqrt{\lambda_k} \leq C^2/T + T\lambda_k$. Thus, for a new constant $C > 0$,

$$\|v\|_{L^2(0,T)} \leq Ce^{C/T}\|y_0\|_{H^{-1}(0,T)}\left(\sum_{k\geq 1} e^{-\lambda_k T}\right)^{1/2} \leq \frac{C}{\sqrt{T}}e^{C/T}\|y_0\|_{H^{-1}(0,T)}.$$

Thus,

1. Control cost for the heat equation

Theorem

There exist $\tau_0 > 0$ *and* $C_1 > 0$ *such that*

$$\mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

We proof uses

1. $\{\lambda_k\}_{k\geq 1} \subset (0, \infty)$ is an increasing sequence and, for $p, \alpha > 0$,

$$p\sqrt{\lambda_k} = k + O(1) \iff \left|p\sqrt{r} - \mathcal{N}(r)\right| \leq \alpha, \quad \forall r > 0,$$

   where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\geq 1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

2. Gap condition: $|\lambda_k - \lambda_n| \geq \rho\left|k^2 - n^2\right|$, for any $k, n \geq 1$ $(\rho > 0)$.

Bibliography: [FATTORINI, RUSSELL] Arch. Rat. Mech. Anal. (1971).

Some generalizations:

1. The complex case: $\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$.

2. No gap condition and separation:

$$\boxed{\inf_{k,n\geq 1, k\neq n} \frac{|\lambda_k - \lambda_n|}{k^2 - n^2} = 0} \quad \text{and} \quad \boxed{\inf_{k,n\geq 1, k\neq n} |\lambda_k - \lambda_n| > 0.}$$

3. or condensation:

$$\boxed{\inf_{k,n\geq 1, k\neq n} |\lambda_k - \lambda_n| = 0.}$$

2. Complex spectrum without condensation

2. The complex case without condensation

Let us consider the 2×2 linear reaction-diffusion system:

$$(2) \qquad \begin{cases} y_t - y_{xx} + Ay = 0 & \text{in } Q = (0,\pi) \times (0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

Here $y = (y_1, y_2)^t$, $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ and

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Existence and uniqueness

For any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, system (3) has a unique solution $y \in L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2))$ defined by transposition.

2. The complex case without condensation

$$(2)\qquad \begin{cases} y_t - y_{xx} + Ay = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

- $\sigma(-\partial_{xx} Id + A) = \{n^2 \pm i\}_{n\geq 1} := \{\lambda_k\}_{k\geq 1}$ ($\lambda_{2n-1} = n^2 - i$, $\lambda_{2n} = n^2 + i, \forall n \geq 1$).
- $\{\Phi_k\}$ a Riesz basis of $H^{-1}(0,\pi;\mathbb{C}^2)$ (Φ_k eigenfunctions of $\boxed{-\partial_{xx} Id + A}$).

$$\boxed{\inf_{k,n\geq 1, k\neq n} \frac{|\lambda_k - \lambda_n|}{k^2 - n^2} = 0} \text{ and } \boxed{\inf_{k,n\geq 1, k\neq n} |\lambda_k - \lambda_n| > 0}.$$

The moment problem

Given $\{c_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying $\boxed{\sum_{k\geq 1} |c_k|^2 \leq C\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)}}$, find $v \in L^2(0,T;\mathbb{C})$ s.t.

$$\boxed{\int_0^T v(T-t)e^{-\overline{\lambda}_k t}\,dt = c_k e^{-\overline{\lambda}_k T}}, \quad \forall k \geq 1.$$

2. The complex case without condensation

Assumptions (existence)

$$\begin{cases} \lambda_k \neq \lambda_n, & \forall k, n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k \geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta |\lambda_k|, & \forall k \geq 1. \end{cases}$$

The previous assumptions imply the set $\{e^{-\lambda_k t}\}_{k \geq 1} \subset L^2(0, T; \mathbb{C})$ is **minimal** (linearly independent) and:

Theorem (Existence)

Under the previous assumptions, $\{e^{-\lambda_k t}\}_{k \geq 1} \subset L^2(0, T; \mathbb{C})$ admits a ***biorthogonal family*** *$\{q_k\}_{k \geq 1}$ in $L^2(0, T; \mathbb{C})$.*

[FATTORINI,RUSSELL] Arch. Rat. Mech. Anal. (1971).
[AMMAR KHODJA, BENABDALLAH, G.-B., DE TERESA] J. Math. Pures Appl. (9) **96** (2011).

2. The complex case without condensation

The moment problem

Given $\{c_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying $\boxed{\sum_{k\geq 1} |c_k|^2 \leq C\|y_0\|_{H^{-1}(0,\pi;\mathbb{R}^2)}}$, find $v \in L^2(0,T;\mathbb{C})$ s.t.

$$\boxed{\int_0^T v(T-t)e^{-\overline{\lambda}_k t}\,dt = c_k e^{-\overline{\lambda}_k T}}, \quad \forall k \geq 1.$$

Again, a formal solution to the previous problem is v given by:

$$\boxed{v(T-t) = \sum_{k\geq 1} c_k e^{-\overline{\lambda}_k T} q_k(t)}.$$

Question: $v \in L^2(0,T;\mathbb{C})$?, i.e., is the previous series convergent in $L^2(0,T;\mathbb{C})$? **In general, it does not** (T must be large enough).

2. The complex case without condensation

Assumptions (estimates)

$\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ such that $\lambda_k \neq \lambda_n, \quad \forall k, n \in \mathbb{N} : k \neq n,$

1. $\boxed{\Re(\lambda_k) > 0}$ and $\exists \beta \geq 0$ s.t. $\boxed{\Im(\lambda_k) \leq \beta\sqrt{\Re(\lambda_k)}}$, $\forall k \geq 1$.
2. **Gap condition without condensation**: for some $\rho, q > 0$,

$$\begin{cases} |\lambda_k - \lambda_n| \geq \rho|k^2 - n^2|, & \forall k, n : |k - n| \geq q. \\ \inf\limits_{k\neq n:|k-n|<q} |\lambda_k - \lambda_n| > 0. \end{cases}$$

3. For some $p, \alpha > 0$,

$$\left|p\sqrt{r} - \mathcal{N}(r)\right| \leq \alpha, \quad \forall r > 0,$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\geq 1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

2. The complex case without condensation

Under the previous assumptions, one has:

Theorem (Explicit estimates w.r.t. T and k)

There exist $\tau_0 > 0$ and $C > 0$ such that, if $T \in (0, \tau_0)$, one has

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C e^{C\sqrt{\Re(\lambda_k)}} e^{C/T}, \quad \forall k \geq 1.$$

[Benabdallah, Boyer, G.-B., Olive], SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001.

Reasoning as before, it is not difficult to see that

$$v(T-t) = \sum_{k\geq 1} c_k e^{-\overline{\lambda}_k T} q_k(t) \in L^2(0,T;\mathbb{C})$$

and there exists $C_1 > 0$ such that

$$\|v\|_{L^2(0,T;\mathbb{C})} \leq \exp\left(\frac{C_1}{T}\right)\left(\sum_{k\geq 1} |c_k|^2\right)^2, \quad \forall T \in (0, \tau_0).$$

In particular,

$$(2)\qquad \begin{cases} y_t - y_{xx} + A_1 y = 0 & \text{in } Q = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0, & \text{in } (0,\pi), \end{cases}$$

$$A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Theorem

System (2) is null controllable at any time $T > 0$ and there exist $\tau_0 > 0$ and $C_0 > 0$ such that

$$\mathcal{K}(T) \le \exp\left(\frac{C_0}{T}\right), \quad \forall T \in (0,\tau_0).$$

2. The complex case without condensation

Remark

When the sequence $\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies

$$\begin{cases} \lambda_k \neq \lambda_n, \quad \forall k, n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta |\lambda_k|, \quad \forall k \geq 1. \end{cases}$$

and condition

$$\boxed{\inf_{k\neq n : |k-n|<q} |\lambda_k - \lambda_n| = 0}$$

the corresponding null controllability problem develops a **minimal time** of control:

"$\exists T_0 \in [0, \infty]$ s.t. the system is not null controllable if $T < T_0$ and is null controllable if $T > T_0$."

Ammar Khodja, Benabdallah, G.-B., de Teresa, J. Funct. Anal. **267** (2014).

3. Complex spectrum with condensation: the main result

3. Sequences with condensation

Assume L is the generator of a C^0-semigroup of contractions on a complex Hilbert space H s. t.

- $\sigma(L) = \Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ and $\lambda_k \neq \lambda_n$ for $k \neq n$,
- the sequence of eigenfunctions $\{\Phi_k\}_{k\geq 1}$ forms a Riesz basis of H.

Assumptions on Λ

$$\begin{cases} \lambda_k \neq \lambda_n, \quad \forall k, n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta|\lambda_k|, \quad \forall k \geq 1. \end{cases}$$

3. Sequences with condensation

Assumptions on Λ

$$\begin{cases} \lambda_k \neq \lambda_n, & \forall k, n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta|\lambda_k|, & \forall k \geq 1. \end{cases}$$

Theorem

Fix $T > 0$. Then $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T;\mathbb{C})$ admits a ***biorthogonal family*** *$\{q_k\}_{k\geq 1}$ in $L^2(0,T;\mathbb{C})$ satisfying: for any $\varepsilon > 0$, $\exists\, C_{1,\varepsilon,T}, C_{2,\varepsilon,T} > 0$ s.t.*

$$C_{1,\varepsilon,T} \frac{e^{-\varepsilon \Re(\lambda_k)}}{|E'(\lambda_k)|} \leq \|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_{2,\varepsilon,T} \frac{e^{\varepsilon \Re(\lambda_k)}}{|E'(\lambda_k)|}, \quad \forall k \geq 1,$$

where $E(z)$ is the interpolating function:

$$E(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right), \qquad \boxed{E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right)}$$

3. Sequences with condensation

Definition

The **index of condensation** of $\Lambda = \{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ is:

$$c(\Lambda) = \limsup_{k\to\infty} \frac{-\log |E'(\lambda_k)|}{\Re(\lambda_k)} \in [0, +\infty] .$$

$$\boxed{E'(\lambda_k) = -\frac{2}{\lambda_k} \prod_{j\neq k}^{\infty} \left(1 - \frac{\lambda_k^2}{\lambda_j^2}\right) .}$$

Ammar Khodja, Benabdallah, G.-B., de Teresa, J. Funct. Anal. **267** (2014)),

Remark

The constants $C_{1,\varepsilon,T}$ and $C_{2,\varepsilon,T}$ depend on $T > 0$. The authors do not provide an explicit dependence of them with respect to T due to the method used: these inequalities are first obtained in $L^2(0,\infty;\mathbb{C})$ and, then, proved in $L^2(0,T;\mathbb{C})$ ($T \in (0,\infty)$) after a contradiction argument.

3. Sequences with condensation

The abstract moment problem

Given $\{c_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying $\boxed{\sum_{k\geq 1} |c_k|^2 \leq C}$, find $v \in L^2(0,T;\mathbb{C})$ s.t.

$$\int_0^T v(T-t)e^{-\overline{\lambda}_k t}\,dt = c_k e^{-\overline{\lambda}_k T}, \quad \forall k \geq 1.$$

A consequence:

Theorem

Under the previous assumptions, given $T > 0$, one has:

1. *If $T > c(\Lambda)$, the previous moment problem has a solution $v \in L^2(0,T;\mathbb{C})$ (**positive null controllability result** at time T).*
2. *Assume $T < c(\Lambda)$. Then, there exists $\{c_k\}_{k\geq 1} \subset \mathbb{C}$ satisfying $\boxed{\sum_{k\geq 1} |c_k|^2 \leq C}$ such that the previous moment problem does not admit a solution $v \in L^2(0,T;\mathbb{C})$ (**negative null controllability result** at time T).*

3. Sequences with condensation

Control cost when $T > c(\Lambda)$???

Explicit estimates of $\|q_k\|_{L^2(0,T;\mathbb{C})}$ w.r.t. T and k ???

3. Sequences with condensation

Let us fix $\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ such that $\lambda_k \neq \lambda_n$ with $k \neq n$ and

Assumptions:

1. $\Re(\lambda_k) > 0$ and $\Im(\lambda_k) \leq \beta\sqrt{\Re(\lambda_k)}$, for any $k \geq 1$ ($\beta \geq 0$).
2. $\exists \rho, q > 0$ such that

$$\boxed{|\lambda_k - \lambda_n| \geq \rho |k^2 - n^2|}, \quad \forall k, n \geq 1 : |k-n| \geq q,$$

   ($\displaystyle\inf_{k\neq n : |k-n|<q} |\lambda_k - \lambda_n|$ could be zero).

3. $\exists p_1, p_2, \alpha > 0$ such that

$$-\alpha + p_1\sqrt{r} \leq \mathcal{N}(r) \leq \alpha + p_2\sqrt{r}, \quad \forall r > 0,$$

   where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\geq 1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

The previous assumptions imply

$$\sum_{k\geq 1} \frac{1}{|\lambda_k|} < \infty; \quad \Re(\lambda_k) \geq \delta|\lambda_k|, \quad \forall k \geq 1.$$

Thus, the set $\{e^{-\lambda_k t}\}_{k\geq 1} \subset L^2(0,T;\mathbb{C})$ is **minimal** and admits a **biorthogonal family**.

3. Sequences with condensation

Theorem (The main result)

Under the previous assumptions, there exists $C_0 > 0$ *s.t. for any* $T > 0$ *there exists a biorthogonal family* $\{q_k\}_{k\geq 1}$ *to* $\{e^{-\lambda_k t}\}_{k\geq 1}$ *in* $L^2(0,T;\mathbb{C})$ *s.t.*

$$\|q_k\|_{L^2(0,T;\mathbb{C})} \leq C_0 e^{C_0\sqrt{\Re(\lambda_k)}} e^{C_0/T} \mathcal{P}_k^{-1}, \quad \forall k \geq 1,$$

where

$$\mathcal{P}_k = \prod_{1\leq|k-n|<q} |\lambda_k - \lambda_n|, \quad \forall k \geq 1.$$

In addition, if for $\nu > 0$ *one has* $|\lambda_k - \lambda_n| \leq \nu \left|k^2 - n^2\right|$, $\forall k, n \geq 1$, *then* $\exists C_1 > 0$ *s.t. any biorthogonal family* $\{s_k\}_{k\geq 1}$ *to* $\{e^{-\lambda_k t}\}_{k\geq 1}$ *in* $L^2(0,T;\mathbb{C})$

$$\|s_k\|_{L^2(0,T;\mathbb{C})} \geq C_1 e^{C_1/T} \frac{(2q-2)!}{T^{2(q-1)}} \left(\frac{4q-3}{2T} + |\lambda_{k+1-q}|\right)^{1/2} \mathcal{P}_k^{-1}, \quad \forall k \geq q.$$

The proof is very technical.

[OUAILI, G.-B.,] to appear.

3. Sequences with condensation

Remarks

1. If $\{\lambda_k\}_{k\geq 1} \subset \mathbb{C}$ satisfies

$$|\lambda_k - \lambda_n| \geq \gamma > 0, \quad \forall k, n \geq 1 : k \neq n,$$

for a constant $\gamma > 0$ (no condensation), then

$$\mathcal{P}_k^{-1} \leq \gamma^{2-2q}, \quad \forall k \geq 1.$$

and the previous theorem generalizes the result in [Benabdallah, Boyer, G.-B., Olive], SIAM J. Control Optim. **52** (2014).

2. It is posible to give explicit dependence of the constants C_0 and C_1 w.r.t. the parameters β, ρ, q, p_1, p_2 and α.

4. Two applications

1. First application: parabolic system with different diffusions

Let us consider the boundary controllability problem for the one-dimensional linear reaction-diffusion system:

$$(3)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T = (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(1, d)$, $\boxed{d > 0}$, $\boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$.

Existence and uniqueness

For any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, system (3) has a unique solution $y \in L^2(Q_T) \cap C^0([0,T]; H^{-1}(0,\pi;\mathbb{R}^2))$ defined by transposition.

Approximate or null controllability properties of system (3)??

$$(3) \quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

If $L_1 = -D\partial_{xx} + A_0$, with $D(L_1) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$, one has

1. $d = 1$ ($D = Id$), $\sigma(L_1) = \sigma(L_1^*) = \{k^2\}_{k\geq 1}$.
2. $d \neq 1$, $\sigma(L_1) = \sigma(L_1^*) = \{n^2, dn^2 : n \geq 1\} = \left\{\lambda_k^{(1)}\right\}_{k\geq 1}$, with $\Lambda_d = \left\{\lambda_k^{(1)}\right\}_{k\geq 1}$, a positive increasing sequence.

$$(3)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Theorem ($d = 1$)

Under the previous conditions, system (3) is ***approximate and null controllable*** *at any time $T > 0$. Moreover, there exist $\tau_0 > 0$ and $C_0, C_1 > 0$ such that*

$$\exp\left(\frac{C_0}{T}\right) \le \mathcal{K}(T) \le \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

The proof is a consequence of [Fernández-Cara, G.-B., de Teresa], J. Funct. Anal. (2010) and [Ammar-Khodja,Benabdallah,G.-B.,de Teresa], J. Math. Pures Appl. (2011). **NC result**.

[Benabdallah, Boyer, G.-B., Olive], SIAM J. Control Optim. **52** (2014). **Control cost**.

(3)
$$\begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \operatorname{diag}(1,d), \quad \boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

Proposition ($d \neq 1$, approximate controllability)

Assume $d \neq 1$. Then, system (3) is ***approximately controllable*** *at time $T > 0$ if and only if (Fattorini-Hautus condition)*

$$\lambda_k^{(1)} \neq \lambda_n^{(1)}, \ \forall k,n \geq 1, \ k \neq n \quad \left(\iff dk^2 \neq n^2, \ \forall k,n \geq 1 \iff \boxed{\sqrt{d} \notin \mathbb{Q}} \right).$$

$$(3) \qquad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \operatorname{diag}(1,d), \boxed{A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

Assumption

In the sequel, $\boxed{D = \operatorname{diag}(1,d)}$ with $\boxed{d \neq 1}$ and $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

Null controllability properties at time $T > 0$ of system (3)??

$$(3)\qquad \begin{cases} y_t - D y_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \operatorname{diag}(1,d), \quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}, \quad \boxed{\sqrt{d} \notin \mathbb{Q}}.$$

Proposition

The sequence Λ_d *satisfies*

$$\begin{cases} \lambda_k^{(1)} \neq \lambda_n^{(1)}, \quad \forall k,n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k^{(1)}|} < \infty; \quad \Re(\lambda_k^{(1)}) \geq \delta |\lambda_k^{(1)}|, \quad \forall k \geq 1. \end{cases}$$

Remark

The condition $\boxed{\exists \rho > 0 : |\lambda_k^{(1)} - \lambda_n^{(1)}| \geq \rho, \quad \forall k,n \geq 1,\ k \neq n}$, in general, does not hold (**condensation**).

$$(3)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$D = \operatorname{diag}(1, d)$, $\Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}$, $\boxed{\sqrt{d} \notin \mathbb{Q}}$.

If $\boxed{T_0(d) = c(\Lambda_d) \in [0,\infty]}$ (**condensation index** of the sequence Λ_d),

then

Corollary

Under the previous conditions, let $\boxed{T_0(d) = c(\Lambda_d) \in [0,\infty]}$ *be the* ***condensation index*** *of the sequence* Λ_d*. Then,*

1. *If* $T > T_0(d)$*, system* (3) *is null controllable at time* T*.*
2. *If* $T < T_0(d)$*, system* (3) *is not null controllable at time* T*.* ∎

AMMAR KHODJA, BENABDALLAH, G.-B., DE TERESA, J. Funct. Anal. **267** (2014).

Assumptions:

1. $\Re(\lambda_k) > 0$ and $\Im(\lambda_k) \le \beta\sqrt{\Re(\lambda_k)}$, for any $k \ge 1$ ($\beta \ge 0$).
2. $\exists \rho, q > 0$ such that

$$\boxed{|\lambda_k - \lambda_n| \ge \rho |k^2 - n^2|}, \quad \forall k, n \ge 1 : |k - n| \ge q,$$

($\inf_{k \ne n : |k-n| < q} |\lambda_k - \lambda_n|$ could be zero).

3. $\exists p_1, p_2, \alpha > 0$ such that

$$-\alpha + p_1\sqrt{r} \le \mathcal{N}(r) \le \alpha + p_2\sqrt{r}, \quad \forall r > 0,$$

where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k\ge1}$:

$$\mathcal{N}(r) = \#\{k : |\lambda_k| \le r\}, \quad \forall r > 0.$$

$$(3) \qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y\big|_{x=0} = Bv, \quad y\big|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$$D = \operatorname{diag}(1, d), \qquad \Lambda_d = \{n^2, dn^2\}_{n\ge1} = \{\lambda_k^{(1)}\}_{k\ge1}, \qquad \boxed{\sqrt{d} \notin \mathbb{Q}}.$$

Proposition

Let us take $p = 1 + 1/\sqrt{d}$*. Then,*

1. $\lambda_k^{(1)} > 0$, $\forall k \ge 1$.
2. $\boxed{|\lambda_k^{(1)} - \lambda_n^{(1)}| \ge \frac{5}{8}\frac{1}{p^2}|k^2 - n^2|}$, $\quad \forall k, n \ge 1 : |k - n| \ge \boxed{2}$.
3. $\boxed{-2 + p\sqrt{r} \le \mathcal{N}(r) \le 2 + p\sqrt{r}}$, $\forall r > 0$.
4. $\boxed{|\lambda_k^{(1)} - \lambda_n^{(1)}| \le \frac{8}{3}\frac{1}{p^2}|k^2 - n^2|}$, $\forall k, n \ge 1$

$D = \text{diag}\,(1, d)$, $\quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}$, $\quad \boxed{\sqrt{d} \notin \mathbb{Q}}$.

Corollary

There exist $C_0 > 0$ s.t. $\forall T > 0$ there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k^{(1)} t}\}_{k\geq 1}$ in $L^2(0,T)$ s.t.

$$\|q_k\|_{L^2(0,T)} \leq C_0 e^{C_0\sqrt{\lambda_k^{(1)}}} e^{C_0/T} \mathcal{P}_k^{-1}, \quad \forall k \geq 1.$$

In addition, $\exists C_1 > 0$ s.t. any biorthogonal family $\{s_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ satisfies

$$\|s_k\|_{L^2(0,T)} \geq C_1 e^{C_1/T} \left(\frac{5}{2T} + \left|\lambda_{k-1}^{(1)}\right|\right)^{1/2} \mathcal{P}_k^{-1}, \quad \forall k \geq 2,$$

with

$$\mathcal{P}_k = \left(\lambda_{k+1}^{(1)} - \lambda_k^{(1)}\right)\left(\lambda_k^{(1)} - \lambda_{k-1}^{(1)}\right).$$

$D = \operatorname{diag}(1, d), \quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}, \quad \boxed{\sqrt{d} \notin \mathbb{Q}}.$

Proposition (Algebraic irrational numbers)

Assume $\sqrt{d}$ is an algebraic irrational number of degree $\ell \geq 2$. Then, $\exists \sigma > 0$ s.t.

$$\lambda_{k+1}^{(1)} - \lambda_k^{(1)} \geq \frac{\sigma}{k^{\ell-2}}, \quad \forall k \geq 1.$$

Remark

The previous result is a consequence of Liouville's Theorem on Diophantine Approximation: if $\nu \in \mathbb{R}$ is an algebraic irrational number of degree $\ell \geq 2$, then $\exists C > 0$ s.t.

$$\left|\nu - \frac{p}{q}\right| \geq \frac{C}{q^\ell}, \quad \forall p, q \in \mathbb{Z}, \quad q \geq 1.$$

It is also valid if ν is an irrational number with finite Liouville-Roth constant.

$D = \mathrm{diag}\,(1, d)$, $\quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}$, $\quad \boxed{\sqrt{d} \notin \mathbb{Q}}$.

Proposition (Algebraic irrational numbers)

Assume $\sqrt{d}$ is an algebraic irrational number of degree $\ell \geq 2$. Then, $\exists \sigma > 0$ s.t.

$$\lambda_{k+1}^{(1)} - \lambda_k^{(1)} \geq \frac{\sigma}{k^{\ell-2}}, \quad \forall k \geq 1.$$

Remark

If $\sqrt{d}$ is a quadratic algebraic irrational number, one has

$$|\lambda_k^{(1)} - \lambda_n^{(1)}| \geq \sigma, \quad \forall k, n \geq 1 : k \neq n.$$

for a constant $\sigma > 0$. It is possible to characterise the coefficients $d > 0$ for which the property holds.

$D = \operatorname{diag}(1, d), \quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}, \quad \boxed{\sqrt{d} \notin \mathbb{Q}}.$

Corollary

Assume $\sqrt{d}$ is an algebraic irrational number of degree $\ell \geq 2$. Then, $\exists \widetilde{C}_0 > 0$ s.t. $\forall T > 0$ there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k^{(1)} t}\}_{k\geq 1}$ in $L^2(0,T)$ s.t.

$$\|q_k\|_{L^2(0,T)} \leq \widetilde{C}_0 e^{\widetilde{C}_0 \sqrt{\lambda_k^{(1)}}} e^{C_0/T}, \quad \forall k \geq 1.$$

$$(3)\qquad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T, \\ y|_{x=0} = Bv, \quad y|_{x=\pi} = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

$D = \text{diag}\,(1, d)$, $\quad \Lambda_d = \{n^2, dn^2\}_{n\geq 1} = \{\lambda_k^{(1)}\}_{k\geq 1}$, $\quad \boxed{\sqrt{d} \notin \mathbb{Q}}$.

Corollary

Assume $\sqrt{d}$ is an algebraic irrational number of degree $\ell \geq 2$. Then, system (3) is null controllable at any time $T > 0$ and $\tau_0, C_0, C_1 > 0$ s.t.

$$\exp\left(\frac{C_0}{T}\right) \leq \mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

Question

Could the control cost $\mathcal{K}(T)$ have a more explosive behavior with respect to T when $T \to 0$?

4. Two applications

2. Second application: a parabolic decoupled system

Let us consider the problem

$$(4) \qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $a \in L^2(0,\pi)$ is a function such that $\boxed{\int_0^\pi a(x)\,dx = 0}$, $y = (y_1, y_2)^t$, $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$, $v \in L^2(0,T)$ and

$$\boxed{A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Existence and uniqueness

For any $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$, system (4) has a unique solution $y \in L^2(Q_T) \cap C^0([0,T];H^{-1}(0,\pi;\mathbb{R}^2))$ defined by transposition.

Approximate or null controllability properties of system (4)??

$$(4)\qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$\boxed{\int_0^\pi a(x)\,dx = 0, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

If $L_2 = -\partial_{xx} Id + a(x)A_1$, with $D(L_2) = H^2(0,\pi;\mathbb{R}^2) \cap H_0^1(0,\pi;\mathbb{R}^2)$, one has

$$\boxed{\sigma(L_2) = \sigma(L_2^*) = \{n^2, n^2 + \beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}} \text{ where } \{\beta_k\}_{k\geq 1} \in \ell^2.$$

$$\boxed{\inf_{k,n\geq 1, k\neq n} \frac{\left|\lambda_k^{(2)} - \lambda_n^{(2)}\right|}{k^2 - n^2} = 0}, \qquad \boxed{\inf_{k,n\geq 1, k\neq n} \left|\lambda_k^{(2)} - \lambda_n^{(2)}\right| = \inf_{k\geq 1} |\beta_k| = 0}$$

(**condensation**).

$$(4) \qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

Proposition (Approximate controllability)

Under the previous conditions, system (4) is ***approximately controllable*** *at time $T > 0$ if and only if*

$$\lambda_k^{(2)} \neq \lambda_n^{(2)}, \ \forall k,n \geq 1, \ k \neq n \quad (\Longleftrightarrow k^2 \neq n^2 + \beta_n, \ \forall k,n \geq 1).$$

Remark

As $\{\beta_k\}_{k\geq 1} \in \ell^2$, the previous condition is equivalent to

$$k^2 \neq n^2 + \beta_n, \ \forall k,n \leq N_0 \quad \text{and} \quad \beta_n \neq 0, \quad \forall n \geq 1,$$

for a positive integer N_0.

$$
(4) \qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}
$$

$$
\boxed{\int_0^\pi a(x)\,dx = 0, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}
$$

Assume $\boxed{k^2 \neq n^2 + \beta_n, \ \forall k, n \geq 1}$.

Proposition

The sequence $\Lambda_a = \{n^2, n^2 + \beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}$, $\{\beta_k\}_{k\geq 1} \in \ell^2$, *satisfies*

$$
\begin{cases} \lambda_k^{(2)} \neq \lambda_n^{(2)}, \quad \forall k, n \in \mathbb{N} : k \neq n, \\ \displaystyle\sum_{k\geq 1} \frac{1}{|\lambda_k^{(2)}|} < \infty; \quad \Re(\lambda_k^{(2)}) \geq \delta|\lambda_k^{(2)}|, \quad \forall k \geq 1. \end{cases}
$$

$$(4) \qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$\boxed{\Lambda_a = \{n^2, n^2 + \beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}}, \quad \{\beta_k\}_{k\geq 1} \in \ell^2, \quad \boxed{k^2 \neq n^2 + \beta_n, \ \forall k, n \geq 1}.$$

If $\boxed{T_0(a) = c(\Lambda_a) \in [0,\infty]}$ (**condensation index** of the sequence Λ_a),

then

Corollary

Under the previous conditions, let $\boxed{T_0(a) = c(\Lambda_a) \in [0,\infty]}$ *be the* ***condensation index*** *of the sequence* Λ_a*. Then,*

1. *If* $T > T_0(a)$*, system (4) is null controllable at time* T*.*
2. *If* $T < T_0(a)$*, system (4) is not null controllable at time* T*.* ∎

[OUAILI], Math. Control Relat. Fields **10** (2020).

The **condensation index** of the sequence $\boxed{\Lambda_a = \{n^2, n^2 + \beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}}$, $\{\beta_k\}_{k\geq 1} \in \ell^2$, is given by

$$\boxed{T_0(a) = \limsup \frac{-\log|\beta_k|}{k^2} \in [0, \infty]} \quad (\beta_k \neq 0, \quad \forall k \geq 1).$$

Assumptions:

1. $\Re(\lambda_k) > 0$ and $\Im(\lambda_k) \leq \beta\sqrt{\Re(\lambda_k)}$, for any $k \geq 1$ ($\beta \geq 0$).
2. $\exists \rho, q > 0$ such that
$$\boxed{|\lambda_k - \lambda_n| \geq \rho|k^2 - n^2|}, \quad \forall k, n \geq 1 : |k - n| \geq q,$$
($\inf_{k \neq n : |k-n| < q} |\lambda_k - \lambda_n|$ could be zero).
3. $\exists p_1, p_2, \alpha > 0$ such that
$$-\alpha + p_1\sqrt{r} \leq \mathcal{N}(r) \leq \alpha + p_2\sqrt{r}, \quad \forall r > 0,$$
where $\mathcal{N}$ is the counting function associated with the sequence $\{\lambda_k\}_{k \geq 1}$:
$$\mathcal{N}(r) = \#\{k : |\lambda_k| \leq r\}, \quad \forall r > 0.$$

$$\boxed{\Lambda_a = \{n^2, n^2 + \beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}}, \ \{\beta_k\}_{k\geq 1} \in \ell^2, \ \boxed{k^2 \neq n^2 + \beta_n, \ \forall k, n \geq 1}.$$

Proposition

Let us take $\varepsilon_0 = \sup_{k\geq 1} |\beta_k|$. *Then,*

1. $\lambda_k^{(2)} > 0, \ \forall k \geq 1.$
2. $\boxed{|\lambda_k^{(2)} - \lambda_n^{(2)}| \geq \frac{1}{6}|k^2 - n^2|}, \quad \forall k, n \geq 1 : |k - n| \geq \boxed{2}.$
3. $\boxed{-\alpha + 2\sqrt{r} \leq \mathcal{N}(r) \leq \alpha + 2\sqrt{r}}, \ \forall r > 0, \ \boxed{\alpha = 2 + \sqrt{\varepsilon_0}}.$
4. $\boxed{|\lambda_k^{(2)} - \lambda_n^{(2)}| \leq \nu |k^2 - n^2|}, \ \forall k, n \geq 1, \ \boxed{\nu = (1 + \varepsilon_0)/2}.$

$$\boxed{\Lambda_a = \{n^2, n^2+\beta_n\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1}}, \ \{\beta_k\}_{k\geq 1} \in \ell^2, \ \boxed{k^2 \neq n^2+\beta_n, \ \forall k, n \geq 1}.$$

Corollary

$\exists C_0, C_1 > 0$ s.t. $\forall T > 0$ there exists a biort. fam. $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k^{(2)} t}\}_{k\geq 1}$ in $L^2(0,T)$ s.t.

$$\|q_k\|_{L^2(0,T)} \leq C_0 e^{C_0 k} e^{C_0/T} \mathcal{P}_k^{-1}, \quad \forall k \geq 1.$$

In addition, $\exists C_1 > 0$ s.t. any biorthogonal family $\{s_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ satisfies

$$\|s_k\|_{L^2(0,T)} \geq C_1 e^{C_1/T} \left(\frac{1}{T}+k\right)^{1/2} \mathcal{P}_k^{-1}, \quad \forall k \geq 2.$$

with $\mathcal{P}_1 = |\beta_1|$ *and*
$$\begin{cases} (2n-2)|\beta_n| \leq \mathcal{P}_{2n-1} \leq (2n-1)|\beta_n|, & \forall n \geq 2, \\ 2n|\beta_n| \leq \mathcal{P}_{2n} \leq (2n+1)|\beta_n|, & \forall n \geq 1. \end{cases}$$

Particular case:

$$(4)\qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$\boxed{\int_0^\pi a(x)\,dx = 0, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Lemma

Given $\gamma \in (0,1)$*, there exists* $a \in L^2(0,\pi)$ *with* $\int_0^\pi a(x)\,dx = 0$ *such that*

$$\sigma(L_2) = \sigma(L_2^*) = \{n^2, n^2 + e^{-n^{2\gamma}}\} = \left\{\lambda_k^{(2)}\right\}_{k\geq 1} \textit{ where } L_2 = -\partial_{xx} + a(x)A_1.$$

From now on, we work with $\gamma \in (0,1)$ and the coefficient $a \in L^2(0,\pi)$ provided by the previous result. Thus, $\boxed{\beta_n = e^{-n^{2\gamma}}}, \quad \forall n \geq 1.$

Particular case:

$$(4)\qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In this case ($\gamma \in (0,1)$),

$$\boxed{T_0(a) = \limsup \frac{-\log|\beta_k|}{k^2} = \limsup \frac{-\log\left(e^{-k^{2\gamma}}\right)}{k^2} = 0},$$

and the system is null controllable at any time $T > 0$. Also, for new constants $c, C > 0$,

$$\begin{cases} k^{\frac{3}{2}} e^{c(1+\frac{1}{T})} \boxed{e^{k^{2\gamma}}} \le \|q_{2k-1}\|_{L^2(0,T)} \le e^{C(1+k+\frac{1}{T})} \boxed{e^{k^{2\gamma}}}, & \forall k \ge 1, \\ k^{\frac{3}{2}} e^{c(1+\frac{1}{T})} \boxed{e^{k^{2\gamma}}} \le \|q_{2k}\|_{L^2(0,T)} \le e^{C(1+k+\frac{1}{T})} \boxed{e^{k^{2\gamma}}}, & \forall k \ge 1. \end{cases}$$

Particular case:

$$(4) \qquad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

In this case $(\sigma(L_2) = \{n^2, n^2 + e^{-n^{2\gamma}}\}, \gamma \in (0,1))$, we can prove,

Theorem

Under the previous conditions, system (4) is null controllable at any time $T > 0$ ($T_0 = 0$) and there exist $\tau_0, C_0, C_1 > 0$ (independent of γ) such that

$$\exp\left(\frac{C_0}{T} + \frac{C_0}{T^{\frac{\gamma}{1-\gamma}}}\right) \le \mathcal{K}(T) \le \exp\left(\frac{C_1}{T} + \frac{C_1}{T^{\frac{\gamma}{1-\gamma}}}\right), \quad \forall T \in (0,\tau_0).$$

[OUAILI, G.-B.,] to appear.

Remark

The previous inequalities are equivalent to:

1. If $\boxed{\gamma \in (0, 1/2]}$, then, there exist three positive constants τ_0, C_0 and C_1 (independent of γ) such that (same estimates as for the heat equation)

$$\exp\left(\frac{C_0}{T}\right) \le \mathcal{K}(T) \le \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0)$$

2. If $\boxed{\gamma \in (1/2, 1)}$, again, there exist three positive constants τ_0, C_0 and C_1 (independent of γ) such that

$$\exp\left(\frac{C_0}{T^{\frac{\gamma}{1-\gamma}}}\right) \le \mathcal{K}(T) \le \exp\left(\frac{C_1}{T^{\frac{\gamma}{1-\gamma}}}\right), \quad \forall T \in (0, \tau_0).$$

The control cost blows up when $\gamma \to 1^-$. This is natural because the minimal time for system (4) when $\gamma = 1$ is $T_0 = 1$ and the system is not null controllable at time T when $T < 1$. ■

Question

What happens if we exert two boundary controls on the previous systems?

4. Two applications

Let us consider

$$(5)\quad \begin{cases} y_t - Dy_{xx} + A_0 y = 0 & \text{in } Q_T = (0,\pi)\times(0,T),\\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T),\\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

$$(6)\quad \begin{cases} y_t - y_{xx} + a(x)A_1 y = 0 & \text{in } (0,\pi)\times(0,T),\\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T),\\ y(\cdot,0) = y_0 & \text{in } (0,\pi), \end{cases}$$

where $D = \operatorname{diag}(1,d)$ $(d>0)$,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$a \in L^2(0,\pi)$, $y = (y_1, y_2)^t$, $y_0 \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v = (v_1, v_2)^t \in L^2(0,T;\mathbb{R}^2)$ (**two boundary controls**).

4. Two applications

Theorem

Under the previous conditions, systems (5) and (6) are ***approximate*** *and* ***null controllable*** *at any time $T > 0$. Moreover, there exist $\tau_0 > 0$ and $C_0, C_1 > 0$ such that*

$$\exp\left(\frac{C_0}{T}\right) \leq \mathcal{K}(T) \leq \exp\left(\frac{C_1}{T}\right), \quad \forall T \in (0, \tau_0).$$

Remark

The previous result is valid in $\mathbb{R}^N$ and for any coupling matrices depending on (x, t): **global Carleman inequalities** for scalar parabolic problems.

4. Some References

1. [Fattorini,Russell] Arch. Rat. Mech. Anal. (1971).
2. [Fernández-Cara, G.-B., de Teresa], J. Funct. Anal. (2010).
3. [Ammar Khodja, Benabdallah, G.-B., de Teresa] J. Math. Pures Appl. (9) **96** (2011).
4. [Benabdallah, Boyer, G.-B., Olive], SIAM J. Control Optim. **52** (2014).
5. [Ammar Khodja, Benabdallah, G.-B., de Teresa], J. Funct. Anal. **267** (2014).
6. [Ouaili], Math. Control Relat. Fields **10** (2020).
7. [Ouaili, G.-B.,] to appear.

Thank you for your attention!!