Quantitative rapid and finite time stabilization of the heat equations

Shengquan Xiang
EPFL

Control in Times of Crisis

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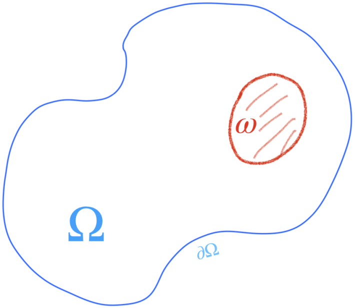
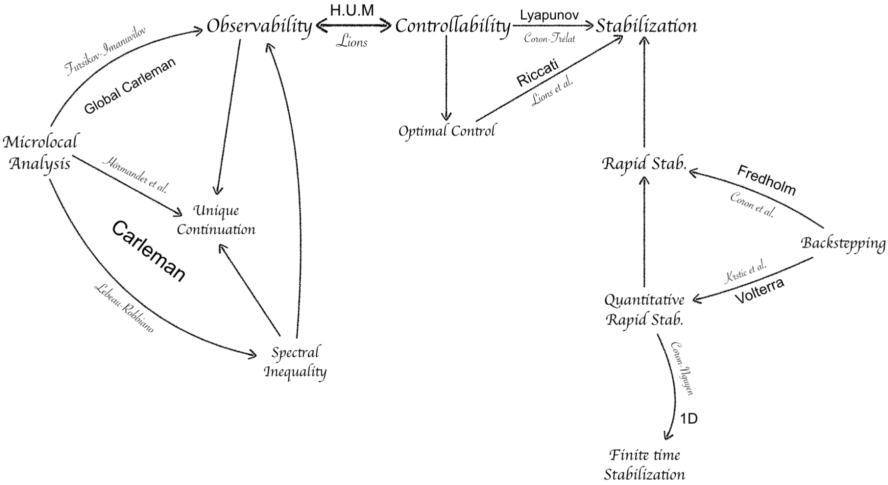
The controlled heat equation

Let Ω be an connected open domain in $\mathbb{R}^d$ with smooth boundary and $\omega \subset \Omega$ an open subset.

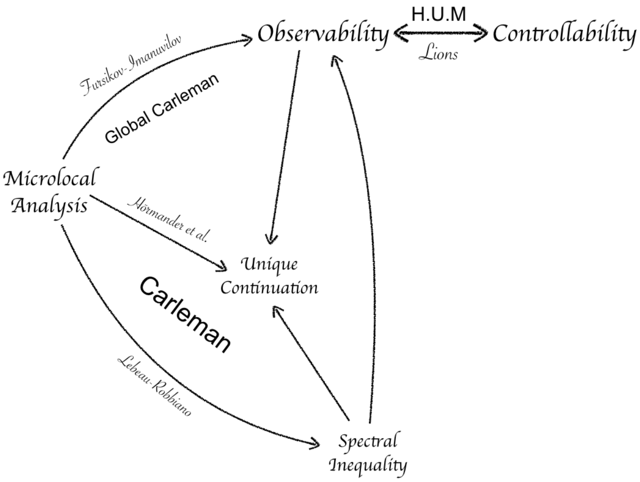
$$\begin{cases} y_t - \Delta y = 1_\omega u, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega. \end{cases}$$

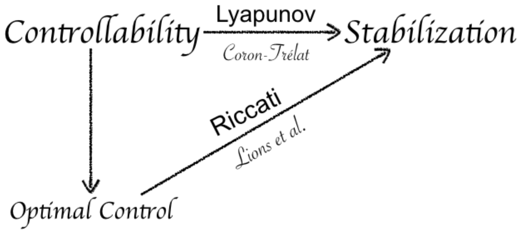


Control theory on the heat equation

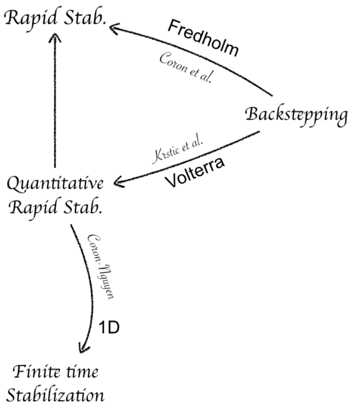


From constructive control to operator theory

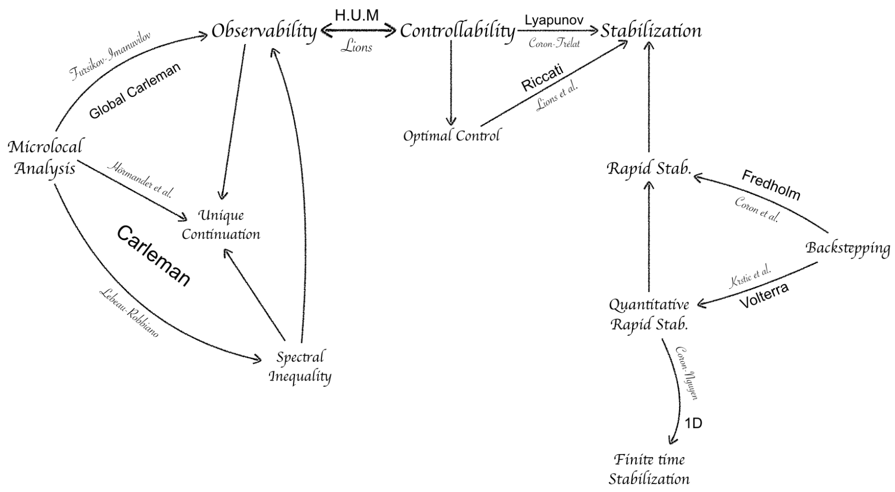


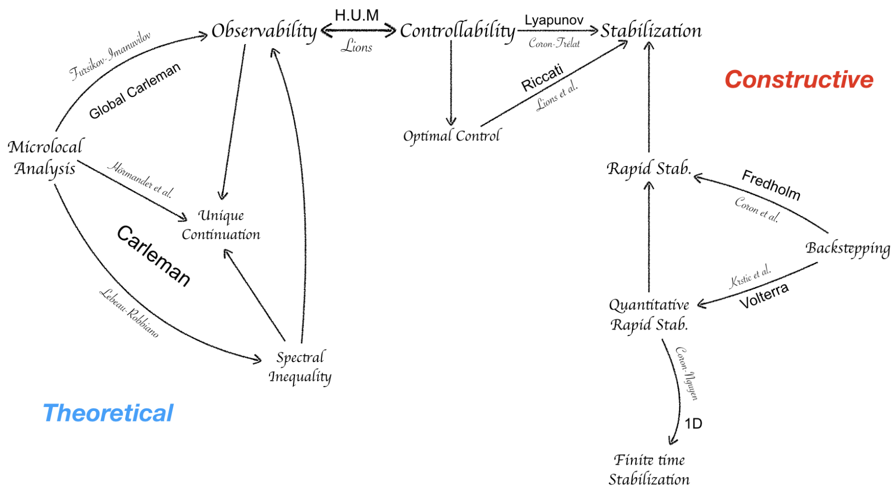

Controllability
Lyapunov
Coron-Trélat
Stabilization
Riccati
Lions et al.
Optimal Control

Finite time stabilization



New connections

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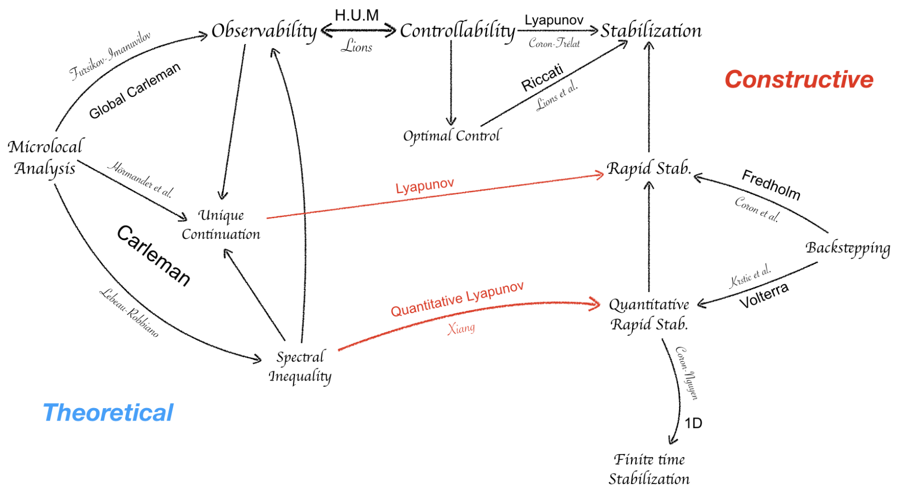
Lebeau-Robbiano
Spectral Inequality
Coron-Nguyen
1D
Finite time Stabilization
Theoretical

New connections



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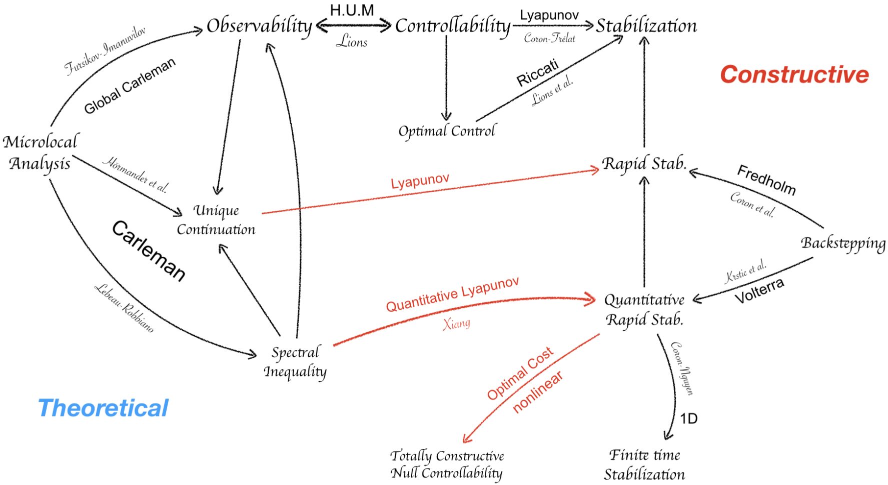

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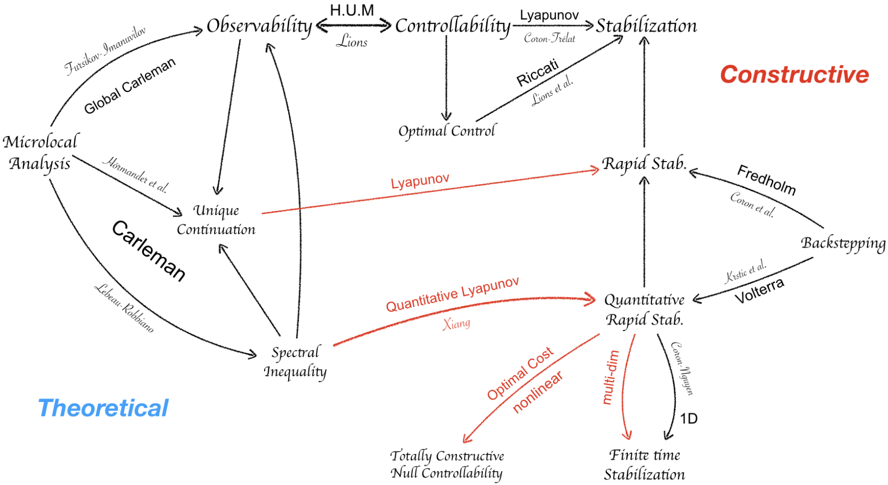

Beyond the classical results

- Rapid stabilization.
- Null controllability with optimal costs.
- Finite time stabilization of the 1D heat equation.

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  Quantitative rapid stabilization via explicit feedback laws.
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  Totally constructive null controllability with optimal costs.
  Also works for Navier–Stokes equations.

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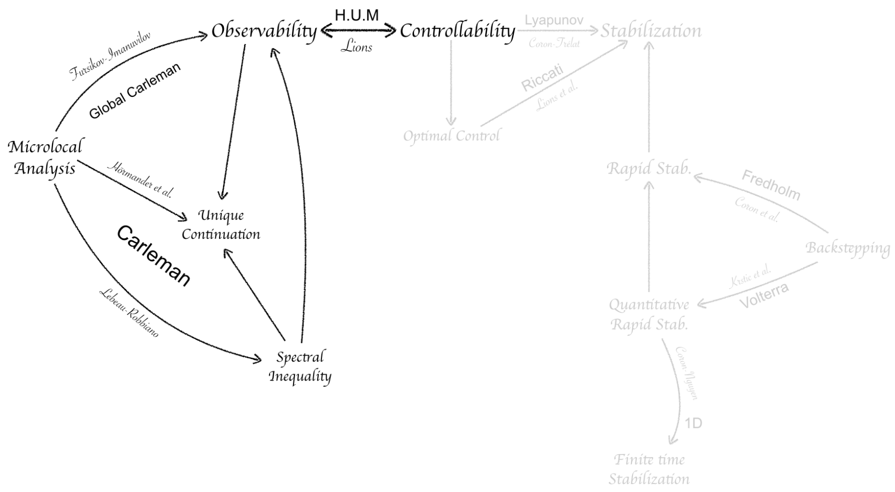
- Finite time stabilization of the 1D heat equation.
  Finite time stabilization of the multidimensional heat equation and the two dimensional Navier–Stokes equations.

Review 1: From constructive control to operator theory



Controllability of

$$\dot{x} = Ax + Bu.$$

Natural constructive approach:
Explicitly solve the system and find the optimal choice.

Lions' observation (H.U.M.):
The exact controllability is equivalent to the observability of the dual operator.

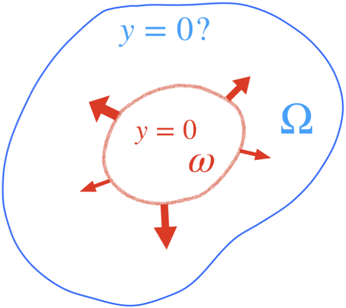
Unique continuation on operators

Given $\omega \subset \Omega$ and a linear differential operator $\mathcal{L}$,

$$\mathcal{L}y = 0 \ \text{ on } \Omega.$$

Observability inequality:
the observation (value) on $y|_\omega$
controls the value of $y|_\Omega$.

Unique continuation:
$y|_\omega = 0$ implies $y|_\Omega = 0$.

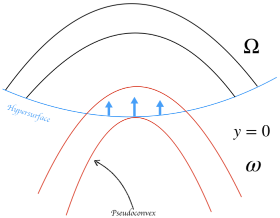


Carleman's innovation

$\mathcal{L}$ be a linear operator,

$$\mathcal{L}y = 0 \;\text{ in } \Omega,$$

$$y|_\omega = 0 \Rightarrow y|_\Omega = 0 \;???$$



- Holmgren: *$\mathcal{L}$ has analytic coefficients.*
- Carleman's idea: *weighted estimates.*
- Calderon: *simple characteristic condition.*
- Hörmander: *peudo-convexity condition.*
- Hörmander-Tataru-Robbiano-Zuily: *conditional peudo-convexity, partially analytic wave type operators.*

Carleman estimates and Observability

Two methods are discovered simultaneously in the 90's.

- ★ Fursikov–Imanuvilov method: *global Carleman*
  nonlinear and non-autonomous systems,
  optimal costs for the nonlinear heat equations.

- ★ Lebeau–Robbiano strategy: *local Carleman (spectral inequality)*
  constructive on linear systems,
  optimal costs for the heat equations and Stokes equations.

Spectral inequality

The Laplace operator has an orthogonal basis of $L^2(\Omega)$,

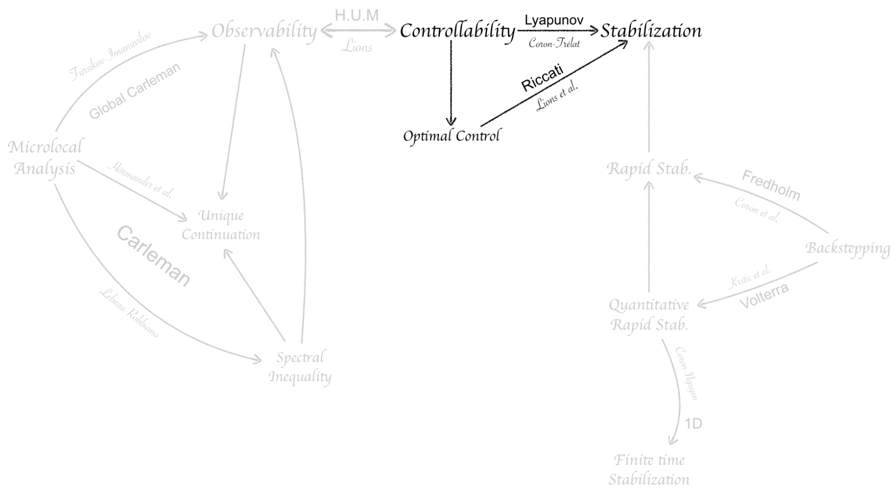
$$\Delta : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega),$$
$$0 < \tau_1 \leq \tau_2 \leq \tau_3 \leq ... \leq \tau_n \leq ...,$$
$$-\Delta e_i = \tau_i e_i \text{ with } e_i|_{\partial\Omega} = 0.$$

Spectral inequality

There exists $C_1 \geq 1$ independent of $\lambda > 0$ such that

$$|| \sum_{\tau_i \leq \lambda} a_i e_i ||^2_{L^2(\omega)} \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} \sum_{\tau_i \leq \lambda} a_i^2.$$

Review 2: exponential stabilization



Center manifold decomposition

$$y_t + Ay = 1_\omega u,$$

with $\{\varphi_k\}_k$ the eigenfunctions.

($\searrow$) H^s, stable manifold, with $y_s = P^s y$ the projection.

($\nearrow$) H^u, unstable manifold, with $y_u = P^u y$ the projection.

$$y_t^s + Ay^s = P^s(1_\omega u) \text{ on } H^s,$$
$$y_t^u + Ay^u = P^u(1_\omega u) \text{ on } H^u.$$

Riccati method

(1) For any y_0^u there exits open loop control $u(t)$,

$$|y^u(t)| + |u(t)| \leq Ce^{-\gamma t}|y_0^u|.$$

(2) For any y_0 there exits $u(t)$,

$$|y^s(t)| + |y^u(t)| + |u(t)| \leq Ce^{-\gamma t}|y_0|.$$

(3) Optimal control problem,

$$Q(y_0) = \min\left\{\frac{1}{2}\int_0^\infty |(-\Delta)^{\frac{3}{4}}y(t)|^2 + |u(t)|^2 dt\right\}.$$

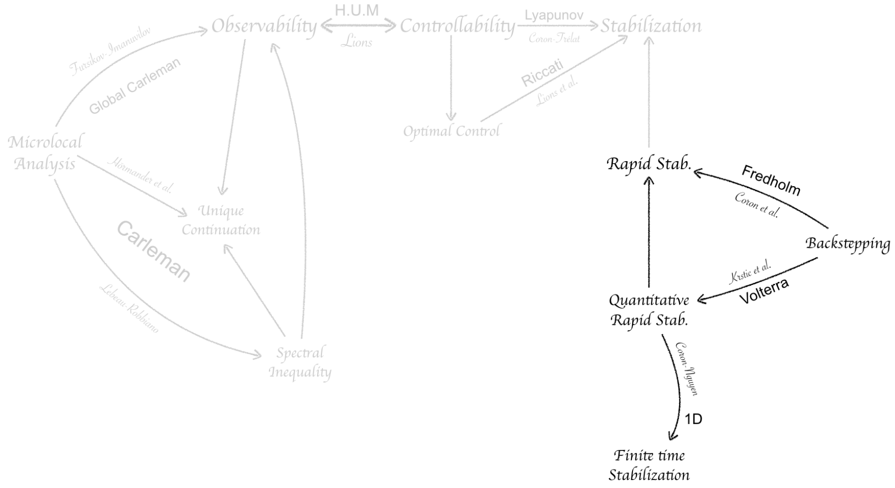
(4) Solve G satisfying nonlinear Riccati equation,

$$(Gy, y) = 2Q(y),$$

$$2(Ay, Gy) + \sum_{1\leq k\leq N}(\varphi_k, Gy)_\omega^2 = |(-\Delta)^{\frac{3}{4}}y|^2.$$

(5) Feedback law $u = \sum(Gy, \varphi_k)\varphi_k$ stabilizes the closed-loop system.

Review 3: finite time stabilization



Finite time stabilization: Let $T > 0$. Find a (time-varying) feedback law such that the system is uniformly stable, moreover, the solutions become 0 after any interval of time greater than T.

Finite time stabilization project

Finite time (global) stabilization for the system which is finite time (global) null controllable.

☺ *Natural*

Finite time stabilization $\xrightarrow{\text{trivial}}$ Finite time null controllability

☹ *Difficult*

Finite time stabilization $\xleftarrow{(???)}$ Finite time null controllability

KdV equation

We consider the KdV system

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = 0 & \text{for } \ (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = a(t) & \text{for } \ t \in (s,+\infty), \\ y(t,L) = 0 & \text{for } \ t \in (s,+\infty), \\ y_x(t,L) = 0 & \text{for } \ t \in (s,+\infty), \end{cases}$$

at time t, the state is $y(t,\cdot) \in L^2(0,L)$ and the control is $a(t)$.

Natural energy dissipation

For the linearized system

$$\begin{cases} y_t + y_{xxx} + y_x = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ y(t,0) = 0 & \text{for } t \in (s,+\infty), \\ y(t,L) = 0 & \text{for } t \in (s,+\infty), \\ y_x(t,L) = 0 & \text{for } t \in (s,+\infty), \end{cases}$$

the energy decays

$$\frac{d}{dt}\|y(t)\|_{L^2}^2 \leqslant 0.$$

A better system with faster decay

We further consider

$$\begin{cases} z_t + z_{xxx} + z_x + \lambda z = 0 & \text{for } (t,x) \in (s,+\infty) \times (0,L), \\ z(t,0) = 0 & \text{for } t \in (s,+\infty), \\ z(t,L) = 0 & \text{for } t \in (s,+\infty), \\ z_x(t,L) = 0 & \text{for } t \in (s,+\infty). \end{cases}$$

Then exponential decay with rate λ,

$$\frac{d}{dt}\|z(t)\|_{L^2}^2 \leqslant -2\lambda\|z(t)\|_{L^2}^2.$$

Rapid stabilization: backstepping (Krstic et al.)

$$\text{Bad} \Leftarrow y \rightsquigarrow z \Rightarrow \text{Good?}$$

$Backstepping$: Find a feedback law and a bounded linear invertible transformation

$$\Pi_\lambda : L^2_y \to L^2_z,$$

such that the flow of y (the solution of KdV with feedback law) is mapped into a flow of z ($z = \Pi_\lambda y$).

$$y(t) = \Pi_\lambda^{-1} S_\lambda(t) \Pi_\lambda y(0),$$

$$\|y(t)\|_{L^2} \leqslant e^{-\lambda t} \|\Pi_\lambda^{-1}\|_{L^2 \to L^2} \|\Pi_\lambda\|_{L^2 \to L^2} \|y(0)\|_{L^2}.$$

Kernel function: Coron–Cerpa (2013)

Volterra transformation

$$z(x) = \Pi_\lambda(y) := y(x) - \int_x^L k(x,r)y(r)dr,$$

with feedback law

$$a(t) = \int_0^L k(0,x)y(t,x)dx.$$

Hence the kernel k verifies

$$\begin{cases} k_{xxx} + k_{yyy} + k_x + k_y + \lambda k = 0 & \text{in } \mathcal{T}, \\ k(x,L) = 0 & \text{on } [0,L], \\ k(x,x) = 0 & \text{on } [0,L], \\ k_x(x,x) = \frac{\lambda}{3}(L-x) & \text{on } [0,L]. \end{cases}$$

Since $\lambda_n > 0$ and

$$\|y(t)\|_{L^2} \leqslant e^{-\lambda_n t} \|\Pi_{\lambda_n}^{-1}\| \|\Pi_{\lambda_n}\| \|y(0)\|_{L^2},$$

there exists t_n such that

$$\|y(t_n)\|_{L^2} \leqslant \frac{1}{2} \|y(0)\|_{L^2}.$$

Null controllability in time T: find $\{\lambda_n\}_n$ such that

$$\sum_n t_n < T < +\infty \Longleftarrow \|\Pi_\lambda\|, \|\Pi_\lambda^{-1}\| \text{ are “well” controlled}.$$

Further modification for finite time stabilization.

“Good enough" estimates

Lemma (Xiang, 2017)

Let $\lambda > 2$. The kernel equation has a unique solution $k_\lambda \in C^3(\mathcal{T})$ satisfying

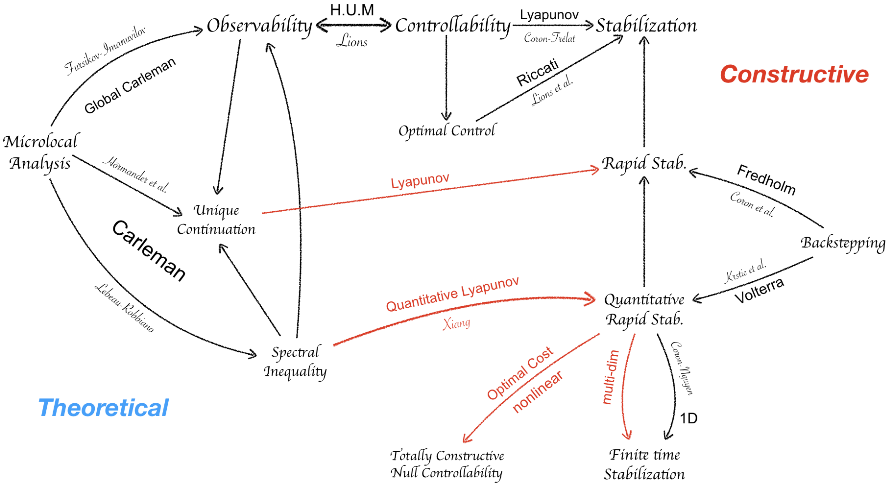
$$\|k_\lambda\|_{C^3(\mathcal{T})} \leqslant e^{(1+L)^2\sqrt{\lambda}}.$$

“$Proof$”:

(1) Construct an explicit solution via iteration.

(2) Give quantitative estimates on this solution.

(3) Prove the uniqueness.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Xiang
Quantitative Rapid Stab.
Spectral Inequality
Optimal Cost
nonlinear
multi-dim
Coron-Nguyen
1D
Theoretical
Totally Constructive Null Controllability
Finite time Stabilization

Motivation: finite time stabilization

$e^{C\sqrt{\lambda}}$ type rapid stabilization

$$\Downarrow$$

finite time stabilization

“Obstruction” on backstepping

Backstepping’s success on 1D models: *linear Schrödinger, viscous Burgers, KdV, linear water tank, parabolic, transport, hyperbolic of conservation laws, Kuramoto-Sivashinsky, degenerate operators, ODE-PDE, PDE-PDE...*
(Krstic, Coron, Christophe Zhang, Amaury Hayat, Bastin, Vazquez, Espitia, Polyakov, Efimov, Perruquetti, Lissy, Liu, Cerpa, Prieur, Hu, Shang, Nguyen, Lü, Olive, Gagnon, Girard, Di Meglio, Morancey, Marx, Steeves, *et al.*)

What about multidimensional models?
☕ "...... the kernel equations seem to be quite complicated......"
—— Coron

Riccati is “complicated”

Nonlinear algebraic Riccati equation

$$Q(y_0) = \min \left\{ \frac{1}{2} \int_0^\infty |(-\Delta)^{\frac{3}{4}} y(t)|^2 + |u(t)|^2 dt \right\},$$
$$2(Ay, Gy) + \sum_{1 \le k \le N} (\varphi_k, Gy)_\omega^2 = |(-\Delta)^{\frac{3}{4}} y|^2,$$
$$(Gy, y) = 2Q(y)...$$

- “Seems that not easy to get quantitative estimates. Regularities..."

—— My feeling

Letters with Trélat

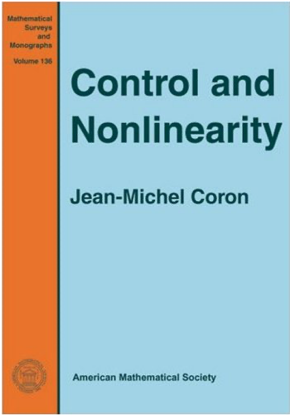
“Cher Emmanuel...... demander un point technique sur l’équation de Riccati...... Amitiés Shengquan"

“Cher Shengquan...... par ailleurs, au cas où ça te serait utile, je t’envoie ce document...... C’est une idée qui remonte à loin : elle est bien expliquée par Russell dans son survey de 1978. Jean-Michel et moi avons utilisé cette idée, en la combinant à une homotopie, pour la contrôlabilité des paraboliques semi-linéaires, dans un article de 2004...... Amitiés Emmanuel"

Coron–Trélat: stabilization for control (quasi static)

Let $L > 0$ and $f \in C^2(\mathbb{R};\mathbb{R})$ with $f(0) = 0$.

$$y_t - y_{xx} - f(y) = 0,\ y(t,0) = 0, y(t,L) = u(t).$$



Jean-Michel Coron,
Control and Nonlinearity,
Mathematical Surveys and Monographs, 136, 2007.

FREELY available from the author's web page.

Coron–Trélat: stabilization for control (quasi static)

Let $L > 0$ and $f \in C^2(\mathbb{R}; \mathbb{R})$ with $f(0) = 0$.

$$y_t - y_{xx} - f(y) = 0,\; y(t,0) = 0, y(t,L) = u(t).$$

After variantion and linearization (simplified), with $A = \Delta + f'(0)id$,

$$z_t - Az = b(x)u,\; z(t,0) = z(t,L) = 0.$$

Let $\{\lambda_i, e_i\}$ be the basis of A with $\lambda_{N+1} < 0 \leq \lambda_N$.

$$z(t,x) = \sum_i z_i(t) e_i(x),$$

$$\dot{z}_i(t) = \lambda_i z_i(t) + b_i u(t).$$

Define $X_N(t) = (z_1(t), ..., z_N(t))^T$,

$$\dot{X}_N(t) = A_N X_N(t) + B_N u(t),$$

with (A_N, B_N) satisfying Kalman condition (via controllability).

$$\exists K_N = (k_1, ..., k_N) \text{ and symmetric positive definite } Q_N,$$
$$Q_N(A_N + B_N K_N) + (A_N + B_N K_N)^T Q_N = -I_N.$$

Take $u := K_N X_N$ and Lyapunov function

$$V(z) = \gamma X_N^T Q_N X_N - (z, Az)_{L^2}.$$

Direct calculation yields

$$\begin{aligned} \dot{V} &= -\gamma |X_N|^2 - 2||Az||^2 - (bK_N X_N, Az), \\ &\leq -\varepsilon V(z), \end{aligned}$$

with suitable choice of γ and small enough ε.

Attempt on the multidimensional heat equation

$$y_t - \Delta y = 1_\omega u,$$
$$0 < \tau_1 \leq ... \leq \tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1} \leq ...,$$
$$-\Delta e_i = \tau_i e_i \text{ with } e_i|_{\partial\Omega} = 0.$$

Observation: the control term $1_\omega u(t)$ have infinite dimensional degree, compared to the degree 1 of 1D model.

$$y(t) := \sum_{i=1}^{\infty} y_i(t) e_i, \quad u(t) := \sum_{i=1}^{N(\lambda)} e_i u_i(t),$$
$$1_\omega e_j = \sum_{i=1}^{\infty} \left(1_\omega e_j, e_i\right)_{L^2(\Omega)} e_i = \sum_{i=1}^{\infty} \left(e_i, e_j\right)_{L^2(\omega)} e_i.$$

Unique continuation implies stabilization

$$X_N := \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ y_N \end{pmatrix}, U_N := \begin{pmatrix} u_1 \\ u_2 \\ \ldots \\ u_N \end{pmatrix}, A_N := \begin{pmatrix} -\tau_1 & & & \\ & -\tau_2 & & \\ & & \ldots & \\ & & & -\tau_N \end{pmatrix}$$

the low frequency satisfies

$$\dot{X}_N(t) = A_N X_N(t) + J_N U_N(t),$$
$$J_N := \left((e_i, e_j)_{L^2(\omega)}\right)_{i,j=1}^N .$$

Thanks to unique continuation, the symmetric matrix J_N is invertible: (A_N, J_N) verifies Kalman condition, by Coron-Trélat method get exponential stabilization.

Quantitative rapid stabilization?

Quantitate matrix J_N

$$J_N = \left((e_i, e_j)_{L^2(\omega)}\right)_{i,j=1}^{N} \ : \ \text{symmetric, positive definite.}$$

We choose $U_N(t) := -\gamma_\lambda X_N(t)$,

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t),$$

and propose Lyapunov function,

$$V(z) = \gamma X_N^T Q_N X_N - (z, Az)_{L^2},$$
$$Q_N(A_N + J_N K_N) + (A_N + J_N K_N)^T Q_N = -I_N.$$

Thus,

$$\frac{1}{2}\frac{d}{dt} X_N^T X_N = X_N^T A_N X_N - \gamma_\lambda X_N^T J_N X_N.$$

Quantitate matrix J_N

$$J_N = \left((e_i, e_j)_{L^2(\omega)}\right)_{i,j=1}^{N} : \text{ symmetric, positive definite.}$$

We choose $U_N(t) := -\gamma_\lambda X_N(t)$,

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t),$$

and propose Lyapunov function,

$$V(y) = \mu_\lambda X_N^T X_N + ||P_N^\perp y||_{L^2}^2 = \mu_\lambda ||P_N y||_{L^2}^2 + ||P_N^\perp y||_{L^2}^2.$$

Thus,

$$\frac{1}{2}\frac{d}{dt} X_N^T X_N = X_N^T A_N X_N - \gamma_\lambda X_N^T J_N X_N.$$

Quadratic form J_N and Spectral inequality

Quantitative estimate of $X_N^T J_N X_N$?

Quadratic form J_N and Spectral inequality

Quantitative estimate of $X_N^T J_N X_N$? looks like $(\sum a_i e_i)^2$...

Quadratic form J_N and Spectral inequality

Quantitative estimate of $X_N^T J_N X_N$? looks like $(\sum a_i e_i)^2$...

Spectral inequality

There exists $C_1 \geq 1$ *independent of* $\lambda > 0$ *such that*

$$|| \sum_{\tau_i \leq \lambda} a_i e_i ||^2_{L^2(\omega)} \geq C_1^{-1} e^{-C_1 \sqrt{\lambda}} \sum_{\tau_i \leq \lambda} a_i^2 .$$

Quadratic form J_N and Spectral inequality

Quantitative estimate of $X_N^T J_N X_N$? looks like $(\sum a_i e_i)^2$...

Spectral inequality

There exists $C_1 \geq 1$ *independent of* $\lambda > 0$ *such that*

$$||\sum_{\tau_i \leq \lambda} a_i e_i||^2_{L^2(\omega)} \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} \sum_{\tau_i \leq \lambda} a_i^2.$$

$$X_N^T J_N X_N \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} ||X_N||_2^2$$

Proof: remind that $N = N(\lambda)$, $i.e.\ \tau_{N(\lambda)} \leq \lambda < \tau_{N(\lambda)+1}$,

$$X_N^T J_N X_N = \sum_{1\leq i,j\leq N} a_i \left(e_i, e_j\right)_{L^2(\omega)} a_j = \left(\sum_{1\leq i\leq N} a_i e_i, \sum_{1\leq j\leq N} a_j e_j\right)_{L^2(\omega)}$$

$$= ||\sum_{1\leq i\leq N} a_i e_i||^2_{L^2(\omega)} \geq C_1^{-1} e^{-C_1\sqrt{\lambda}} ||X_N||_2^2.$$

Quantitative rapid stabilization

Let $\lambda > 0$. For γ_λ and μ_λ to be chosen later on,

$$U_N(t) := \gamma_\lambda X_N(t), \; V(y) = \mu_\lambda ||X_N||_2^2 + ||P_N^\perp y||_{L^2}^2.$$

We have

$$\dot{X}_N(t) = A_N X_N(t) - \gamma_\lambda J_N X_N(t),$$
$$y_t = \Delta y - \gamma_\lambda 1_\omega \left(\sum_{i=1}^{N} e_i y_i(t) \right) \text{ in } \Omega,$$

thus

$$\frac{d}{dt} V\left(y(t)\right) = 2\mu_\lambda X_N^T \dot{X}_N + 2 \left\langle P_N^\perp y, \frac{d}{dt} y \right\rangle_{H_0^1(\Omega) \times H^{-1}(\Omega)}.$$

On the one hand,

$$2\mu_\lambda X_N^T \dot{X}_N = 2\mu_\lambda X_N^T \big(A_N - \gamma_\lambda J_N\big) X_N \leq -2\mu_\lambda \gamma_\lambda C_1^{-1} e^{-C_1\sqrt{\lambda}} ||X_N||_2^2.$$

On the other hand

$$\begin{aligned}
2\left\langle P_N^\perp y, \frac{d}{dt} y \right\rangle &= 2\left\langle P_N^\perp y, \Delta y \right\rangle - 2\gamma_\lambda \left(P_N^\perp y, 1_\omega \left(\sum_{i=1}^{N} e_i y_i(t) \right) \right)_{L^2(\Omega)}, \\
&= -2 \sum_{i=N+1}^{\infty} \tau_i y_i^2 - 2\gamma_\lambda \left(P_N^\perp y, 1_\omega (P_N y) \right)_{L^2(\Omega)}, \\
&\leq -2\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + 2\gamma_\lambda ||P_N^\perp y||_{L^2(\Omega)} ||P_N y||_{L^2(\Omega)}, \\
&\leq -2\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda} ||P_N y||_{L^2(\Omega)}^2, \\
&= -\lambda ||P_N^\perp y||_{L^2(\Omega)}^2 + \frac{\gamma_\lambda^2}{\lambda} ||X_N||_2^2.
\end{aligned}$$

On the choice of γ_λ and μ_λ

$$V(y) = \mu_\lambda ||X_N||_2^2 + ||P_N^\perp y||^2_{L^2(\Omega)}.$$

Take

$$\gamma_\lambda := C_1 e^{C_1\sqrt{\lambda}}\lambda, \quad \mu_\lambda := \frac{\gamma_\lambda^2}{\lambda^2} = C_1^2 e^{2C_1\sqrt{\lambda}}.$$

Then,

$$\begin{aligned}
\frac{d}{dt}V\left(y(t)\right) &\leq -2\mu_\lambda\gamma_\lambda C_1^{-1}e^{-C_1\sqrt{\lambda}}||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)} + \frac{\gamma_\lambda^2}{\lambda}||X_N||_2^2, \\
&= -2\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)} + \mu_\lambda\lambda||X_N||_2^2, \\
&= -\mu_\lambda\lambda||X_N||_2^2 - \lambda||P_N^\perp y||^2_{L^2(\Omega)} = -\lambda V(y(t)).
\end{aligned}$$

Quantitative rapid stabilization:

$$||y(t)||^2_{L^2} \leq V(y(t)) \leq e^{-\lambda t}V(y(0)) \leq C_1^2 e^{2C_1\sqrt{\lambda}} e^{-\lambda t}||y(0)||^2_{L^2}!!!$$

Theorem (Xiang, 2020)

For any $\lambda > 0$ the closed-loop system

$$y_t - \Delta y = -\gamma_\lambda 1_\omega \left(P_{N(\lambda)} y\right) \ \textit{in } \Omega,$$
$$y = 0 \ \textit{ on } \partial\Omega,$$

is exponentially stable:

$$||y(t)||_{L^2(\Omega)} \leq C_1 e^{C_1\sqrt{\lambda}} e^{-\frac{\lambda t}{2}} ||y_0||_{L^2(\Omega)}.$$

$Remark$: without the spectral inequality, by a good choice of γ and μ we can still get rapid stabilization, but no estimate of the form $e^{C\sqrt{\lambda}}$.

Beyond the classical results

- Rapid stabilization.
  Quantitative rapid stabilization via explicit feedback laws.

- Null controllability with optimal costs.
  Totally constructive null controllability with optimal costs.
  Also works for Navier–Stokes equations.

- Finite time stabilization of the 1D heat equation.
  Finite time stabilization of the multidimensional heat equation and the two dimensional Navier–Stokes equations.

Constructive null controllability

For any $\lambda_n > 0$, there exists t_n such that

$$\|y(t)\|_{L^2}^2 \leq C_1^2 e^{2C_1\sqrt{\lambda_n}} e^{-\lambda_n t} \|y(0)\|_{L^2}^2,$$
$$\|y(t_n)\|_{L^2} \leq \frac{1}{2} \|y(0)\|_{L^2}.$$

Constructive null controllability by selecting $\{\lambda_n\}_n$ such that

$$\sum_n t_n < T < +\infty.$$

An advice from Fernández-Cara

☕ "...... dans les problèmes que tu as considérés, il serait très intéressant de déduire des estimations du coût du contrôle, c'est-à-dire, de la plus petite norme d'un contrôle qui fait le travail......"

—— Fernández-Cara

Optimal cost for null controllability

There exists $C > 0$ such that the cost of null controllability satisfies,

$$||u||_{L^2(0,T;L^2(\omega))} \leq Ce^{\frac{C}{T}}, \ \forall T \in (0,1),$$

where the form $\frac{1}{T}$ is optimal.

- Spectral inequality (Seidman, Miller, *et al.*)
  the heat equation, also $\frac{1}{T}$ for Stokes.
- Global Carleman (Imanuvilov, Fernández-Cara, Puel, Zuazua, Guerrero, *et al.*)
  nonconstructive, nonlinear heat systems,
  $\frac{1}{T^4}$ for Stokes, $\frac{1}{T^9}$ for Navier–Stokes.

Totally constructive null control with optimal costs

Remember C_1 from spectral inequality. Take $C_2 > C_1$ such that

$$8(1+\lambda)C_1^2 e^{2C_1\sqrt{\lambda}} \leq C_2 e^{C_2\sqrt{\lambda}}, \ \forall\, \lambda > 0.$$

Select $Q > 0$ such that

$$C_2 e^{C_2 Q m} \leq e^{\frac{Q^2}{64} m}, \ \forall\, m \geq 1.$$

Let $1/T \in (2^{n_0-1}, 2^{n_0}]$ with $n_0 \in N^*$. For every $n \in \mathbb{N}$, we define

$$T_n := 2^{-n_0}\left(1 - \frac{1}{2^n}\right), I_n := [T_n, T_{n+1}), \lambda_n := Q^2 2^{2(n_0+n)}.$$

Define $C_3 = Q^2/32$, the cost is given by

$$||u||_{L^\infty(0,T;L^2(\omega))} \leq C_3 e^{\frac{C_3}{T}}, \ \forall T \in (0,1).$$

Optimal cost for null controllability

Totally constructive approach:

$$||u||_{L^\infty(0,T;L^2(\omega))} \leq C_3 e^{\frac{C_3}{T}}, \ \forall T \in (0,1).$$

Optimal cost for null controllability

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- nonlinear heat equations, $e^{C/T}$;
- Navier–Stokes equations, $e^{C/T}$;
- to be discovered...

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Finite time stabilization

Some modification on the constructive null controllability.

Theorem (Xiang, 2020)

For any $\Lambda \geq 1$, for any $T > 0$, we construct an explicit T-periodic proper feedback law U that stabilizes the heat equation in finite time:

(i) *($2T$ stabilization)* $\Phi(2T + t, t; y_0) = 0, \ \ \forall t \in \mathbb{R}, \ \forall ||y_0|| \leq \Lambda.$

(ii) *(Uniform stability) For every $\delta > 0$ there exists an effectively computable $\eta > 0$ such that*

$$(||(y_0||_{L^2} \leq \eta) \Rightarrow \left(||\Phi(t, t'; y_0)||_{L^2} \leq \delta, \ \forall t' \in \mathbb{R}, \ \forall t \in (t', +\infty)\right).$$

The same for two dimensional incompressible internal controlled Navier–Stokes equations. (Xiang, 2020)

Thank you for your attention!