Radial basis function methods with hybrid kernels applied to control problems

Webinar "Control in Times of Crisis"

Instituto de matemáticas, UNAM

Jorge Zavaleta Sánchez

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Outline

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- Optimal control problem
- Null controllability of the Stokes equation

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Introduction

Radial basis functions (RBF) are highly effective meshfree methods for the solution of PDEs problems. These methods can be divided into global and local techniques and have been applied to several fields, including:

- **Convection-diffusion**, Chandhini (2007), Stevens (2009),
- **Naiver-Stokes**, Chinchapatnam (2009),
- **atmospheric global electric circuit**, Bayona (2010),
- **shallow water simulation**, Flyer (2012),
- **reaction-diffusion on surfaces**, Shankar (2015),
- **time-domain elastic wave propagation in 2D isotropic media**, Buhmann (2015),
- **heat flow**, Buhmann (2017), among others.

Introduction

Control problems

Within the context of the RBF theory, few works address the solution to these problems:

- Optimal control
  - Pearson*. **Poisson** - Global collocation methods.
  - We†. **Poisson** and **convection-diffusion** - Local RBF methods with extended precision.
  - Guan, Wang, Zhu. **Elliptic PDEs.** - Global collocation methods.
- Controllabillity
  - Breton, González Casanova, Montoya‡. **Stokes** - Global collocation and local methods.

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*Pearson, "A radial basis function method for solving PDE-constrained optimization problems", *Num. Alg. 64-3 (2013).*

†González Casanova, Gout, Zavaleta, "Radial basis function methods for optimal control of the convection–diffusion equation: A numerical study", *Eng. Anal. Bound. Elem. 108 (2019).*

‡Breton, González Casanova, Montoya, "RBF collocation and hybrid–LHI methods for Stokes systems and its application to controllability problems", *J. Comput. Appl. Math., Accepted, (2020).*

RBFs methods have the following advantages and limitations:

- They are meshfree methods (*scattered data in* $\mathbb{R}^d$*, able to handle complex boundaries, reduce numerical complexity*).
- Exponential convergence for some kerneles (*high order*).
- Gram matrix have the same structure for all $\mathbb{R}^d$ (*easy to program*).
- Exponential convergence implies exponential increase of the condition number (**uncertainty principle**).
  - Extended precision (Kansa, Shaback).
  - Polyharmonic splines + polynomials (Bayona).
  - Hybrid Kernels (Mishra).

Introduction

- In this talk, we are going to give some insight into RBF methods with hybrid kernels.

- We are going to give some examples of their application to the numerical solution of control problems, specifically:

  - Distributed optimal control problems.

  - Null controllability of Stokes equation.

Outline

RBFs and hybrid kernels

Radial basis functions

- Let $\mathcal{X} = \{\boldsymbol{x}_k\}_{k=1}^{n} \subset \mathcal{D} \subset \mathbb{R}^d$. An RBF interpolant is defined by

$$\sigma(\boldsymbol{x}) = \sum_{k=1}^{n} \lambda_k \phi(r(\boldsymbol{x} - \boldsymbol{x}_k)) + \sum_{\ell=1}^{m} \gamma_\ell p_\ell(\boldsymbol{x}),$$

  with $r := r(\boldsymbol{x}) = \|\boldsymbol{x}\|$ the euclidean norm $\mathbb{R}^d$, $\phi : [0, \infty) \to \mathbb{R}$ a RBF. Also we could have $\Phi(\boldsymbol{x}) = \phi(\|\boldsymbol{x}\|)$.
- Imposing the conditions

$$\sigma(\boldsymbol{x}_k) = f_k, \qquad \text{(interpolation)}$$

$$\sum_{k=1}^{n} \lambda_k p_\ell(\boldsymbol{x}_k) = 0, \ \ell = 1, \ldots, m, \qquad \text{(moment)}$$

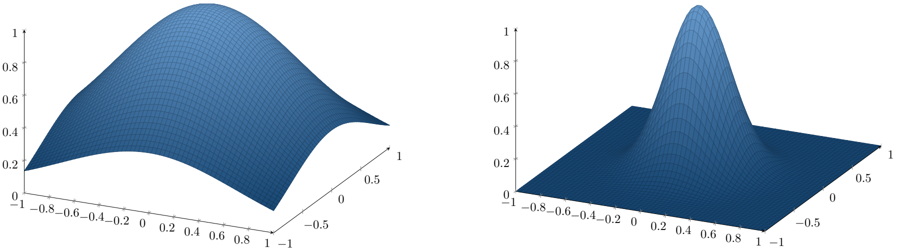
- We get the **algebraic system**

$$A\omega = \begin{pmatrix} G & P \\ P^T & O \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \gamma \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix} = \boldsymbol{b}.$$

Type of basis function	Radial function $\phi(r)$
Polyharmonic splines (PHS)	
Radial powers (Odd)	$r^{2\alpha-1}, \quad 0 < \alpha \in \mathbb{N}$
Thin plate spline (Even)	$r^{2\alpha} \log r \quad 0 < \alpha \in \mathbb{N}$
Gaussian (GA)	$e^{-(\varepsilon r)^2}$
Multiquadric (MQ)	$\sqrt{(\varepsilon r)^2 + 1}$
Inverse multiquadric (IMQ)	$\dfrac{1}{\sqrt{(\varepsilon r)^2 + 1}}$
RBF with compact support	several formulas

ε *is the shape parameter*

RBFs and hybrid kernels
Radial basis functions



$\varepsilon = 1$ $\varepsilon = 3$

Figure: Effect of the shape parameter on a Gaussian kernel

RBFs and hybrid kernels

Radial basis functions

The following table, was taken from Koupaei et al.[§], and shows a review of the proposed strategies to find an "optimal" shape parameter.

Table 2 Some approaches about shape parameter of RBF.

Authors	The shape parameter	Year
Hardy [1]	c = 0.815d, $d=\frac{1}{N}\sum_{i=1}^{N} d_i$, d_i is the distance between the i th data point and its nearest neighbor	1971
Franke [2]	$c = 1.25\frac{D}{\sqrt{N}}$, D is the smallest of the diameter of circle containing all data points	1982
Wahba [5]	He introduced Generalized cross validation (GCV)	1990
Carlson and Foley [6]	This method uses RMS error to determine c parameter	1991
Stein [7]	The method maximizes the likelihood function	1999
Wang and Liu [8]	On the optimal shape parameters of radial basis functions used for 2-D mesh-less methods was proposed in this paper	2002
Afiatdoust and Esmaeilbeigi [9]	They proposed the genetic algorithm to determine good variable shape parameters of RBF for the solution of ODE equations	2014
Uddin [10]	The author extended Rippa's algorithm for selecting good values of shape parameter c in solving time-dependent partial differential equations using radial basis functions	2014
Biazar and Hosami [11]	A new algorithm is suggested to obtain a valid interval in variable shape parameter of RBF	2016

Figure: Strategies to find an "optimal" shape parameter

[§] Koupaei et al., "Finding a good shape parameter of RBF to solve PDEs based on the particle swarm optimization algorithm", *Alex. Eng. J. 57 (2018)*.

RBFs and hybrid kernels
Hybrid kernels

Mishra et al.[¶] introduced hybrid kernels (HK) which are a combination of Gaussian kernels (GA) with radial powers (odd PHS), namely

$$\Phi_H(\boldsymbol{x}) = \Phi_H(\boldsymbol{x}; \varepsilon, \gamma) = e^{-(\varepsilon\|\boldsymbol{x}\|)^2} + \gamma\|\boldsymbol{x}\|^{2\alpha-1}.$$

- The Gaussian component contribute to attain **exponential convergence**
- while the polyharmonic part control the **stability**, namely the grow of the condition number, of the scheme.

[¶] Mishra et al., "Hybrid Gaussian-cubic radial basis functions for scattered data interpolation", *Comput. Geosci. 22-5 (2018)*.

Theorem

Suppose $\Phi : \mathbb{R}^d \to \mathbb{C}$ *is continuous, slowly increasing, and possesses a generalized Fourier transform* $\hat{\Phi}$ *of order* m*, which is continuous on* $\mathbb{R}^d \backslash \{0\}$*. Then* Φ *is conditionally positive definite of order* m *if and only if* $\hat{\Phi}$ *is nonnegative and nonvanishing.*

Theorem (G-C, Zavaleta (2020))

The function $\Phi_H(\mathbf{x}) = \Phi_G(\mathbf{x}) + \gamma\Phi_{PH}(\mathbf{x})$*, where* $\gamma \in (0,1)$*;* $\Phi_G(\mathbf{x}) = \exp^{-a\|\mathbf{x}\|^2}$*;* $\Phi_{PH}(\mathbf{x}) = \|\mathbf{x}\|^b,\ b >,\ b \notin 2\mathbb{N}$*, has the generalized Fourier transform*

$$\hat{\Phi}_H(\omega) = \frac{1}{(\sqrt{2}a)^2}\exp^{-\frac{\|\omega\|^2}{4a^2}} + \gamma\frac{2^{b+d/2}(d+b)/2)}{\Gamma(-b/2)}\|\omega\|_2^{-b-d}, \quad \omega \neq 0$$

which is nonnegative and nonvanishing.

Problem

How to choose parameters ε and γ in such a way that there is a balance between accuracy and conditioning?

Consider $\mathcal{D} \subset \mathbb{R}^d$, $\mathcal{X} = \{\boldsymbol{x}_k\}_{k=1}^n$ in $\mathcal{D}$, and a scaled domain $\mathcal{D}_h$, defined as

$$\mathcal{D}_h = \{\boldsymbol{y} : \boldsymbol{y} = h\boldsymbol{x} \text{ for } \boldsymbol{x} \in \mathcal{D} \text{ and } 0 < h \in \mathbb{R}\}.$$

where h is a scale factor.

We define the following interpolant for $\mathcal{Y} = \{\boldsymbol{y}_k\}_{k=1}^n$ (with $\boldsymbol{y}_k = h\boldsymbol{x}_k \ \forall k$) in $\mathcal{D}_h$ using hybrid kernels

$$\widehat{\sigma}(\boldsymbol{y}) = \sum_{k=1}^{n} \widehat{\lambda}_k \Phi_H(\boldsymbol{y} - \boldsymbol{y}_k; \hat{\varepsilon}, \hat{\gamma})$$

RBFs and hybrid kernels

Hybrid kernels

Setting $\hat{\gamma} = 1$ and $\hat{\varepsilon}$ free, we consider:

$$\widehat{\sigma}(\boldsymbol{y}) = \sum_{k=1}^{n} \widehat{\lambda}_k \Phi_H(\boldsymbol{y} - \boldsymbol{y}_k; \hat{\varepsilon}, 1) \tag{1}$$

What we want to do is characterize the interpolant σ given by (2)

$$\sigma(\boldsymbol{x}) = \sum_{k=1}^{n} \lambda_k \Phi_H(\boldsymbol{x} - \boldsymbol{x}_k; \varepsilon, \gamma) \tag{2}$$

through $\hat{\sigma}$ given by (1).

Proposition (G-C, Zavaleta (2020))

Let σ be the interpolant defined in $\mathcal{D}$ over $\mathcal{X}$ given in (2) and $\hat{\sigma}$ the interpolant defined in $\mathcal{D}_h$ over $\mathcal{Y}$ given in (1). Assume that

$$h = \sqrt[2\alpha-1]{\gamma}, \quad \hat{\varepsilon} = \frac{\varepsilon}{\sqrt[2\alpha-1]{\gamma}} \quad y \quad \lambda_k = \widehat{\lambda}_k \, \forall \, k = 1, \ldots, n,$$

then

$$\hat{\sigma}(\boldsymbol{y}) = \sigma(\boldsymbol{x})$$

for $\boldsymbol{x} \in \mathcal{D}$ and $\boldsymbol{y} = h\boldsymbol{x} \in \mathcal{D}_h$.

Moreover, if $A_{\Phi_H(\cdot;\varepsilon,\gamma),\mathcal{X}}$ and $A_{\Phi_H(\cdot;\hat{\varepsilon},1),\mathcal{Y}}$ are the corresponding Gram matrices of using (2) and (1), it holds that

$$\kappa(A_{\Phi_H(\cdot;\varepsilon,\gamma),\mathcal{X}}) = \kappa(A_{\Phi_H(\cdot;\hat{\varepsilon},1),\mathcal{Y}})$$

RBFs and hybrid kernels

Hybrid kernels

Conjecture (Zavaleta (2020))

Let $\mathcal{X} = \{\mathbf{x}_k\}_{k=1}^{n} \subset \mathcal{D} \subset \mathbb{R}^2$ and consider the interpolant

$$\sigma(\mathbf{x}) = \sum_{k=1}^{n} \lambda_k \Phi_H(\mathbf{x} - \mathbf{x}_k; \varepsilon, \gamma) = \sum_{k=1}^{n} \lambda_k \left(e^{-(\varepsilon \|\mathbf{x}-\mathbf{x}_k\|)^2} + \gamma \|\mathbf{x} - \mathbf{x}_k\|^{2\alpha - 1} \right)$$

Then, for some ω_n, which depends on n, if we take

$$\varepsilon = \frac{\sqrt[2\alpha-1]{\frac{n}{\omega_n}}}{R_{\mathcal{X}}}$$

where $R_{\mathcal{X}}$ is the radii of the of the smallest disk enclosing $\mathcal{X}$, there are constants $c_1, c_2 \in \mathbb{R}$, such that for all $\gamma < c_1$, we have $\kappa(A) < c_2$ and the approximation error is acceptable.

Theorem (G-C, Zavaleta (2020))

For the interpolation with $\Phi_H(\mathbf{x})$ the condition number of the interpolation matrix can be bounded by

$$\kappa\left(A_{\Phi_H,\mathcal{X}}\right) = \frac{\lambda_{max}(A_{\Phi_H,\mathcal{X}})}{\lambda_{min}(A_{\Phi_H,\mathcal{X}})} \leq \frac{C_{\Phi_H} h_{\mathcal{X},\mathcal{D}}^{-d}}{C_d(2a)^{-d/2}\exp^{-M_d^2/(q_{\mathcal{X}}^2 a)} q_{\mathcal{X}}^{-d} + \gamma C_d C_b (2M_d)^{-d-b} q_{\mathcal{X}}^{b}}$$

where $C_{\Phi_H} = \Phi_H(0)$; $\gamma \in (0,1)$, n the total number of nodes, $h_{\mathcal{X},\mathcal{D}}$ is the fill distance and $q_{\mathcal{X}}$ the separation distance and for any $M_d > 0$ satisfying

$$M_d \geq \frac{6.38d}{q_{\mathcal{X}}}.$$

Outline

Optimal control problem

We are interested in finding the solution to the following distributed optimal control problem

$$\min_{y,u} \mathcal{J}_\beta(y,u) = \min_{y,u} \frac{1}{2}\|y-\hat{y}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{s. t. } \mathcal{E}y = u \text{ in } \Omega, \quad \mathcal{B}y = g \text{ on } \partial\Omega$$

Incorporating PDE constraints by means of Lagrange multipliers (**Lagrangian formulation**),

$$\mathcal{L}(y,u,p_1,p_2) = \mathcal{J}_\beta(y,u) + \int_\Omega (\mathcal{E}y-u)\,p_1 + \int_{\partial\Omega} (\mathcal{B}y-g)\,p_2$$

Optimal control problem

Taking Frechet derivative we obtain the following **Euler-Lagrange equations**

$$
\begin{array}{ll|ll}
\mathcal{E}y = u & \text{in } \Omega & \beta\mathcal{E}^{*}u = \hat{y} - y & \text{in } \Omega \\
\mathcal{B}y = g & \text{over } \partial\Omega & u = 0 & \text{over } \partial\Omega
\end{array}
$$

Within the context of the RBF theory, there are few works that address the solution of this problem:

- Pearson. **Poisson** - Global collocation methods.
- We. **Poisson** and **convection-diffusion** - Local RBF methods with extended precision.

Optimal control problem

The methods that have been used to solve the equations were:

- Global asymmetric collocation (MQ,GA,**HK**).

- Local methods.

  - Local asymmetric method with differential quadrature - **LAM-DQ** (**MQ**,GA).

  - Local asymmetric applied twice - **LAM-LAM** (**MQ**,GA).

  - Radial basis-finite difference - **FBR-DF** (**PHS + poly**,**HK**,**HK + poly**)

$$\mathcal{E}y|_{\boldsymbol{x}_k} \approx \sum_{j=1}^{n^{(k)}} w_j y_j, \quad y_j = y(\boldsymbol{x}_j).$$

Optimal control problem

Numerical results

Consider the Poisson control problem,

$$-\triangle y = u, \quad -\beta \triangle u = \hat{y} - y \qquad \text{in } \Omega$$

$$y = g, \qquad u = 0 \qquad \text{on } \partial\Omega$$

$$\hat{y} = \sin \pi x_1 \sin \pi x_2$$

$$g = 0$$

with exact solution given by

$$y_\beta(x_1, x_2) = \frac{1}{1 + 4\beta(\pi)^4} \sin \pi x_1 \sin \pi x_2$$

$$u_\beta(x_1, x_2) = \frac{2\pi^2}{1 + 4\beta(\pi)^4} \sin \pi x_1 \sin \pi x_2$$

Optimal control problem

Numerical results

Method	ε/p	RE_y	RE_u	κ
AC HK	$7.5/-$	7.35E-6	4.32E-4	1.83E14
AC HK+poly	$7.5/7$	2.06E-7	7.52E-6	3.73E14
FBR-DF PHS+poly	$-/7$	1.49E-9	1.31E-7	6.46E12
FBR-DF HK	$-/-$	5.28E-6	5.09E-4	2.90E14
FBR-DF HK+poly	$-/6$	4.35E-9	2.81E-7	3.17E14

Table: Comparison with RBF-FD for $\beta = 10^{-6}$ and $n^{(k)} = 50$ for the local systems. Using RBF-FD with PHS+poly $\Phi_{PH} + p$ $p \in \mathcal{P}_8(\mathbb{R}^2)$, with HK Φ_H and with HK + poly $\Phi_H + p$ $p \in \mathcal{P}_8(\mathbb{R}^2)$, in all cases $\gamma = 10^{-8}$ and ε is variable for each local system calculated with $\varepsilon_k = \dfrac{\sqrt[3]{n^{(k)}}}{4.0846 \cdot R_{D_k}}$. In all cases $n = 622$.

Null controllability of the Stokes equation

Mathematical formulation of the problem

The velocity vector y and pressure p is model via the time dependent **Stokes equations**:

$$\begin{cases} \mathbf{y}_t - \mu\Delta y + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = 0 & \text{on} \quad \Sigma, \\ \mathbf{y}(\cdot,0) = \mathbf{y}_0(\cdot) & \text{in} \quad Q, \end{cases}$$

with $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

Given $\mathbf{y}_0 \in H$ we want to find a control $\mathbf{v} = \mathbf{v}(\mathbf{y}_0) \in L^2(0,T;L^2(\omega)^N)$ with $v_j \equiv 0$ for some $j \in \{1,\ldots,N\}$ such that $\mathbf{y}$ is the solution of the time dependent Stokes equations and:

$$\mathbf{y}(\cdot,T) = 0 \quad \text{in} \quad \Omega.$$

where: $H := \{\mathbf{u} \in L^2(\Omega)^N : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega; \;\; \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$

Null controllability of the Stokes equation

Theoretical description

Let $T > 0$ and $\omega \subset \Omega$ a (small) nonempty open subset which is the control domain and $Q := \Omega \times (0, T)$.

- For $\mathbf{y}_0 \in H$, we consider the **cost functional** J defined by

$$J(\mathbf{v}) := \frac{1}{2} \iint\limits_{\omega \times (0,T)} |\mathbf{v}|^2 \, dx \, dt + \frac{1}{2c_1} \|\mathbf{y}(., T)\|^2_{L^2(\Omega)} \, dx + \frac{1}{2c_2} \iint\limits_{\omega \times (0,T)} |v_j|^2 \, dx \, dt$$

  To calculate the gradient we **introduce the Lagrangian formulation** defined by:

- $\mathcal{L} : [H^1_0(\Omega)^2]^2 \times [L^2_0(\Omega)]^2 \times L^2(Q)^2 \to \mathbb{R}$

$$\begin{aligned}\mathcal{L}(\mathbf{y}, \mathbf{w}, q, p, \mathbf{v}) := \quad & \frac{1}{2} \iint\limits_{\omega \times (0,T)} |\mathbf{v}|^2 \, dx \, dt + \frac{1}{2c_1} \|\mathbf{y}(., T)\|^2_{L^2(\Omega)} \, dx + \frac{1}{2c_2} \iint\limits_{\omega \times (0,T)} |v_j|^2 \, dx \, dt \\ & + \iint\limits_{Q} \mathbf{y}_t \cdot \mathbf{w} + \nabla y \cdot \nabla \mathbf{w} - div(\mathbf{w}) p - div(\mathbf{w}) q \, dx \, dt \\ & - \iint\limits_{\omega \times (0,T)} \mathbf{v} \cdot \mathbf{w} \, dx \, dt + \int_{\Omega} (\mathbf{y}(\cdot, 0) - \mathbf{y}_0) \cdot \mathbf{w}(\cdot, 0) \, dx.\end{aligned}$$

where y is solution of the Stokes system, c_1, c_2 are arbitrary positive numbers associated respectively to the final condition $\mathbf{y}(\cdot, T) = 0$ and the internal control.

Null controllability of the Stokes equation

Adjoint problem and Frechet derivative

It is easy to verify that the Fréchet derivate of J with respect to $\mathbf{v}$ is:

$$\frac{\partial J}{\partial \mathbf{v}}(\mathbf{v}) = v_i - w_i \text{ if } i \neq j \quad \text{and} \quad \frac{\partial J}{\partial \mathbf{v}}(\mathbf{v}) = \frac{1}{2c_2} v_j - w_j, \quad \text{in } \omega \times (0, T).$$

where $\mathbf{w} \in V$ is the solution of the **adjoint system** of Stokes equations:

$$\begin{cases} -\mathbf{w}_t - \nabla \cdot D\mathbf{w} + \nabla q = 0 & \text{in} \quad Q, \\ \nabla \cdot \mathbf{w} = 0 & \text{in} \quad Q, \\ \mathbf{w} = 0 & \text{on} \quad \Sigma, \\ \mathbf{w}(\cdot, T) = -\dfrac{1}{c_1}\mathbf{y}(\cdot, T) & \text{in} \quad \Omega, \end{cases}$$

Numerical solution of the control problem:

Using the conjugate gradient method and LHI Div-free-Hybrid RBF techniques for the Stokes and the adjoint systems we are able to numerical optimized the cost function J.

Wendland[‖] provided the basis for a new discretization scheme for Stokes equations using analytically divergence-free approximation spaces.

- It works in infinite dimensional spaces (Native Spaces), and can produce exponential order approximations.
- Avoid saddle point problems for speed and pressure.
- Use a discretization space for velocity and pressure simultaneously.
- This allows a collocation meshfree method to be used to approximate the PDE solution using positive defined divergence-free matrix-valued kernels.

$$\boldsymbol{\phi}_{Div} = \nabla \times \Delta \times \phi = \{-\Delta I + \nabla\nabla^{T}\}\phi$$

[‖] Wendland, "Divergence-free kernel methods for approximating the Stokes problem", *SIAM J. Numer. Anal. 47-4* (2009).

Null controllability of the Stokes equation

Numerical results

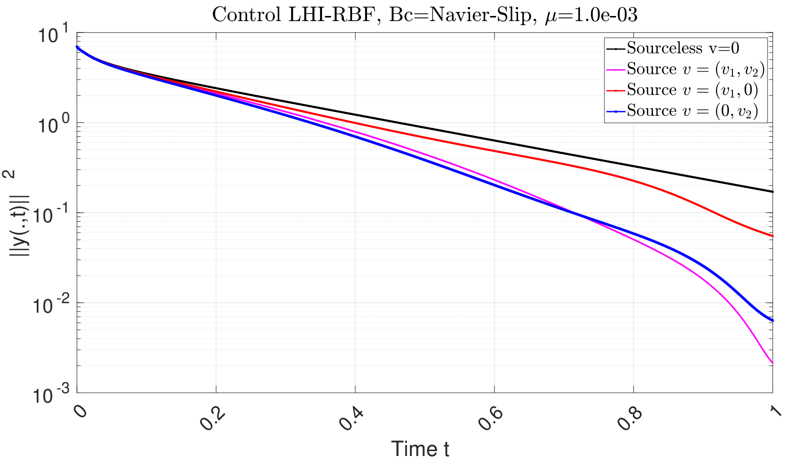


Figure: L^2-norm square solution of the velocity field (as a function of time) for the approximate control problem with controls $\boldsymbol{v} = \mathbf{0}$ (black), $\boldsymbol{v} = (v_1, v_2)$ (pink), $\boldsymbol{v} = (v_1, 0)$ (red) and $\boldsymbol{v} = (0, v_2)$ (blue) with Navier-slip boundary condition. LHI-RBF hybrid kernel, with parameters $\gamma_1 = 1\text{E-}3$, $\gamma_2 = 1\text{E-}8$, $\varepsilon_1 = 1.0$, $\varepsilon_2 = 5\text{E-}10$.

Null controllability of the Stokes equation

Numerical results

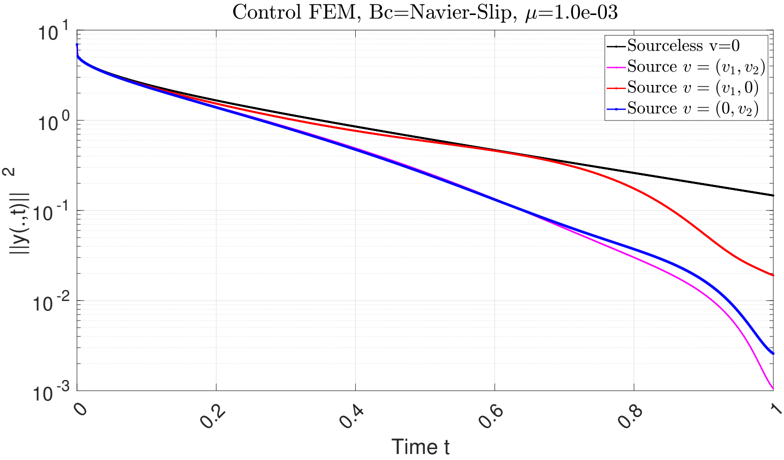


Figure: L^2-norm square solution of the velocity field (as a function of time) for the approximate control problem with controls $\boldsymbol{v} = \mathbf{0}$ (black), $\boldsymbol{v} = (v_1, v_2)$ (pink), $\boldsymbol{v} = (v_1, 0)$ (red) and $\boldsymbol{v} = (0, v_2)$ (blue) with Navier-slip boundary condition. FEM

Outline

Conclusions and future work

- The application of RBFs methods to the solution of test control problems has showed good results, and gives the possibility to extend them to other control problems.

- No LBB or inf/sup condition is necessary. Exponential convergence can be obtained for smooth solutions.

- Navier-slip boundary conditions can be easily incorporated to complex boundaries.

- Also, during the study of the numerical solution of control problems we gave some theoretical and practical insight of how can we apply these methods to these problems.

- Work is in progress to solve the local exact controllability of the Navier-Stokes equation by these methods.

Thank you for your attention