Stabilization of controllable systems: application to the linearized water tank

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The general question

For large classes of systems:

$$\text{Controllability} \implies \text{Stabilizability}$$

When/how can this be used **constructively**?

Summary

1. Controllability and stabilization
   - Poleshifting
   - F-equivalence
   - F-equivalence and stabilization methods
   - F-equivalence and backstepping
2. Application to the water tank
   - System and results
   - Virtual system, target system
   - Backstepping (or F-equivalence)

Classical pole shifting

Consider the finite-dimensional **controllable** control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}).$$

Kalman condition: $rank\{A^n B \mid n = 0, \cdots, n-1\} = n.$

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Poleshifting: $\forall P,\ \exists K \in \mathcal{M}_{1,n}(\mathbb{C}), \quad \chi(A+BK) = P$.

Idea: Brunovski normal form Idea: write (A, B) in **canonical form**

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_1 & \cdots & \cdots & \cdots & -a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

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Brunovski's ideas

The "real" notion: **F-equivalence**.

Definition

(A, B) and $(\widetilde{A}, \widetilde{B})$ are F-equivalent if there exist invertible matrices T, P and a feedback matrix K such that

$$\widetilde{A} = T(A + BK)T^{-1}, \quad \widetilde{B} = TBP.$$

Focus on the first equation (no drift): conjugation!

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Stabilization: "weak" **F-equivalence** with a stable system.

Solving the equation for specific $\widetilde{A} \to$ explicit feedbacks?

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Gramian and F-equivalence

Gramian method (Komornik 1997, Urquiza 2005, Vest 2013)

Take the particular case

$$\widetilde{A} = A^* - 2\lambda I.$$

Find solutions to this equation by using its symmetries:

$$\widetilde{A}T - TA = TBK$$

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$$A^*T - TA - 2\lambda T = -TBB^*T, \quad K = -B^*T^* = -B^*T.$$

Now assume that T is invertible, and $T^* = T$:

$$T^{-1}A^* - AT^{-1} - 2\lambda T^{-1} = -BB^*$$

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$$e^{-At}T^{-1}A^*e^{(A^*-2\lambda I)t} - e^{-At}AT^{-1}e^{(A^*-2\lambda I)t} - 2\lambda e^{-At}T^{-1}e^{(A^*-2\lambda I)t}$$
$$= -e^{-At}BB^*e^{(A^*-2\lambda I)t}$$

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If $\lambda > 0$ is large enough:

$$T^{-1} = \int_0^\infty e^{-At}BB^*e^{(A^*-2\lambda I)t}dt \ \in S_n(\mathbb{R}).$$

The Gramian matrix is a transformation to a stable system!

One last word on poleshifting

Now take $\widetilde{A} = A - \lambda I$.

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Put (A, B) in canonical form: $A' = PAP^{-1}, \quad (0, \cdots, 1)^\top = B' = PB.$

$$TP^{-1}(A' + B'(KP^{-1}))PT^{-1} = A - \lambda I$$

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Put $(A - \lambda I, B)$ in canonical form: $A'' = Q(A - \lambda I)Q^{-1}, \quad B' = QB$.

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Solution: $T = Q^{-1}P.$

$$TB = B.$$

$TB = B$ and the backstepping equations

Variables not separated:

$$\widetilde{A}T - TA = TBK,$$

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$$\begin{aligned} \widetilde{A}T - TA &= TBK, \\ TB &= B. \end{aligned} \tag{*}$$

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$$\begin{aligned} \widetilde{A}T - TA &= BK, \\ TB &= B. \end{aligned} \qquad (*)$$

Gives a *nice form* to the problem (and actually natural condition!): we recover definition of F-equivalence (with $\widetilde{B} = B$)!

$TB = B$ and the backstepping equations

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Proposition

If (A, B) ***and*** $(\widetilde{A}, B)$ ***are controllable****, then there exists a unique pair* (T, K) *satisfying the backstepping equations.*

Follows from Brunovski in finite dimension. Infinite dimension?

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Krstic's example

Unstable heat equation:

$$\begin{cases} u_t - u_{xx} = \lambda u, \\ u(0) = 0, \quad u(1) = U(t). \end{cases}$$

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Ansatz for T: Volterra Transformation. Always invertible!

$$Tu(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \quad B^*T^* = B^*.$$

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Feedback control design: $U(t) = \int_0^1 k(1,y)u(t,y)dy.$

Boskovic, Balogh, Krstic, MCSS 2003.

Backstepping and F-equivalence

Backstepping for PDEs: transform the system into a stable target system using an invertible transformation.

This is F-equivalence! But...

- No controllability assumption: no canonical form, no Gramian, etc...
- Very strong Ansatz for T replaces this assumption and leads to

$$\widetilde{A}T - TA = BK,$$
$$TB = B.$$

Trade-off: weaker Ansatz, but controllability assumption.

Integro-differential hyperbolic equations (J.-M. Coron, L. Hu, G. Olive 2016), linearized bilinear Schrödinger equation (J.-M. Coron, L. Gagnon, M. Morancey 2018)...

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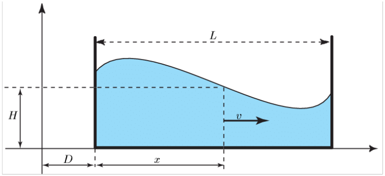
The water tank

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = -u(t), \\ V(t,0) = V(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$



Equilibria:

$(H_0, 0, 0)$ locally "controllable"

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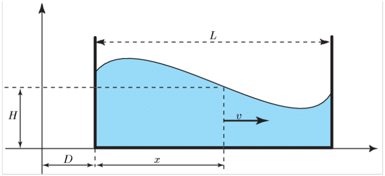

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Explicit feedbacks? First linearize around:

$$(H^\gamma, V^\gamma, \gamma) := (H_0 - \gamma x, 0, \gamma).$$

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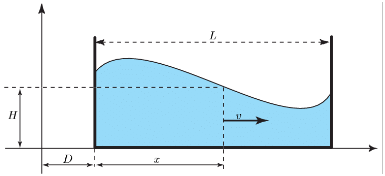

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Result

Theorem (J.-M.Coron, A.Hayat, S.-Q.Xiang, CZ, 2020)

There exists $\gamma_0 > 0$ and $C > 0$ such that, for all $\gamma \leq \gamma_0$, for all $\mu \leq -C\ln(\gamma)$ there exists a feedback law that stabilizes the linearized water tank around (H^γ, V^γ) exponentially, with decay rate μ.

Not a simple linear feedback: Proportional Integral (dynamic extension of the system) control.

$$u(t) = \left\langle \begin{pmatrix} h \\ v \end{pmatrix}(t,\cdot), F_1^\gamma \right\rangle + u_2(t),$$

$$u_2'(t) = \frac{\nu L}{L^\gamma} \langle (h_0^\gamma \; v_0^\gamma)^T, F_1^\gamma \rangle \left(u_2(t) + \left\langle \begin{pmatrix} h \\ v \end{pmatrix}(t,\cdot), F_1^\gamma \right\rangle \right)$$

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Why? Conservation of mass.*

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A really controllable system

After several variable changes (and **vector** notation):

$$\begin{cases} \partial_t \zeta + \Lambda \partial_x \zeta + \delta(x) J \zeta = u(t) \mathcal{I}, \\ \quad \zeta_1(t,0) = -\zeta_2(t,0), \quad \zeta_2(t,1) = -\zeta_1(t,1). \end{cases}$$

$$\delta(x) \sim \gamma/(1-\gamma x), \; J = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \mathcal{I} = e^{\int_0^x \delta(s) ds} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathcal{A} = \Lambda \partial_x + \delta(x) J, \quad f_n, \mu_n, \quad \mu_0 = 0. \tag{1}$$

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Artificial control component:

System is potentially controllable.

Mass no longer conserved.

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A controllable target system

Target:

$$\begin{cases} \partial_t z + \Lambda \partial_x z + \delta(x) J z = 0, \\ z_1(t,0) = -e^{-2\mu L} z_2(t,0), \\ z_2(t,L) = -z_1(t,L). \end{cases}$$

Exponentially stable (proof with Lyapunov function).

Dissipation at the boundary: a technicality?

Important and difficult lemma: choose component νf_0 so that

Lemma

The water tank system, for γ small enough, and the target system, if $\mu \leq -C\ln(\gamma)$, are controllable with the controller $\mathcal{I}_\nu$.

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Backstepping

$$\mathcal{A} := \Lambda \partial_x + \delta(x) J, \quad f_n,\ \mu_n\ (\sim n).$$
$$\widetilde{\mathcal{A}} := \Lambda \partial_x + \delta(x) J, \quad \tilde{f}_n, \widetilde{\phi}_n, \quad \tilde{\mu}_n\ (\sim \mu + i\pi n/L).$$

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For a given feedback K, build T on the basis (f_n).

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$$T\alpha = -\sum_{n \in \mathbb{Z}} \alpha_n \langle f_n, K \rangle \sum_{p \in \mathbb{Z}} \left\langle \mathcal{I}_\nu, \widetilde{\phi}_p \right\rangle / (\widetilde{\mu}_p - \mu_n) \tilde{f}_p, \quad \alpha \in D(\mathcal{A}).$$

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$$\widetilde{\mathcal{A}} := \Lambda\partial_x + \delta(x)J, \quad \tilde{f}_n, \widetilde{\phi}_n, \quad \tilde{\mu}_n\ (\sim \mu + i\pi n/L).$$

$$\widetilde{\mathcal{A}}Tf_n - T\mathcal{A}f_n = BKf_n, \quad TB = B.$$

For a given feedback K, build T on the basis (f_n).

$$g_n := Tf_n, \quad \begin{cases} \Lambda\partial_x g_n + \delta(x)Jg_n - \mu_n g_n = -\langle f_n, K\rangle \mathcal{I}_\nu, \\ g_n^1(0) + e^{-2\mu L}g_n^2(0) = 0,\ g_n^1(L) + g_n^2(L) = 0. \end{cases}$$

$$T\alpha = -\sum_{n\in\mathbb{Z}} \alpha_n \langle f_n, K\rangle \sum_{p\in\mathbb{Z}} \left\langle \mathcal{I}_\nu, \widetilde{\phi}_p \right\rangle / (\widetilde{\mu}_p - \mu_n)\tilde{f}_p, \quad \alpha \in D(\mathcal{A}).$$

Controllability of **target system** $\rightarrow$ Riesz basis of $D(\widetilde{\mathcal{A}})$.

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T invertible iff $|\langle f_n, K\rangle| \sim n$ $(\mu_n\alpha_n \in \ell^2)$.

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Dirichlet-type convergence theorem:

$$\boxed{\langle f_n, K\rangle := -2\tanh(\mu L)\frac{(f_n^1(0))^2}{\langle \mathcal{I}_\nu, f_n\rangle}, \quad \forall n \in \mathbb{Z}.}$$

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Reminder: artificial f_0 component (mass component) has to be removed!

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$$z_0(0) = 0, \begin{cases} \dot{z}_0 = \nu \langle \tilde{\zeta}, K \rangle - 2 z_0 \tanh(\mu L)(f_0^1(0))^2, \\ \partial_t \tilde{\zeta} + \mathcal{A}\tilde{\zeta} = \left\{ \langle \tilde{\zeta}, K \rangle - 2 z_0 \tanh(\mu L)(f_0^1(0))^2 \right\} \mathcal{I}. \end{cases}$$

z_0 is a sort of output/integrator, “virtual spilled/added water”.

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- Underlying idea is old, technical developments are new.

Prospects

For the water tank:

- Stabilization speed is bounded but there's no real obstruction: technicality?
- Stabilization of the nonlinear system for $\gamma = 0$.
- Implementation?

For PDE stabilization in general:

- How do we choose the type of backstepping?
- Would it work with approximate controllability?
- In higher dimensions?

Thank you!

Papers (for the transport equation):

- *Internal rapid stabilization of a 1-D linear transport equation with a scalar feedback*, CZ, accepted by MCRF
- *Finite-time time internal stabilization of a 1-D linear transport equation*,CZ, Systems & Control Letters, Volume 133, 2018.
- Water tank: coming soon...