Hierarchical control of the semilinear heat equation
Webinar Control in Time of Crisis

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Cooperative games

1. The notion of game theory began with Gabriel Cramer in 1782 and Daniel Bernoulli in 1738.
2. Publications of John Nash, Von Newman and notion of cooperation between participants in a game.
3. The idea of Hierarchical control is that a participant called follower reacts by the actions of another participant called leader.

Classical hierarchical control problem [J.L. Lions]

Let ω and $\mathcal{O}$ open sets in Ω.

$$\begin{array}{ll} y_t - \Delta y = f 1_\omega + v 1_{\mathcal{O}} & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y^0(x) & \text{in } \Omega \end{array} \tag{1}$$

Functions f and v are called follower and leader respectively.

1. The leader control must control the solution y approximate to zero, i.e $\|y(T)\|_{L^2(\Omega)} \leq \varepsilon$.
2. The follower control should minimise a functional.

Objectives

1. Optimal control : $y \approx y_d$ in $\mathcal{O}_d \times (0,T)$

$$\min_{v} \frac{1}{2} \int_{\mathcal{O}_d \times (0,T)} |y - y_d|^2 + \frac{\beta}{2} \int_{\omega \times (0,T)} |f|^2, \quad \beta > 0.$$

2. Aproximate controllability : for each $\epsilon > 0$ find v such that $||y(T)||_{L^2(\Omega)} < \epsilon$.

Hierarchical control, with control leader satisfying :

1. Lions (1994) proved that there exist a control v such that the solution is approximately controllable – $\|y(T)\|_{L^2(\Omega)} \leq \varepsilon$.
2. Araruna et al. (2015) proved when $\omega \cap \mathcal{O}_d \neq \emptyset$ and

$$\boxed{\iint_Q \rho^2 |y_d|^2 < \infty} \qquad \text{for } \rho \to \infty \text{ as } t \to T,$$

then the solution is null controllable.

Inverted roles in hierarchical control problem

. Let Ω, ω and $\mathcal{O} \subset \Omega$ open sets

$$\begin{array}{ll} y_t - \Delta y = f1_\omega + v1_{\mathcal{O}} & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y(x,0) = y_0(x) & \text{in } \Omega \end{array} \tag{2}$$

1. The follower control f steers the solution to zero in positive time, i.e. $y(T) = 0$.
2. The leader control v must minimise a functional.

Stackelberg strategy

1. Given $v \in \mathcal{V}$ find $f[v]$ such that

$$\begin{cases} \text{Minimize} S(v; f) := \frac{1}{2} \int_Q \rho^2 |y|^2 + \frac{1}{2} \int_{\omega \times (0,T)} \rho_0^2 |f|^2 \\ \text{Subject to } f \in \mathcal{F}. \end{cases} \tag{3}$$

2. Then, we look for admissible controls $\hat{v} \in \mathcal{U}$ satisfying

$$P(\hat{v}; f[\hat{v}]) = \min_v P(v; f[v]) \tag{4}$$

where

$$P(v; f) = \frac{\alpha}{2} \int_{\mathcal{O}_d \times (0,T)} |y - y_d|^2 + \frac{\mu}{2} \int_{\mathcal{O} \times (0,T)} |v|^2, \quad \beta > 0.$$

Basic definitions

Let $\eta_0 = \eta_0(x)$ be a function satisfying

$$\eta_0 \in C^2(\overline{\Omega}), \eta_0 > 0 \text{ in } \Omega, \eta_0 = 0 \text{ on } \partial\Omega, \quad |\nabla \eta_0| > 0 \text{ in } \overline{\Omega} \setminus \omega.$$

Introduce the weight functions

$$\sigma(x,t) := \frac{e^{4\lambda\|\eta^0\|_\infty} - e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)}, \quad \xi(x,t) := \frac{e^{\lambda(2\|\eta^0\|_\infty + \eta^0(x))}}{\ell(t)},$$

$$\rho := e^{s\sigma}, \quad \rho_0 := (s\xi)^{-3/2}\lambda^{-2}\rho, \quad \rho_1 := (s\xi)^{-1/2}\lambda^{-1}\rho, \quad \rho_2 := (s\xi)^{1/2}\rho,$$

where $\ell \in C^\infty([0,T])$ satisfies
$\ell(t) \geq T/2$ in $[0, T/2]$ and $\ell(t) = t(T-t)$ in $[T/2, T]$ and
$\lambda, s > 0$ are large enough.

Weighted spaces

For the weight ρ_0 define

$$\mathcal{U} := \{\, v : \rho_0 v \in L^2(\omega \times (0,T)) \,\}, \tag{5}$$

$$\mathcal{Y} := \{\, y : \rho y \in L^2(Q) \,\}, \quad \mathcal{F} := \{\, f : \rho_0 f \in L^2(\mathcal{O} \times (0,T)) \,\}.$$

$\mathcal{P}_0 := \{\, p \in C^2(\overline{Q}) : p = 0 \;\text{ on }\; \Sigma \,\}.$

$$m(a; p, p') := \iint_Q \left(\rho^{-2} L_a^* p \, L_a^* p' + 1_{\mathcal{O}} \rho_0^{-2} p \, p'\right).$$

The linear case

$F(y) = a(t,x)y$ with a in $L^\infty(Q)$.

Let ω and $\mathcal{O}$ open sets in Ω

$$\begin{cases} y_t - \Delta y + F(y) = f1_{\mathcal{O}} + v1_\omega \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \\ y(\cdot, 0) = y_0 \text{ in } \Omega \end{cases} \tag{6}$$

with the sets $\mathcal{O}_d \cap \omega \neq \emptyset$.

Characterisation of the follower

Proposition

Let v be given in $\mathcal{U}$. Then, there exists exactly one solution $f[v] \in \mathcal{F}$ to the null controllability problem, where S is given and y is the solution. Furthermore, one has

$$f[v] = -\rho_0^{-2} p\big|_{\mathcal{O}\times(0,T)}, \quad y = \rho^{-2} L_a^* p, \tag{7}$$

where $p \in \mathcal{P}$ is the unique solution to the linear problem

$$m(a; p, p') = \ell(v; p'), \quad \forall p' \in \mathcal{P} \tag{8}$$

and we have used the notation

$$\ell(v;p) := \iint_{\omega\times(0,T)} v\, p' + \int_\Omega y_0(x)\, p'(x,0)\, dx.$$

Carleman inequalities

Theorem

There exist positive constants λ_0, s_0 and C_0, only depending on Ω, $\mathcal{O}$ and T, such that, if we take $\lambda = \lambda_0$ and $s = s_0$, any $p \in \mathcal{P}$ satisfies

$$\iint_Q \left[\rho_2^{-2}(|p_t|^2 + |\Delta p|^2) + \rho_1^{-2}|\nabla p|^2 + \rho_0^{-2}|p|^2\right] \leq C_0\, m(0; p, p). \tag{9}$$

Furthermore, λ_0 and s_0 can be found arbitrarily large.

Sketch of the proof (Characterisation of the follower)

The function $f \mapsto S(v; f)$ is coercive, lower semi-continuous, convex and proper in $\mathcal{F}$.

Then $p 1_{[0,T/2]} \in C^0([0,T/2]; H_0^1(\Omega))$
Therefore, the linear mapping $p \mapsto p(\cdot\,,0)$ is well defined and continuous.

The fourth order equation

In other words, p solves (8) if and only if $p \in \mathcal{P}$ and one has

$$\begin{cases} L_a(\rho^{-2} L_a^* p) + \rho_0^{-2} p 1_{\mathcal{O}\times(0,T)} = v 1_{\omega\times(0,T)} \text{ in } Q, \\ p = 0, \quad \rho^{-2} L_a^* p = 0 \text{ on } \Sigma, \\ \rho^{-2} L_a^* p\big|_{t=0} = y_0, \quad \rho^{-2} L_a^* p\big|_{t=T} = 0 \text{ in } \Omega. \end{cases} \tag{10}$$

Characterisation of leader

Theorem

The unique solution $\hat{v}$ to (4) satisfies, together with the associated $\hat{y}$, $\hat{p}$, $\hat{\phi}$ and $\hat{\psi}$, the following optimally system :

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + a(x,t)\hat{y} = f[\hat{v}]1_{\mathcal{O}} + \hat{v}1_\omega \text{ in } Q, \\ \hat{y} = 0 \text{ on } \Sigma, \quad \hat{y}(\cdot,0) = y_0 \text{ in } \Omega, \end{cases} \tag{11}$$

$$f[\hat{v}] = -\rho_0^{-2}\hat{p}\big|_{\mathcal{O}\times(0,T)}, \quad \hat{y} = \rho^{-2}L_a^*\hat{p}, \quad \hat{p}\in\mathcal{P}, \tag{12}$$

$$m(a;\hat{p},p') = \iint_{\omega\times(0,T)} \hat{v}\,p' + \int_\Omega y_0(x)\,p'(x,0)\,dx, \quad \forall p'\in\mathcal{P}, \tag{13}$$

Characterisation of leader

$$\begin{cases} -\hat{\phi}_t - \Delta\hat{\phi} + a(x,t)\hat{\phi} = \alpha(\hat{y} - y_d)1_{\mathcal{O}_d} \text{ in } Q, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot, T) = 0 \text{ in } \Omega, \end{cases} \tag{14}$$

$$m(a; p', \hat{\psi}) = -\iint_{\mathcal{O}\times(0,T)} \rho_0^{-2}\hat{\phi}\, p' \quad \forall p' \in \mathcal{P}, \quad \hat{\psi} \in \mathcal{P}, \tag{15}$$

$$\hat{v} = -\frac{1}{\mu}\rho_0^{-2}(\hat{\psi} + \hat{\phi})\big|_{\omega\times(0,T)}. \tag{16}$$

Sketch of the proof

Introduce this systems

$$\begin{cases} z_t - \Delta z + a(x,t)z = g1_{\mathcal{O}} + w1_{\omega} \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \quad z(\cdot,0) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -\phi_t - \Delta \phi + a(x,t)\phi = \alpha(y - y_d)1_{\mathcal{O}_d} \text{ in } Q, \\ \phi = 0 \text{ on } \Sigma, \quad \phi(\cdot,T) = 0 \text{ in } \Omega, \end{cases}$$

Sketch of the proof

$$\frac{d}{d\varepsilon} P(f[v+\varepsilon w]; v+\varepsilon w)\Big|_{\varepsilon=0} = \alpha \iint_{\mathcal{O}_d\times(0,T)} (y-y_d)\, z + \mu \iint_{\omega\times(0,T)} \rho_0^2\, v\, w = \iint_{\mathcal{O}\times(0,T)} \phi\, g + \iint_{\omega\times(0,T)} \left(\phi + \mu\rho_0^2 v\right) w,$$

Note that $g = -\rho_0^{-2} q\big|_{\omega\times(0,T)}$, whence

$$\iint_{\mathcal{O}\times(0,T)} \phi\, g = -\iint_{\mathcal{O}\times(0,T)} \rho_0^{-2} \phi\, q = m(a; q, \psi) = \iint_{\omega\times(0,T)} \psi\, w.$$

Indeed, let us denote by F_0 the function given by

$$F_0(\xi) = \frac{F(\xi)}{\xi} \text{ if } \xi \neq 0, \quad F_0(0) = F'(0).$$

Obviously, $F_0(\xi)$ is uniformly bounded in $\mathbb{R}$

Semilinear case : linelization

For each $z \in L^2(Q)$, we will denote by $\Lambda(z)$ the unique solution y_z to the linear problem

$$\begin{cases} y_t - \Delta y + F_0(z)y = f1_{\mathcal{O}} + v1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(\cdot, 0) = y_0 & \text{in} \quad \Omega, \end{cases} \tag{17}$$

where f is the unique minimise to S with $a = F_0(z)$. Let us denote this control by f_z. Define

$$\Lambda(z) = y, \quad \text{(corresponding to } f_z\text{)}.$$

Solved by Calsavara, Fdez- Cara, de Teresa, J.A.V. (2020).

Null controlability of heat equation.

Let $\Sigma = \partial\Omega \times (0,T)$, $\omega \subset \Omega$ open and $\gamma \subset \Sigma$ open in the relative topology.

$$\begin{cases} y_t - \Delta y + a(x,t)y = v1_\omega \text{ in } Q, \\ y = f1_\gamma \text{ on } \Sigma, \\ y(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases} \tag{18}$$

Define the spaces

$$\mathcal{U}^* := \{v : \varrho_0 v \in L^2(\omega \times (0,T))\},$$
$$\mathcal{Y}^* := \{y : \varrho y \in L^2(Q)\} \quad \text{and}$$
$$\mathcal{F}^* := \{f : \varrho_2 f \in L^2(\gamma \times (0,T))\}.$$

Hierarchical control

1. For each leader $v \in \mathcal{U}^*$ we associate the unique solution $f[v]$ to the extreme problem

$$\begin{cases} \text{Min}\, S^*(v;f) = \frac{1}{2}\int_Q \varrho^2|y|^2 + \frac{1}{2}\int_{\gamma\times(0,T)} \varrho_2^2|f|^2 \\ \text{subject to } f \in \mathcal{F}^*. \end{cases} \tag{19}$$

2. Then, we consider the functional $v \mapsto P^*(v;f[v])$ and we try to find $\hat{v}$ satisfying

$$\begin{cases} \min P(v;f) = \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} |y-y_d|^2 + \frac{\mu}{2}\int_{\omega\times(0,T)} \varrho_0^2|v|^2 \\ f \in \mathcal{V}^* \end{cases} \tag{20}$$

Bi linear form

Define the space $\mathcal{P}_0$

$$b(a; p, \tilde{p}) = \iint_Q \varrho^{-2} L_a^* p L_a^* p' + \iint_{\gamma \times (0,T)} \varrho_2^{-2} \frac{\partial p}{\partial \nu} \frac{\partial p'}{\partial \nu} \, d\Gamma \, dt. \quad (21)$$

In view of the unique continuation property, $b(a; \cdot\,, \cdot)$ is a norm in $\mathcal{P}_0$

Let the function $\tilde{\eta}_0$ be such that

$$\tilde{\eta}_0 \in C^2(\overline{\Omega}), \;\; \tilde{\eta}_0 \geq 0 \qquad \nabla \tilde{\eta}_0 \neq 0 \;\text{ in } \Omega, \; \text{and } \frac{\partial \tilde{\eta}_0}{\partial \nu} \leq 0 \;\text{ on }\; \partial\Omega \backslash \gamma,$$

let $\tilde{\sigma}$ and $\tilde{\xi}$ be the analog of the functions σ and ξ in (**??**) with η replaced by $\tilde{\eta}_0$ and let us introduce the weights $\varrho = e^{s\tilde{\sigma}}$, $\varrho_0 = (s\tilde{\xi})^{-3/2}\lambda^{-2}\varrho$, $\varrho_1 = (s\tilde{\xi})^{-1/2}\lambda^{-1}\varrho$, $\varrho_2 = (s\tilde{\xi})^{-1/2}\,\varrho$ and $\varrho_3 = (s\tilde{\xi})^{1/2}\,\varrho$.

Carleman stimates

Theorem

There exist positive constants λ_1, s_1 *and* C_1, *only depending on* Ω, γ *and* T, *such that, if we take* $\lambda = \lambda_1$ *and* $s = s_1$, *any* $p \in \mathcal{B}$ *satisfies*

$$\iint_Q \left[\varrho_3^{-2}(|p_t|^2 + |\Delta p|^2) + \varrho_1^{-2}|\nabla p|^2 + \varrho_0^{-2}|p|^2\right] \leq C_1\, b(0;p,p).$$

Furthermore, λ_1 *and* s_1 *can be found arbitrarily large.*

Theorem

For each $v \in \mathcal{U}^*$*, there exists exactly one solution* $f[v]$ *to* (18)*. Furthermore, the follower* $f[v]$ *and the associated state* y *satisfy*

$$f[v] = \varrho_2^{-2} \frac{\partial p}{\partial \nu}\bigg|_{\gamma \times (0,T)}, \quad y = \varrho^{-2} L_a^* p, \tag{22}$$

where $p \in \mathcal{B}$ *is the unique solution to the problem*

$$b(a; p, p') = \iint_{\omega \times (0,T)} vp' + \int_\Omega y_0(x) p'(x,0)\, dx \quad \forall p' \in \mathcal{B}, . \tag{23}$$

Sketch of proof

In order to apply the Lax -Milgram theorem is necessary to prove the continuity of the functional $\tilde{p} \mapsto \int_\Omega y_0(x)\tilde{p}(0,x)dx$
Then

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |p|^2 dx\, dt \leq C \int\int_{\gamma\times(0,T)} \rho_*^{-2} \left|\frac{\partial p}{\partial \nu}\right|^2 dx\, dt \qquad (24)$$

then $\|p(0)\|^2 \leq C\|p\|_{\mathcal{B}}$

Theorem

Given a test function ψ in $\mathcal{B}$, the function $f_\psi : \mathcal{B} \longrightarrow \mathbb{R}$ given by

$$f_\psi(\varphi) = \int_{\gamma\times(0,T)} \rho_*^{-2} \frac{\partial\psi}{\partial\nu}\frac{\partial\varphi}{\partial\nu} dxdt, \tag{25}$$

is continuous in $\mathcal{B}$

Sketch of the proof

$$\begin{aligned} |f_\psi(\varphi)| &\leq \left\| \rho_*^{-1} \frac{\partial\psi}{\partial\nu} \right\|_{L^2(\gamma\times(0,T))} \left\| \rho_*^{-1} \frac{\partial\varphi}{\partial\nu} \right\|_{L^2(\gamma\times(0,T))} \\ &\leq C\|\varphi\|_{\mathcal{B}} \end{aligned} \tag{26}$$

Explicit form of the leader

Theorem

$\hat{v}$ satisfies, together with the associated state $\hat{y}$ and some $\hat{p}$, $\hat{\phi}$ and $\hat{\psi}$, the following :

$$\begin{cases} \hat{y}_t - \Delta\hat{y} + a(x,t)\hat{y} = \hat{v}1_\omega \text{ in } \Omega \\ \hat{y} = f[\hat{v}]1_\gamma \text{ on } \Sigma, \quad \hat{y}(\cdot\,,0) = y_0 \text{ in } \Omega, \end{cases} \tag{27}$$

$$f[\hat{v}] = \varrho_2^{-2}\frac{\partial\hat{p}}{\partial\nu}\Big|_\gamma, \quad \hat{y} = \varrho^{-2}L_a^*\hat{p}, \hat{p} \in \mathcal{B} \tag{28}$$

$$b(a;\hat{p},p') = \iint_{\omega\times(0,T)} \hat{v}p' + \iint_\Omega y_0(x)p'(x,0)\,dx \quad \forall p' \in \mathcal{B}, \tag{29}$$

$$\begin{cases} -\hat{\phi}_t - \Delta\hat{\phi} + a(x,t)\hat{\phi} = \alpha(\hat{y} - y_d)1_{\mathcal{O}_d} \text{ in}\Omega, \\ \hat{\phi} = 0 \text{ on } \Sigma, \quad \hat{\phi}(\cdot\,,T) = 0 \text{ in } \Omega, \end{cases} \tag{30}$$

$$\hat{v} = -\frac{1}{\mu}\varrho_0^{-2}(\hat{\phi} + \hat{\psi})\big|_{\omega\times(0,T)}, \hat{\psi} \in \mathcal{B} \tag{31}$$

$$b(a; p'.\hat{\psi}) = \iint_{\gamma\times(0,T)} \varrho_2^{-2}\frac{\partial\hat{\phi}}{\partial\nu}\frac{\partial p'}{\partial\nu}\, d\Gamma\, dt, \quad \forall p' \in \mathcal{B}, \quad . \tag{32}$$

Both controls in the boundary

Let γ and σ disjoint open subsets of the boundary of Ω, $\mathcal{O}_d$ an open subset in Ω where a objective function y_d is defined .
Define the initial value problem

$$\left\{ \begin{array}{lll} y_t - \Delta y + ay = 0 & \text{in} & \Omega, \\ y = f1_\gamma + v1_\sigma & \text{in} & \Sigma, \\ y(0) = y_0 & \text{in} & \Omega, \end{array} \right. \tag{33}$$

Weights

$$\begin{aligned} \beta(t) &= l(t)s^{-1}e^{-2s\alpha_*(t)-3\lambda\|\eta_0\|_\infty} \\ \alpha_*(t) &= \max_{x\in\bar{\omega}} \alpha(t,x). \end{aligned} \tag{34}$$

Functionals

$$\begin{aligned} P(v;f) &= \frac{\alpha}{2}\int_{\mathcal{O}_d\times(0,T)} (y-y_d)^2 + \frac{1}{2}\int_{\sigma\times(0,T)} \beta^2|v|^2, \\ S(v;f) &= \frac{1}{2}\int_{\Omega\times(0,T)} \rho^2|y|^2 + \frac{\mu}{2}\int_{\gamma\times(0,T)} \rho_*^2|f|^2. \end{aligned} \tag{35}$$

Proposition

Let v an chose in $\tilde{\mathcal{U}}$. Then there exist a follower control $f[v]$ in $\mathcal{F}^*$ such that the problem for the follower is satisfied and y is a solution to (33). Moreover, is possible to get explicit solutions

$$f[v] = \rho_*^{-2} \frac{\partial p}{\partial \nu}\bigg|_\gamma ; \quad y = \rho^{-2} L_a^*(p), \tag{36}$$

where p is a solution to the problem

$$\begin{cases} b(a,p,p') = \int_{\sigma\times(0,T)} v\frac{\partial p'}{\partial \nu} + \int_{\Omega\times(0,T)} y_0(x)p'(0,x)dx, \\ \text{for all } p \in \mathcal{B}. \end{cases} \tag{37}$$

Sketch of the proof

By Trace Theorem, Carleman estimate of Proposition 3 and Hölder inequality

$$\begin{aligned}\int_{\sigma\times(0,T)} v\frac{\partial p}{\partial \nu} &\leq \|\beta v\|_{L^2(\sigma\times(0,T))}\|\beta^{-1}\partial_\nu p\|_{L^2(\sigma\times(0,T))} \\ &\leq \|\beta v\|_{L^2(\omega\times(0,T))}\|p\|_{\mathcal{B}}.\end{aligned} \tag{38}$$

Explicit form of the leader

Proposition

The unique solution v in $\tilde{\mathcal{U}}$ to the control problem satisfies together with the follower control $f[v]$ in $\tilde{\mathcal{F}}$ and the associated state y the following optimal system

$$\begin{cases} y_t - \Delta y + ay = 0 & \text{in} \quad \Omega, \\ y = f[v]1_\gamma + v1_\sigma & \text{on} \quad \Sigma, \\ y(0) = y_0 & \text{in} \quad \Omega, \end{cases}$$

$$f[v] = \rho_*^{-2} \frac{\partial p}{\partial \nu}\bigg|_\gamma, \quad y = \rho^{-2} L_a^*(p),$$

$$\begin{cases} b(a,p,p') = \int_{\sigma\times(0,T)} v\frac{\partial p'}{\partial \nu} + \int_{\Omega\times(0,T)} y_0(x)p'(0,x)dx, \\ \text{for all } p \in \mathcal{B} \end{cases}$$

$$\begin{cases} -q_t - \Delta q + aq = \alpha(y - y_d)1_{\mathcal{O}_d} & \text{in} \quad \Omega, \\ q = 0 & \text{on} \quad \Sigma, \\ q(T) = 0 & \text{in} \quad \Omega, \end{cases}$$

$$v = -\beta^{-2} \left(\frac{\partial \psi}{\partial \nu} + \frac{\partial q}{\partial \nu} \right) \Big|_{\sigma} \tag{40}$$

$$\begin{cases} m(a, \psi, p') = \displaystyle\int_{\gamma \times (0,T)} \rho_*^{-2} \frac{\partial q}{\partial \nu} \frac{\partial p'}{\partial \nu}, \\ \text{for all } p' \in \mathcal{B} \end{cases}$$

Work to do!

1. Solve the problem for the superlinear case when the leader control is in the boundary and the follower in inner.
2. When the follower is in the boundary some difficulties arise from the fourth order system. Some work about regularity remains to do.

Gracias!
Thank you!