Backstepping for PDEs:

Fundamentals and some recent results

Rafael Vazquez (Univ. Sevilla, Spain)

Miroslav Krstic (Univ. California San Diego, USA)

Florent Di Meglio (MINES ParisTech, France)

Long Hu (Shandong University, China)

Gustavo de Andrade (Universidade Federal de Santa Catarina, Brazil)

Jing Zang (Donghua University, China)

Guangwei Chen (Zhejiang University, Chine)

and others...

Control in Times of Crisis. Online seminar.

February 11, 2021

Backstepping for PDEs

In ODEs, a particular approach to stabilization of dynamic systems with “triangular” structure.

Wildly successful in the area of ODE nonlinear control.

Backstepping for PDEs

In ODEs, a particular approach to stabilization of dynamic systems with “triangular” structure.

Wildly successful in the area of ODE nonlinear control.

For PDEs, roughly speaking, backstepping is a *constructive* method that achieves Lyapunov stabilization by **transforming** the system into a stable “target system,” which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues.

Backstepping allows this task can be achieved in a rather elegant way where the control gains are easy to compute, symbolically, numerically, or even explicitly.

Backstepping for PDEs

In ODEs, a particular approach to stabilization of dynamic systems with “triangular” structure.

Wildly successful in the area of ODE nonlinear control.

For PDEs, roughly speaking, backstepping is a *constructive* method that achieves Lyapunov stabilization by transforming the system into a stable “target system,” which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues.

Backstepping allows this task can be achieved in a rather elegant way where the control gains are easy to compute, symbolically, numerically, and in some cases even explicitly.

sometimes... !

Backstepping for PDEs

1. Identify the undesirable terms in the PDE.

Backstepping for PDEs

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

Backstepping for PDEs

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

3. Find the state transformation typically as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x (rather than from 0 to 1).
   A Volterra transformation is “triangular” or “spatially causal.”

Backstepping for PDEs

1. Identify the undesirable terms in the PDE.

2. Choose a target system in which the undesirable terms are to be eliminated by state transformation and feedback, as in feedback linearization.

3. Find the state transformation typically as *identity minus a Volterra operator* (in x).
   Volterra operator = integral operator from 0 up to x (rather than from 0 to 1).
   A Volterra transformation is “triangular” or “spatially causal.”

4. Obtain boundary feedback from the Volterra transformation. The transformation alone cannot eliminate the undesirable terms, but the transformation brings them to the boundary, so control can cancel them.

Backstepping for PDEs

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

Backstepping for PDEs

Gain fcn of boundary controller = kernel of Volterra transformation.

Volterra kernel satisfies a *linear* PDE.

Backstepping is not “one-size-fits-all.” Requires structure-specific effort by designer.

Reward: elegant controller, clear (more or less) closed-loop behavior.

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls
- Symmetric disk
- Rijke Tube
- Mixed systems

Backstepping for PDEs—An example

Start with one of the simplest unstable PDEs, a (constant-coefficient) reaction-diffusion equation:

$$
\begin{aligned}
u_t(x,t) &= u_{xx}(x,t) + \boxed{\lambda u(x,t)} \qquad (1)\\
u(0,t) &= 0 \qquad (2)\\
u(1,t) &= U(t) = \text{control} \qquad (3)
\end{aligned}
$$

The open-loop system (1), (2) (with $u(1,t) = 0$) is unstable with arbitrarily many unstable eigenvalues for sufficiently large $\lambda > 0$.

Since the term λu is the source of instability, the natural objective for a boundary feedback is to “eliminate” this term.

Backstepping solution presented in Smyshlyaev & Krstic, IEEE TAC 2004

Backstepping for PDEs—An example

Target system (exp. stable)

$$
\begin{aligned}
w_t(x,t) &= w_{xx}(x,t) && (4)\\
w(0,t) &= 0 && (5)\\
w(1,t) &= 0 && (6)
\end{aligned}
$$

State transformation

$$\boxed{w(x,t) = u(x,t) - \int_0^x k(x,y)u(y,t)\,dy} \qquad (7)$$

Feedback control

$$u(1,t) = \int_0^1 k(1,y)u(y,t)\,dy \qquad (8)$$

Task: find kernel $k(x,y)$.

Backstepping for PDEs—An example

Task: find the function $k(x,y)$ (which we call “gain kernel”) that makes the plant (1), (2) with the controller (8) equivalent to the target system (4)–(6).

We introduce the following notation:

$$
\begin{aligned}
k_x(x,x) &= \frac{\partial}{\partial x} k(x,y)|_{y=x} \\
k_y(x,x) &= \frac{\partial}{\partial y} k(x,y)|_{y=x} \\
\frac{d}{dx} k(x,x) &= k_x(x,x) + k_y(x,x).
\end{aligned}
$$

Backstepping for PDEs—An example

Differentiating the transformation (7) with respect to x gives

$$
\begin{aligned}
w_x(x) &= u_x(x) - k(x,x)u(x) - \int_0^x k_x(x,y)u(y)\,dy \\
w_{xx}(x) &= u_{xx}(x) - \frac{d}{dx}(k(x,x)u(x)) - k_x(x,x)u(x) - \int_0^x k_{xx}(x,y)u(y)\,dy \\
&= u_{xx}(x) - u(x)\frac{d}{dx}k(x,x) - k(x,x)u_x(x) - k_x(x,x)u(x) \\
&\quad - \int_0^x k_{xx}(x,y)u(y)\,dy\,. \qquad (9)
\end{aligned}
$$

Backstepping for PDEs—An example

Next, we differentiate the transformation (7) with respect to time:

$$
\begin{aligned}
w_t(x) &= u_t(x) - \int_0^x k(x,y)u_t(y)dy \\
&= u_{xx}(x) + \lambda u(x) - \int_0^x k(x,y)\left(u_{yy}(y) + \lambda u(y)\right) dy \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) \\
&\quad + \int_0^x k_y(x,y)u_y(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy \quad \text{(integration by parts)} \\
&= u_{xx}(x) + \lambda u(x) - k(x,x)u_x(x) + k(x,0)u_x(0) + k_y(x,x)u(x) - k_y(x,0)u(0) \\
&\quad - \int_0^x k_{yy}(x,y)u(y)\,dy - \int_0^x \lambda k(x,y)u(y)\,dy. \quad \text{(integration by parts)} \qquad (10)
\end{aligned}
$$

Subtracting (9) from (10), we get

$$
\boxed{
\begin{aligned}
w_t - w_{xx} &= \left[\lambda + 2\frac{d}{dx}k(x,x)\right] u(x) + k(x,0)u_x(0) \\
&\quad + \int_0^x \left(k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y)\right) u(y)\,dy \\
&= 0
\end{aligned}
}
$$

Backstepping for PDEs—An example

For this to hold for all u, three conditions have to be satisfied:

$$
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) - \lambda k(x,y) &= 0 && (11)\\
k(x,0) &= 0 && (12)\\
\lambda + 2\frac{d}{dx}k(x,x) &= 0. && (13)
\end{aligned}
$$

We simplify (13) by integrating it with respect to x and noting from (12) that $k(0,0) = 0$, which gives us

$$
\boxed{
\begin{aligned}
k_{xx}(x,y) - k_{yy}(x,y) &= \lambda k(x,y)\\
k(x,0) &= 0\\
k(x,x) &= -\frac{\lambda}{2}x
\end{aligned}
}
\qquad (14)
$$

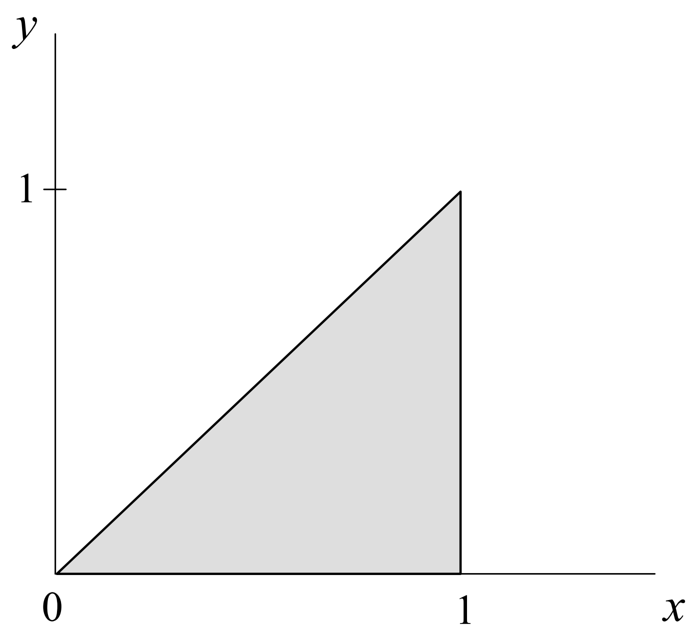
Backstepping for PDEs—An example

These three conditions form a well posed PDE of hyperbolic type in the “Goursat form.”

One can think of the k-PDE as a wave equation with an extra term λk.

x plays the role of time and y of space.

In quantum physics such PDEs are called Klein-Gordon PDEs.



Domain of the PDE for gain kernel $k(x,y)$.

The boundary conditions are prescribed on hypotenuse and the lower cathetus of the triangle.

The value of $k(x,y)$ on the vertical cathetus gives us the control gain $k(1,y)$.

Backstepping for PDEs—An example

To find a solution of the k-PDE (14) we first convert it into an integral equation.

Introducing the change of variables

$$\boxed{\xi = x + y, \qquad \eta = x - y} \tag{15}$$

we have

$$\begin{aligned} k(x,y) &= G(\xi,\eta) \\ k_x &= G_\xi + G_\eta \\ k_{xx} &= G_{\xi\xi} + 2G_{\xi\eta} + G_{\eta\eta} \\ k_y &= G_\xi - G_\eta \\ k_{yy} &= G_{\xi\xi} - 2G_{\xi\eta} + G_{\eta\eta}\,. \end{aligned}$$

Backstepping for PDEs—An example

Thus, the gain kernel PDE becomes

$$G_{\xi\eta}(\xi,\eta) = \frac{\lambda}{4}G(\xi,\eta) \tag{16}$$

$$G(\xi,\xi) = 0 \tag{17}$$

$$G(\xi,0) = -\frac{\lambda}{4}\xi. \tag{18}$$

Integrating (16) with respect to η from 0 to η, we get

$$G_\xi(\xi,\eta) = G_\xi(\xi,0) + \int_0^\eta \frac{\lambda}{4}G(\xi,s)\,ds = -\frac{\lambda}{4} + \int_0^\eta \frac{\lambda}{4}G(\xi,s)\,ds\,. \tag{19}$$

Next, we integrate (19) with respect to ξ from η to ξ to get the integral equation

$$\boxed{G(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G(\tau,s)\,ds\,d\tau} \tag{20}$$

The G-integral eqn is easier to analyze than the k-PDE.

Backstepping for PDEs—An example

Start with an initial guess

$$G^0(\xi,\eta) = 0 \tag{21}$$

and set up the recursive formula for (20) as follows:

$$\boxed{G^{n+1}(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta) + \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta G^n(\tau,s)\,ds\,d\tau} \tag{22}$$

If this functional iteration converges, we can write the solution $G(\xi,\eta)$ as

$$G(\xi,\eta) = \lim_{n\to\infty} G^n(\xi,\eta)\,. \tag{23}$$

Backstepping for PDEs—An example

Let us denote the difference between two consecutive terms as

$$\Delta G^n(\xi,\eta) = G^{n+1}(\xi,\eta) - G^n(\xi,\eta)\,. \tag{24}$$

Then

$$\Delta G^{n+1}(\xi,\eta) = \frac{\lambda}{4}\int_\eta^\xi \int_0^\eta \Delta G^n(\tau,s)\,ds\,d\tau \tag{25}$$

and (23) can be alternatively written as

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G^n(\xi,\eta)\,. \tag{26}$$

Computing ΔG^n from (25) starting with

$$\Delta G^0 = G^1(\xi,\eta) = -\frac{\lambda}{4}(\xi-\eta)\,, \tag{27}$$

we can observe the pattern which leads to the following formula:

$$\boxed{\Delta G^n(\xi,\eta) = -\frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1}} \tag{28}$$

This formula can be verified by induction.

Backstepping for PDEs—An example

The solution to the integral equation is given by

$$G(\xi,\eta) = -\sum_{n=0}^{\infty} \frac{(\xi-\eta)\xi^n\eta^n}{n!(n+1)!}\left(\frac{\lambda}{4}\right)^{n+1} . \tag{29}$$

To compute the series (29), note that a first order modified Bessel function of the first kind can be represented as

$$I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!} . \tag{30}$$

Backstepping for PDEs—An example

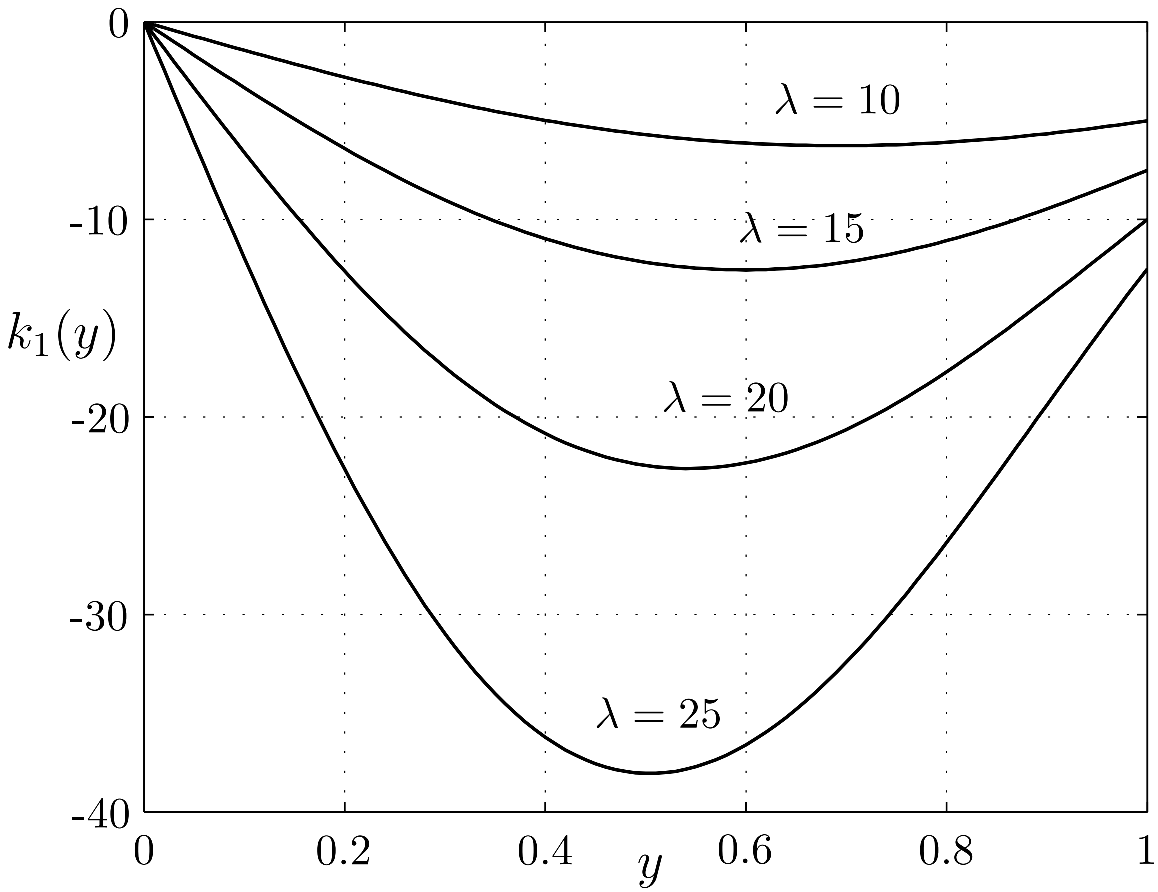
Comparing (30) with (29) we obtain

$$G(\xi,\eta) = -\frac{\lambda}{2}(\xi-\eta)\frac{I_1(\sqrt{\lambda\xi\eta})}{\sqrt{\lambda\xi\eta}} \tag{31}$$

or, returning to the original x, y variables,

$$\boxed{k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}} \tag{32}$$

Backstepping for PDEs—An example



Control gain $k(1, y)$ for different values of λ

As λ gets larger, the plant becomes more unstable which requires more control effort.

Low gain near the boundaries is logical: near $x = 0$ the state is small even without control because of the boundary condition $u(0) = 0$; near $x = 1$ the control has the most impact.

Backstepping for PDEs—An example

We need to establish that stability of the w-target system (4)–(6) implies stability of the u-closed-loop system (1), (2), (8), by showing that the transformation $u \mapsto w$ is invertible.

Invertibility is obvious by seeing the backstepping transformation as an integral equation in u.

Postulate an inverse transformation in the form

$$u(x) = w(x) + \int_0^x l(x,y)w(y)\,dy\,, \tag{33}$$

where $l(x,y)$ is the transformation kernel.

Given the direct transformation (7) and the inverse transformation (33), the kernels $k(x,y)$ and $l(x,y)$ satisfy

$$\boxed{l(x,y) = k(x,y) + \int_y^x k(x,\xi)l(\xi,y)\,d\xi} \tag{34}$$

Backstepping for PDEs—An example

One can find also kernel equations for $l(x,y)$:

$$\boxed{\begin{aligned} l_{xx}(x,y) - l_{yy}(x,y) &= -\lambda l(x,y) \\ l(x,0) &= 0 \\ l(x,x) &= -\frac{\lambda}{2}x \end{aligned}} \tag{35}$$

Comparing this PDE with the PDE (14) for $k(x,y)$, we see that

$$l(x,y;\lambda) = -k(x,y;-\lambda)\,. \tag{36}$$

Backstepping for PDEs—An example

From (32) we have

$$l(x,y) = -\lambda y \frac{I_1\left(\sqrt{-\lambda(x^2-y^2)}\right)}{\sqrt{-\lambda(x^2-y^2)}} = -\lambda y \frac{I_1\left(j\sqrt{\lambda(x^2-y^2)}\right)}{j\sqrt{\lambda(x^2-y^2)}},$$

or, using the properties of I_1,

$$\boxed{l(x,y) = -\lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}} \tag{37}$$

Summary of control design for the reaction-diffusion equation

Plant	$u_t = u_{xx} + \lambda u$	(38)
	$u(0) = 0$	(39)
Controller	$u(1) = -\int_0^1 y\lambda \frac{I_1\left(\sqrt{\lambda(1-y^2)}\right)}{\sqrt{\lambda(1-y^2)}} u(y)\,dy$	(40)
Transformation	$w(x) = u(x) + \int_0^x \lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} u(y)\,dy$	(41)
	$u(x) = w(x) - \int_0^x \lambda y \frac{J_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}} w(y)\,dy$	(42)
Target system	$w_t = w_{xx}$	(43)
	$w(0) = 0$	(44)
	$w(1) = 0$	(45)

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls & Symmetric disk
- Rijke Tube
- Mixed systems
- Some open problems

Backstepping control of coupled hyperbolic 1-D systems

$$u_t(t,x) + \Sigma^+ u_x(t,x) = \Lambda^{++} u(t,x) + \Lambda^{+-} v(t,x)$$
$$v_t(t,x) - \Sigma^- v_x(t,x) = \Lambda^{-+} u(t,x) + \Lambda^{--} v(t,x)$$

with the following boundary conditions

$$u(t,0) = 0, \qquad v(t,1) = U(t)$$

where

$$u = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}^T, \qquad v = \begin{pmatrix} v_1 & \cdots & v_m \end{pmatrix}^T$$

$$\Sigma^+ = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}, \qquad \Sigma^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix}$$

with

$$-\mu_1 < \cdots < -\mu_m < 0 < \varepsilon_1 \leq \cdots \leq \varepsilon_n$$

Backstepping control of coupled hyperbolic 1-D systems

Backstepping transformation

$$\alpha(t,x) = u(t,x)$$
$$\beta(t,x) = v(t,x) - \int_0^x [L(x,\xi)u(\xi) + K(x,\xi)v(\xi)]\, d\xi$$

L and K defined on the triangular domain $\mathcal{T}$.

Target system

$$\alpha_t(t,x) + \Sigma^+\alpha_x(t,x) = \Lambda^{++}\alpha(t,x) + \Lambda^{+-}\beta(t,x) + \int_0^x D^+(x,\xi)\alpha(\xi)d\xi + \int_0^x D^-(x,\xi)\beta(\xi)d\xi$$
$$\beta_t(t,x) - \Sigma^-\beta_x(t,x) = G(x)\beta(0)$$

with boundary conditions

$$\alpha(t,0) = \beta(t,1) = 0$$

Backstepping control of coupled hyperbolic 1-D systems

Structure of G is **lower-diagonal with diagonal of zeros**

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}$$

It can be shown to make stable

$$\beta_t(t,x) - \Sigma^- \beta_x(t,x) = G(x)\beta(0)$$

From there follows target system stability.

Backstepping control of coupled hyperbolic 1-D systems

Structure of G is **lower-diagonal with diagonal of zeros**

$$G(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \cdots & g_{m,m-1}(x) & 0 \end{pmatrix}$$

It can be shown to make stable

$$\beta_t(t,x) - \Sigma^- \beta_x(t,x) = G(x)\beta(0)$$

From there follows target system stability.

G(x) is not chosen, but computed from the kernels.

Backstepping control of coupled hyperbolic 1-D systems

Kernel equations

$$
\begin{aligned}
0 =& \Sigma^{-} L_x(x,\xi) - L_\xi(x,\xi)\Sigma^{+} - L(x,\xi)\Lambda^{++} - K(x,\xi)\Lambda^{-+} \\
0 =& \Sigma^{-} K_x(x,\xi) + K_\xi(x,\xi)\Sigma^{-} - K(x,\xi)\Lambda^{--} - L(x,\xi)\Lambda^{+-}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
0 =& L(x,x)\Sigma^{+} + \Sigma^{-} L(x,x) + \Lambda^{-+} \\
0 =& \Sigma^{-} K(x,x) - K(x,x)\Sigma^{-} + \Lambda^{--} \\
0 =& G(x) - K(x,0)\Sigma^{-}
\end{aligned}
$$

Too many boundary conditions?

Backstepping control of coupled hyperbolic 1-D systems

Kernel equations

$$
\begin{aligned}
0 =& \Sigma^- L_x(x,\xi) - L_\xi(x,\xi)\Sigma^+ - L(x,\xi)\Lambda^{++} - K(x,\xi)\Lambda^{-+} \\
0 =& \Sigma^- K_x(x,\xi) + K_\xi(x,\xi)\Sigma^- - K(x,\xi)\Lambda^{--} - L(x,\xi)\Lambda^{+-}
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
0 =& L(x,x)\Sigma^+ + \Sigma^- L(x,x) + \Lambda^{-+} \\
0 =& \Sigma^- K(x,x) - K(x,x)\Sigma^- + \Lambda^{--} \\
0 =& G(x) - K(x,0)\Sigma^-
\end{aligned}
$$

Again, too many boundary conditions?

No, in fact more boundary conditions are needed $\longrightarrow$ Nonuniqueness!

Backstepping control of coupled hyperbolic 1-D systems

Developing the equations:

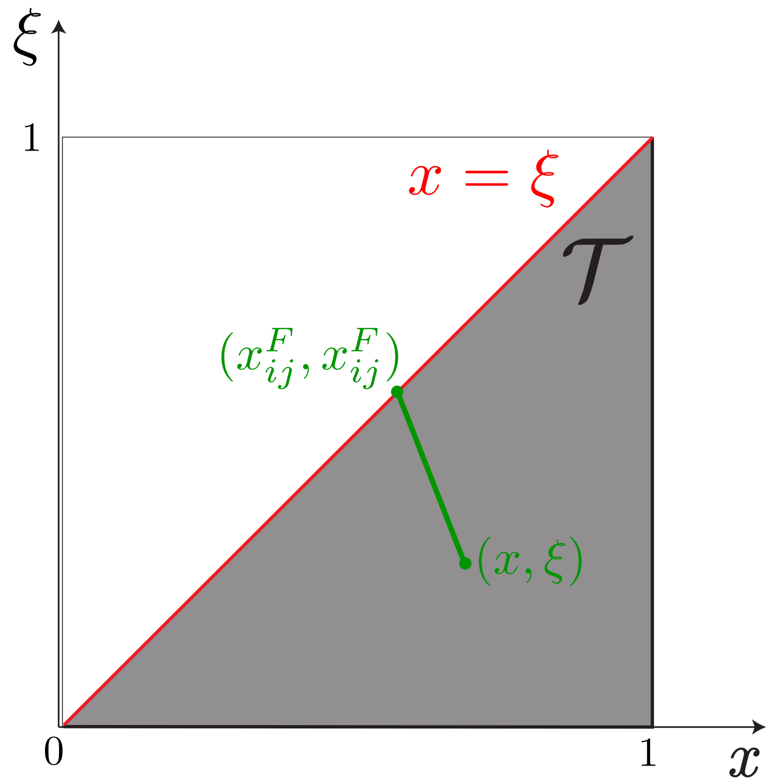
$$\begin{aligned}
\mu_i \partial_x L_{ij}(x,\xi) - \varepsilon_j \partial_\xi L_{ij}(x,\xi) &= \sum_{k=1}^{n} \lambda_{kj}^{++} L_{ik}(x,\xi) + \sum_{p=1}^{m} \lambda_{pj}^{-+} K_{ip}(x,\xi) \\
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda_{pj}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{kj}^{+-} L_{ik}(x,\xi)
\end{aligned}$$

with boundary conditions:

$$\begin{aligned}
\forall 1 \le i \le m, 1 \le j \le n, \quad & L_{ij}(x,x) = -\frac{\lambda_{ij}^{-+}}{\mu_i + \varepsilon_j} \\
\forall 1 \le i, j \le m, i \ne j, \quad & K_{ij}(x,x) = -\frac{\lambda_{ij}^{--}}{\mu_i - \mu_j} \\
\forall 1 \le i \le j \le m, \quad & K_{ij}(x,0) = 0 \\
\forall 1 \le j < i \le m, \quad & K_{ij}(1,\xi) = l_{ij} \\
\forall 1 \le j < i \le m, \quad & g_{ij}(x) = \mu_j K_{ij}(x,0)
\end{aligned}$$

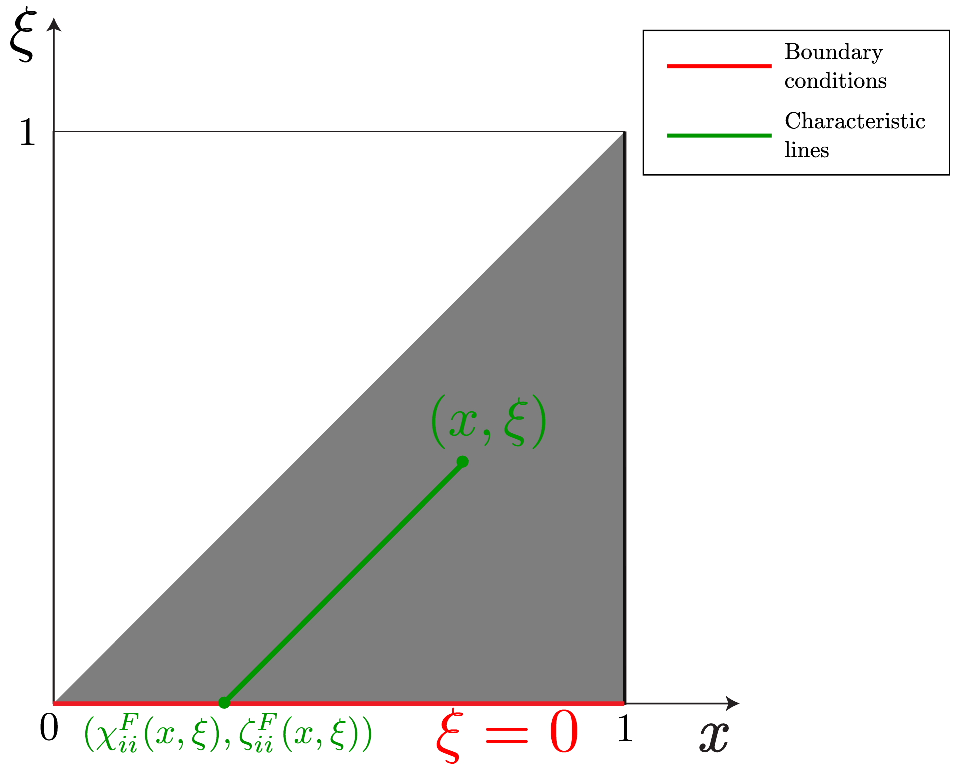
Well-posedness depends on the characteristics!

Characteristics for L_{ij}



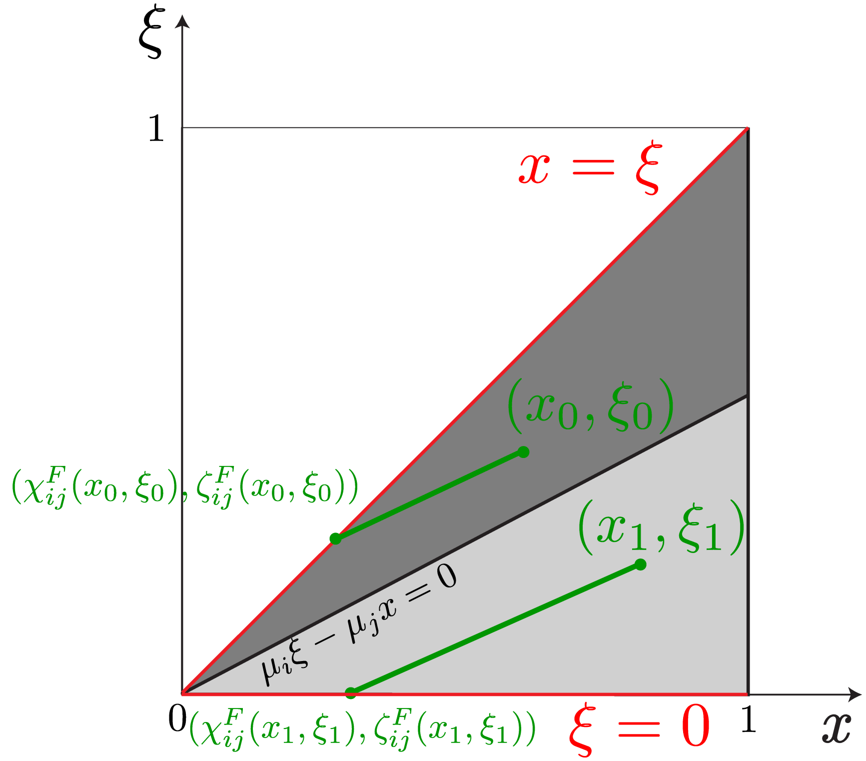
$$
\begin{aligned}
\mu_i \partial_x L_{ij}(x,\xi) - \varepsilon_j \partial_\xi L_{ij}(x,\xi) &= \sum_{k=1}^{n} \lambda_{kj}^{++} L_{ik}(x,\xi) + \sum_{p=1}^{m} \lambda_{pj}^{-+} K_{ip}(x,\xi) \\
L_{ij}(x,x) &= -\frac{\lambda_{ij}^{-+}}{\mu_i + \varepsilon_j}
\end{aligned}
$$

Characteristics for K_{ii}



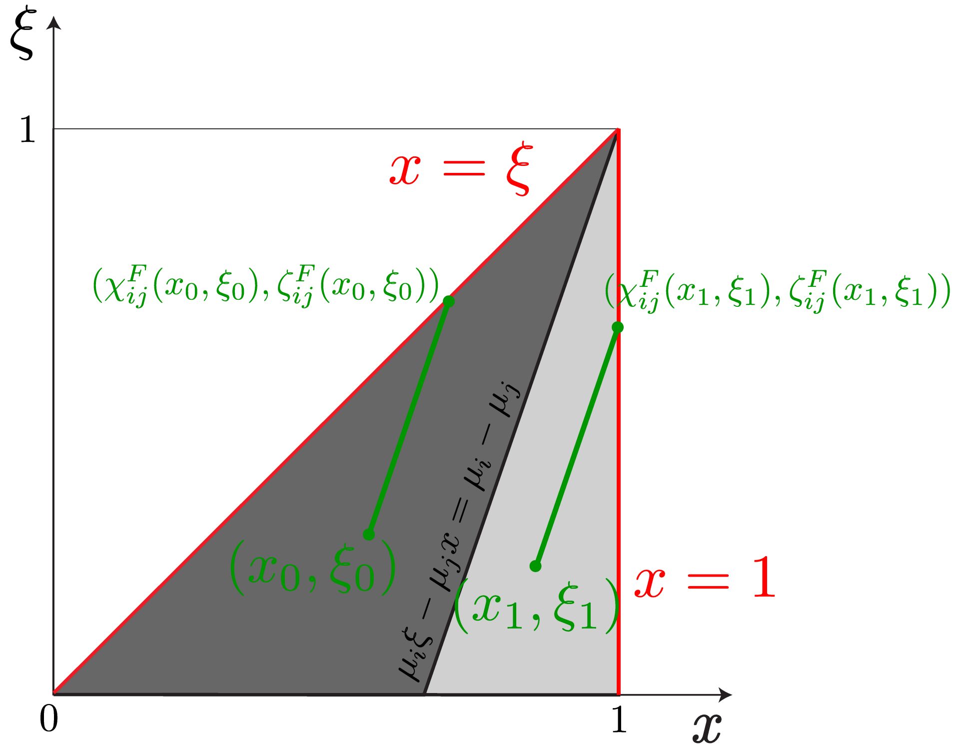
$$
\begin{aligned}
\mu_i\partial_x K_{ii}(x,\xi)+\mu_i\partial_\xi K_{ii}(x,\xi) &= \sum_{p=1}^{m}\lambda_{pi}^{--}K_{ip}(x,\xi)+\sum_{k=1}^{n}\lambda_{ki}^{+-}L_{ik}(x,\xi)\\
K_{ii}(x,0) &= 0
\end{aligned}
$$

Characteristics for $K_{ij}, i < j$



$$
\begin{aligned}
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda_{pj}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{kj}^{+-} L_{ik}(x,\xi) \\
K_{ij}(x,x) &= -\frac{\lambda_{ij}^{--}}{\mu_i - \mu_j} \\
K_{ij}(x,0) &= 0
\end{aligned}
$$

Characteristics for $K_{ij}, i > j$



$$
\begin{aligned}
\mu_i \partial_x K_{ij}(x,\xi) + \mu_j \partial_\xi K_{ij}(x,\xi) &= \sum_{p=1}^{m} \lambda_{pj}^{--} K_{ip}(x,\xi) + \sum_{k=1}^{n} \lambda_{kj}^{+-} L_{ik}(x,\xi) \\
K_{ij}(x,x) &= -\frac{\lambda_{ij}^{--}}{\mu_i - \mu_j} \\
K_{ij}(1,\xi) &= l_{ij} \\
g_{ij}(x) &= \mu_j K_{ij}(x,0)
\end{aligned}
$$

Backstepping control of coupled hyperbolic 1-D systems

The presented approach produces piecewise continuous and differentiable kernels.

Backstepping control of coupled hyperbolic 1-D systems

The presented approach produces piecewise continuous and differentiable kernels.

There are potential lines of discontinuity, which complicate kernel calculation, but do not affect the stability result.

Backstepping control of coupled hyperbolic 1-D systems

The presented approach produces piecewise continuous and differentiable kernels.

There are potential lines of discontinuity, which complicate kernel calculation, but do not affect the stability result.

Next we see how we can produce a strikingly similar result for reaction-diffusion equations.

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls & Symmetric disk
- Rijke Tube
- Mixed systems
- Some open problems

Coupled parabolic systems

Consider

$$u_t(t,x) = \Sigma u_{xx}(t,x) + \Lambda(x)u(t,x)$$

$x \in [0,1]$, $t > 0$, $u \in \mathbb{R}^n$

$$\Sigma = \begin{bmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{bmatrix}, \quad \Lambda(x) = \begin{bmatrix} \lambda_{11}(x) & \lambda_{12}(x) & \dots & \lambda_{1n}(x) \\ \lambda_{21}(x) & \lambda_{22}(x) & \dots & \lambda_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1}(x) & \lambda_{n2}(x) & \dots & \lambda_{nn}(x) \end{bmatrix}$$

with $\varepsilon_i > 0$ ordered, i.e., $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_n > 0$, and boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= U(t) \end{aligned}$$

with $U \in \mathbb{R}^n$.

Backstepping approach

Consider the Backstepping Transformation :

$$w(t,x) = u(t,x) - \int_0^x K(x,\xi)u(t,\xi)d\xi$$

with $K(x,\xi)$ a $n \times n$ matrix of kernels, and w verifies the Target System :

$$w_t(t,x) = \Sigma w_{xx}(t,x) - Cw(t,x) - G(x)w_x(0,t),$$

with C and $G(x)$:

$$C = \begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ g_{21}(x) & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(n-1)1}(x) & g_{(n-1)2}(x) & \dots & 0 & 0 \\ g_{n1}(x) & g_{n2}(x) & \dots & g_{n(n-1)}(x) & 0 \end{bmatrix}$$

where $c_1, c_2, \dots, c_n > 0$. Control law is then

$$U(t) = \int_0^1 K(1,\xi)u(t,\xi)d\xi$$

The challenge is to prove that $K(x,\xi)$ exists and has good properties $\longrightarrow$ Kernel equations

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx}K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma\frac{d}{dx}K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

First b.c. with structure of G becomes:

$$K_{ij}(x,0) = 0, \qquad \forall j \geq i,$$

and

$$g_{ij}(x) = -K_{ij}(x,0)\varepsilon_j, \qquad \forall j < i,$$

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx} K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Second b.c. is:

$$K_{ij}(x,x) = 0, \qquad \forall j \neq i,$$

(no boundary condition for $K_{ii}(x,x)$)

Kernel equations

$$\Sigma K_{xx} - K_{\xi\xi}\Sigma = K\Lambda(\xi) + CK,$$

with b.c.

$$\begin{aligned} G(x) &= -K(x,0)\Sigma, \\ K(x,x)\Sigma &= \Sigma K(x,x), \\ C + \Lambda(x) &= -\Sigma K_x(x,x) - \Sigma \frac{d}{dx} K(x,x) - K_\xi(x,x)\Sigma. \end{aligned}$$

Third boundary condition:

$$0 = \lambda_{ij}(x) + \delta_{ij}c_i + K_{ij\xi}(x,x)\varepsilon_j + \varepsilon_i K_{ijx}(x,x) + \varepsilon_i \frac{d}{dx}\left(K_{ij}(x,x)\right),$$

Duplicating the kernel equations

Key idea ("duplication"): define

$$L(x,\xi) \quad = \quad \sqrt{\Sigma}K_x(x,\xi)+K_\xi(x,\xi)\sqrt{\Sigma} \longrightarrow L_{ij}(x,x) = \sqrt{\varepsilon_i}K_{ijx}(x,x)+\sqrt{\varepsilon_j}K_{ij\xi}(x,x)$$

Then we can rewrite the "duplicated" kernel equations as

$$\begin{aligned} \sqrt{\Sigma}K_x + K_\xi\sqrt{\Sigma} &= L \\ \sqrt{\Sigma}L_x - L_\xi\sqrt{\Sigma} &= K\Lambda(\xi)+CK \end{aligned}$$

Same structure as in the coupled hyperbolic result!

Third boundary condition becomes:

$$i=j\text{: } 0=\lambda_{ii}(x)+c_i+2\varepsilon_i(K_{iix}(x,x)+K_{ii\xi}(x,x)) \longrightarrow L_{ii}(x,x) = -\frac{\lambda_{ii}(x)+c_i}{2\sqrt{\varepsilon_i}}$$

$$i\neq j\text{: } 0=\lambda_{ij}(x)+(\varepsilon_i-\varepsilon_j)K_{ijx}(x,x) \longrightarrow L_{ij}(x,x) = -\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i}+\sqrt{\varepsilon_j}}$$

Duplicating the kernel equations

The boundary conditions therefore are:

- If $i = j$

$$\begin{aligned} L_{ii}(x,x) &= -\frac{\lambda_{ii}(x) + c_i}{2\sqrt{\varepsilon_i}} \\ K_{ii}(x,0) &= 0 \end{aligned}$$

- If $i < j$

$$\begin{aligned} K_{ij}(x,x) &= K_{ij}(x,0) = 0 \\ L_{ij}(x,x) &= -\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i} + \sqrt{\varepsilon_j}} \end{aligned}$$

- Finally if $i > j$

$$\begin{aligned} K_{ij}(x,x) &= 0 \\ K_{ij}(1,\xi) &= l_{ij}(\xi) \\ L_{ij}(x,x) &= -\frac{\lambda_{ij}(x)}{\sqrt{\varepsilon_i} + \sqrt{\varepsilon_j}} \end{aligned}$$

  and the additional condition $g_{ij}(x) = -K_{ij}(x,0)\varepsilon_j$

Same structure as in the coupled hyperbolic result!

Extension to reaction-advection-diffusion systems with spatially-varying coefficients

The method can be extended to

$$u_t = \partial_x (\Sigma(x) u_x) + \Phi(x) u_x + \Lambda(x) u$$

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls &
- Rijke Tube
- Mixed systems
- Some open problems

Reaction-diffusion equation on an n-dimensional ball

Let the state $u = u(t, \vec{x})$, with $\vec{x} = [x_1, x_2, \ldots, x_n]^T$, verify

$$\frac{\partial u}{\partial t} = \varepsilon \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} \right) + \lambda u = \varepsilon \triangle_n u + \lambda u,$$

for constant $\varepsilon > 0$, $\lambda(r, \vec{\theta})$, and for $t > 0$, in the n-ball $B^n(R)$ defined as

$$B^n(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < R\},$$

with b.c. on the boundary of $B^n(R)$, the $(n-1)$-sphere $S^{n-1}(R)$:

$$S^{n-1}(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = R\}.$$

The b.c. is of Dirichlet type:

$$u(t, \vec{x})\Big|_{\vec{x} \in S^{n-1}(R)} = U(t, \vec{x})$$

where $U(t, \vec{x})$ is the actuation variable.

Ultraspherical coordinates

The n-ball domain is well described in n-dimensional spherical coordinates, also known as ultraspherical coordinates:

- one radial coordinate r, $r \in [0, R)$.
- $n-1$ angular coordinates: $\vec{\theta} = [\theta_1, \theta_2, \ldots, \theta_{n-1}]^T$, with $\theta_1 \in [0, 2\pi)$ and $\theta_i \in [0, \pi]$ for $2 \leq i \leq n-1$.

Definition:

$$
\begin{aligned}
x_1 &= r \cos\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_2 &= r \sin\theta_1 \sin\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
x_3 &= r \cos\theta_2 \sin\theta_3 \ldots \sin\theta_{n-1}, \\
&\vdots \\
x_{n-1} &= r \cos\theta_{n-2} \sin\theta_{n-1}, \\
x_n &= r \cos\theta_{n-1}.
\end{aligned}
$$

Laplacian in ultraspherical coordinates

Writing the reaction diffusion equation in ultraspherical coordinates

$$\begin{aligned} u_t &= \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u\right)+\frac{1}{r^2}\triangle^*_{n-1}u+\lambda u, \\ u(t,R,\vec{\theta}\,) &= U(t,\vec{\theta}\,), \end{aligned}$$

where $\triangle^*_{n-1}$ is called the Laplace-Beltrami operator and represents the Laplacian over the $(n-1)$-sphere.

It is defined recursively as

$$\begin{aligned} \triangle^*_1 &= \frac{\partial^2}{\partial\theta_1^2}, \\ \triangle^*_n &= \frac{1}{\sin^{n-1}\theta_n}\frac{\partial}{\partial\theta_n}\left(\sin^{n-1}\theta_n\frac{\partial}{\partial\theta_n}\right)+\frac{\triangle^*_{n-1}}{\sin^2\theta_n}, \end{aligned}$$

Example:

$$\triangle^*_2 = \frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)+\frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\theta_1^2}.$$

Designing a boundary feedback law

- Exploit periodicity in $\vec{\theta}$ by using Spherical Harmonics

- Apply the backstepping method to each harmonic coefficient

- Solve the backstepping kernel equations to find a feedback law for each harmonic

- Re-assemble the feedback law in Spherical Harmonics back to physical space

Spherical Harmonics

Develop u and U in term of Spherical Harmonics coefficients u_l^m and U_l^m:

$$u(t,r,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} u_l^m(r,t) Y_{lm}^n(\vec{\theta}), \quad U(t,\vec{\theta}) = \sum_{l=0}^{l=\infty} \sum_{m=0}^{m=N(l,n)-1} U_l^m(t) Y_{lm}^n(\vec{\theta}),$$

$N(l,n)$: number of (linearly independent) n-dimensional spherical harmonics of degree l

$$N(l,n) = \frac{2l+n-2}{l} \binom{l+n-3}{l-1}, \quad l > 0; \qquad N(0,n) = 1$$

$Y_{lm}^n(\vec{\theta})$: m-th order n-dimensional spherical harmonic of degree l

Coefficients are defined as:

$$u_l^m(r,t) = \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} u(t,r,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \ldots \sin\theta_2 d\vec{\theta},$$

$$U_l^m(t) = \int_0^{\pi} \ldots \int_0^{\pi} \int_0^{2\pi} U(t,\vec{\theta}) \bar{Y}_{lm}^n(\vec{\theta}) \sin^{n-2}\theta_{n-1} \sin^{n-3}\theta_{n-2} \ldots \sin\theta_2 d\vec{\theta},$$

($d\vec{\theta} = d\theta_{n-1} d\theta_{n-2} \ldots d\theta_2 d\theta_1$, $\bar{Y}_{lm}^n$ is the complex conjugate of Y_{lm}^n)

Spherical Harmonics

The n-dimensional spherical harmonics are **eigenfunctions** for the Laplacian $\triangle^*_{n-1}$:

$$\triangle^*_{n-1} Y^n_{lm} = -l(l+n-2) Y^n_{lm}.$$

Thus, each harmonic coefficient $u^m_l(t,r)$ for $l \in \mathbb{N}$ and $0 \leq m \leq N(l,n)$, verifies

$$\partial_t u^m_l = \frac{\varepsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u^m_l \right) - l(l+n-2) \frac{\varepsilon}{r^2} u^m_l + \lambda u^m_l,$$

evolving in $r \in [0,R]$, $t > 0$, with boundary conditions

$$u^m_l(t,R) \quad = \quad U^m_l(t),$$

The PDEs for the harmonics are not coupled: we can independently design each U^m_l and later assemble all of the them to find an expression for U.

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m \;=\; \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m \;=\; \frac{\varepsilon}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) \;=\; 0$$

The transformation is

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho)u_l^m(t,\rho)d\rho$$

with kernels K_{lm}^n to be found.

Backstepping control of Spherical Harmonics coefficients

To design $U_l^m(t)$ seek transformation of

$$\partial_t u_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r \left(r^{n-1}\partial_r u_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}u_l^m + \lambda u_l^m$$

into the (stable) target system

$$\partial_t w_l^m = \frac{\varepsilon}{r^{n-1}}\partial_r \left(r^{n-1}\partial_r w_l^m\right) - l(l+n-2)\frac{\varepsilon}{r^2}w_l^m$$

with boundary conditions

$$w_l^m(t,R) = 0$$

The transformation is

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho)u_l^m(t,\rho)d\rho$$

with kernels K_{lm}^n to be found.

Substituting at $r = R$ we find $U_l^m(t)$ as

$$U_l^m(t)(t) = \int_0^R K_{lm}^n(R,\rho)u_l^m(t,\rho)d\rho$$

Kernel equation

The control kernels $K_{lm}^n(r,\rho)$ are found, for a given $n \geq 2$ and each l, m, from

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K_{lm}^n\right) - \partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K_{lm}^n}{\rho^{n-1}}\right)\right) - l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K_{lm}^n = \frac{\lambda}{\varepsilon}K_{lm}^n.$$

with BC

$$\begin{aligned}
\lambda + 2\varepsilon\frac{d}{dr}\left(K_{lm}^n(r,r)\right) &= 0 \\
K_{lm}^n(r,0) &= 0 \\
(n-2)\partial_\rho K_{lm}^n(r,\rho)|_{\rho=0} &= 0
\end{aligned}$$

Kernel equation

The control kernels $K_{lm}^n(r,\rho)$ are found, for a given $n \geq 2$ and each l, m, from

$$\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r K_{lm}^n\right) - \partial_\rho\left(\rho^{n-1}\partial_\rho\left(\frac{K_{lm}^n}{\rho^{n-1}}\right)\right) - l(l+n-2)\left(\frac{1}{r^2}-\frac{1}{\rho^2}\right)K_{lm}^n = \frac{\lambda}{\varepsilon}K_{lm}^n.$$

with BC

$$\begin{aligned}
\lambda + 2\varepsilon\frac{d}{dr}\left(K_{lm}^n(r,r)\right) &= 0 \\
K_{lm}^n(r,0) &= 0 \\
(n-2)\partial_\rho K_{lm}^n(r,\rho)|_{\rho=0} &= 0
\end{aligned}$$

The first BC integrates (using $K_{lm}^n(0,0) = 0$) to

$$K_{lm}^n(r,r) = -\int_0^r \frac{\lambda}{2\varepsilon}d\rho = -\frac{\lambda r}{2\varepsilon}$$

Explicit Kernel equation solution and feedback law

It is found that

$$K_{lm}^{n}(r,\rho) = -\rho\left(\frac{\rho}{r}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}$$

Thus the feedback law for each spherical harmonic is

$$U_l^m(t) = \int_0^R K_{lm}^n(R,\rho)u_l^m(t,\rho)d\rho = \int_0^R -\rho\left(\frac{\rho}{R}\right)^{l+n-2}\frac{\lambda}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}}u_l^m(t,\rho)d\rho$$

Explicit feedback law

Using some spherical harmonics machinery one obtains an explicit feedback law

$$
\begin{aligned}
U(t,\theta) &= -\frac{\lambda}{\varepsilon}\int_0^R \rho \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(R^2-\rho^2)}} \\
&\quad \times \left[\int_0^{\pi}\!\!\cdots\int_0^{\pi}\int_0^{2\pi} P(R,\rho,\vec{\theta},\vec{\phi})u(t,\rho,\vec{\phi})\rho^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\ldots\sin\phi_2 d\vec{\phi}\right] d\rho
\end{aligned}
$$

where $P(R,\rho,\vec{\theta},\vec{\phi})$ is the Poisson kernel for the n-ball.

Back in rectangular coordinates

$$
U(t,\vec{x}) = -\frac{1}{\mathrm{Area}(S^{n-1})}\sqrt{\frac{\lambda}{\varepsilon}}\int_{B^n(R)} \mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(R^2-\|\vec{\xi}\|^2)}\right]\frac{\sqrt{R^2-\|\vec{\xi}\|^2}}{\|\vec{x}-\vec{\xi}\|^n}u(t,\vec{\xi})d\vec{\xi},
$$

where the integral is extended to the complete n-ball $B^n(R)$ and $\vec{x}\in S^{n-1}(R)$.

Extension to spatially-varying λ

Consider now the same problem but with spatially-varying coefficient λ:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \varepsilon \triangle_n u + \lambda(\vec{x}) u, \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x})\end{aligned}$$

the question is: can backstepping still be applied?

Extension to spatially-varying λ

Consider now the same problem but with spatially-varying coefficient $\lambda(\vec{x})$:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \varepsilon \triangle_n u + \lambda(\vec{x}) u, \\ u(t,\vec{x})\Big|_{\vec{x}\in S^{n-1}(R)} &= U(t,\vec{x})\end{aligned}$$

the question is: can backstepping still be applied?

Consider two cases:

- Sphere

- Disk (harder!)

Under a simplifying assumption, we solve the problem and get a taste of the challenges

Revolution symmetry condition

Revolution Symmetry Condition : if the initial conditions are symmetric (do not depend on the angle or angles in 3-D), and U is chosen constant (do not depend on the position in the boundary) nothing depends on the angle.

Typical engineering simplification. Equations becomes 1-D in radius, with singularities.

Disk: $u_t = \frac{\varepsilon}{r}(ru_r)_r + \lambda(r)u$

Sphere: $u_t = \frac{\varepsilon}{r^2}\left(r^2 u_r\right)_r + \lambda(r)u$

We apply the method as before but only one kernel (corresponding to the constant Fourier mode or Spherical Harmonic) is needed.

3-D case—revolution symmetry

Kernel equation is:

$$\begin{aligned}
K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} &= \frac{\lambda(r)}{\varepsilon}K \\
K(r,0) &= K_\rho(r,0) = 0, \\
K(r,r) &= -\frac{\lambda r}{2\varepsilon},
\end{aligned}$$

3-D case—revolution symmetry

Kernel equation is:

$$\begin{aligned} K_{rr} + 2\frac{K_r}{r} - K_{\rho\rho} + 2\frac{K_\rho}{\rho} - 2\frac{K}{\rho^2} &= \frac{\lambda(r)}{\varepsilon} K \\ K(r,0) &= K_\rho(r,0) = 0, \\ K(r,r) &= -\frac{\lambda r}{2\varepsilon}, \end{aligned}$$

Define $K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho)$. Then:

$$\begin{aligned} \bar{K}_{rr} - \bar{K}_{\rho\rho} &= \frac{\lambda(r)}{\varepsilon}\bar{K} \\ \bar{K}(r,0) &= 0, \\ \bar{K}(r,r) &= -\frac{\lambda r}{2\varepsilon}, \end{aligned}$$

which is the 1-D backstepping equation! Can be proved solvable by successive approximations (classical backstepping papers).

3-D case—revolution symmetry

For instance if λ is constant we directly get:

$$K(r,\rho) = \frac{\rho}{r}\bar{K}(r,\rho) = \frac{\rho^2}{r}\frac{c}{\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}\right]}{\sqrt{\frac{c}{\varepsilon}\left(r^2-\rho^2\right)}}$$

2-D case—revolution symmetry

Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$
\begin{aligned}
K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon}K, \\
K(r,0) &= 0, \\
K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon}d\rho
\end{aligned}
$$

2-D case—revolution symmetry

Interestingly, the 2-D case is harder than the 3-D case. Kernel equations are

$$\begin{aligned} K_{rr} + \frac{K_r}{r} - K_{\rho\rho} + \frac{K_\rho}{\rho} - \frac{K}{\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} K, \\ K(r,0) &= 0, \\ K(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho \end{aligned}$$

Define $G = \sqrt{\frac{r}{\rho}} K$. Then, for G we have:

$$\begin{aligned} G_{rr} - G_{\rho\rho} + \frac{G}{4r^2} - \frac{G}{4\rho^2} &= \frac{\lambda(\rho)}{\varepsilon} G \\ G(r,0) &= 0, \\ G(r,r) &= -\int_0^r \frac{\lambda(\rho)}{2\varepsilon} d\rho. \end{aligned}$$

and we can try to prove existence & uniqueness of a solution by using the classical successive approximation method.

2-D case—revolution symmetry

Define new variables $\alpha = r + \rho$, $\beta = r - \rho$. The G equations become

$$
\begin{aligned}
4G_{\alpha\beta} + \frac{G}{(\alpha+\beta)^2} - \frac{G}{(\alpha-\beta)^2} &= \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{\varepsilon} G \\
G(\beta,\beta) &= 0, \\
G(\alpha,0) &= -\int_0^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho.
\end{aligned}
$$

2-D case—revolution symmetry

Define new variables $\alpha = r + \rho$, $\beta = r - \rho$. The G equations become

$$
\begin{aligned}
4G_{\alpha\beta} + \frac{G}{(\alpha+\beta)^2} - \frac{G}{(\alpha-\beta)^2} &= \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{\varepsilon} G \\
G(\beta,\beta) &= 0, \\
G(\alpha,0) &= -\int_0^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho.
\end{aligned}
$$

This can be transformed into the (singular) integral equation

$$
\begin{aligned}
G(\alpha,\beta) &= -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho + \int_\beta^\alpha \int_0^\beta \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G(\eta,\sigma)\, d\sigma\, d\eta \\
&\quad + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G(\eta,\sigma)\, d\sigma\, d\eta
\end{aligned}
$$

2-D case—revolution symmetry

Try the successive approximations scheme, by defining

$$G_0(\alpha,\beta) \;=\; -\int_{\beta/2}^{\alpha/2} \frac{\lambda(\rho)}{2\varepsilon} d\rho$$

and for $k > 0$,

$$G_k(\alpha,\beta) \;=\; \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\lambda\left(\frac{\eta-\sigma}{2}\right)}{4\varepsilon} G_{k-1}(\eta,\sigma)\,d\sigma d\eta + \int_{\beta}^{\alpha}\int_{0}^{\beta} \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} G_{k-1}(\eta,\sigma)\,d\sigma d\eta$$

then, the solution to the integral equation would be

$$G = \sum_{k=0}^{\infty} G_k(\alpha,\beta)$$

if the series converges.

2-D case—revolution symmetry

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \leq \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform.

2-D case—revolution symmetry

Call $\bar{\lambda} = \max_{(\alpha,\beta)\in\mathcal{T}'} \left| \frac{\lambda\left(\frac{\alpha-\beta}{2}\right)}{4\varepsilon} \right|$.

Then one clearly obtains $|G_0(\alpha,\beta)| \leq \bar{\lambda}(\alpha-\beta)$.

However when trying to substitute in G_1 even the first integral is not so easy to perform. We use an alternative approach based on the following Lemma:

Define, for $n \geq 0, k \geq 0$,

$$F_{nk}(\alpha,\beta) = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta)\frac{\log^k\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{k!}.$$

and $F_{nk} = 0$ if $n < 0$ or $k < 0$. Then F_{nk} is well-defined and nonnegative in the integration domain for all n,k, $F_{nk}(\beta,\beta) = 0$ for all n and k, $F_{nk}(\alpha,0) = 0$ if $n \geq 1$ or $k \geq 1$ and $F_{00}(\alpha,0) = \alpha$, and we have the following identity valid for $n \geq 1$ or $k \geq 1$.

$$F_{nk} = \int_\beta^\alpha \int_0^\beta \bar{\lambda}F_{(n-1)k}(\eta,\sigma)d\sigma d\eta + 4\int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2}\left(F_{n(k-1)}(\eta,\sigma) - F_{n(k-2)}(\eta,\sigma)\right)d\sigma d\eta$$

2-D case—revolution symmetry

We use the lemma to try to find estimates for the terms in the successive approximation series:

$$|G_0| \le F_{00}$$

next

$$|G_1| \le \int_\beta^\alpha \int_0^\beta \bar{\lambda} F_{00}(\eta,\sigma)\,d\sigma\,d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F_{00}(\eta,\sigma)\,d\sigma\,d\eta = F_{10} + \frac{F_{01}}{4}$$

where we have used the formulas of the lemma. The next term is

$$\begin{aligned}
|G_2| &\le \int_\beta^\alpha \int_0^\beta \bar{\lambda}\left(F_{10}+\frac{F_{01}}{4}\right) d\sigma\,d\eta + \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2}\left(F_{10}+\frac{F_{01}}{4}\right) d\sigma\,d\eta \\
&= F_{20} + \frac{F_{11}}{4} + \frac{F_{01}+F_{02}}{16}
\end{aligned}$$

If we keep going we find

$$|G_3| \le F_{30} + \frac{F_{21}}{4} + \frac{F_{11}+F_{12}}{16} + \frac{2F_{01}+2F_{02}+F_{03}}{64}$$

2-D case—revolution symmetry

The key to find these numbers is the following. Call:

$$\begin{aligned} I_1[F] &= \int_\beta^\alpha \int_0^\beta \bar{\lambda} F(\eta,\sigma)\,d\sigma\,d\eta \\ I_2[F] &= \int_\beta^\alpha \int_0^\beta \frac{\eta\sigma}{(\eta^2-\sigma^2)^2} F(\eta,\sigma)\,d\sigma\,d\eta \end{aligned}$$

For instance, to find a bound on G_4 we find the following:

$$\begin{aligned} I_1[F_{30}] &= F_{40} \\ I_2[F_{30}] + \frac{I_1[F_{21}]}{4} &= \frac{F_{31}}{4} \\ \frac{I_2[F_{21}]}{4} + \frac{I_1[F_{11}+F_{12}]}{16} &= \frac{F_{21}+F_{22}}{16} \\ \frac{I_2[F_{11}+F_{12}]}{16} + \frac{I_1[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{2F_{11}+2F_{12}+F_{13}}{64} \\ \frac{I_2[2F_{01}+2F_{02}+F_{03}]}{64} &= \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256} \end{aligned}$$

Thus,

$$|G_4| \leq F_{40} + \frac{F_{31}}{4} + \frac{F_{21}+F_{22}}{16} + \frac{2F_{11}+2F_{12}+F_{13}}{64} + \frac{5F_{01}+5F_{02}+3F_{03}+F_{04}}{256}$$

2-D case—revolution symmetry

Based on this structure, we propose the following recursive formula for $n > 0$:

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

where C_{ij} verifies $C_{ij} = C_{(i-1)(j-1)} + C_{i(j+1)}$, taking $C_{11} = 1$, $C_{i0} = 0$, and $C_{ij} = 0$ if $j > i$, for all i. This set of numbers, known as the “Catalan’s Triangle”, verifies many interesting properties.

In particular it can be shown

$$\begin{aligned} C_{ii} &= 1. \\ C_{ij} &= \sum_{k=j-1}^{i-1} C_{(i-1)k}. \end{aligned}$$

which allows us to write the recursive formula

2-D case—revolution symmetry

Let us show in a table the first few numbers.

C_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
$i=1$	1									
$i=2$	1	1								
$i=3$	2	2	1							
$i=4$	5	5	3	1						
$i=5$	14	14	9	4	1					
$i=6$	42	42	28	14	5	1				
$i=7$	132	132	90	48	20	6	1			
$i=8$	429	429	297	165	75	27	7	1		
$i=9$	1430	1430	1001	572	275	110	35	8	1	
$i=10$	4862	4862	3432	2002	1001	429	154	44	9	1

Catalan's Triangle

2-D case—revolution symmetry

Now, since the solution verifies

$$|G| \leq \sum_{n=0}^{\infty} |G_n(\alpha, \beta)|$$

and we found

$$|G_n| \leq F_{n0} + \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

We get

$$|G| \leq \sum_{n=0}^{\infty} F_{n0} + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \sum_{j=1}^{j=n-i} \frac{C_{(n-i)j}}{4^{n-i}} F_{ij}$$

and we only need to prove convergence of this series.

2-D case—revolution symmetry

First term of the series:

$$\sum_{n=0}^{\infty} F_{n0} = \frac{\bar{\lambda}^{n+1}\alpha^n\beta^n}{n!(n+1)!}(\alpha-\beta) = \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

For the next term, we use the fact that

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} H(n,i) = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty} H(l+i,i)$$

Therefore

$$\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{j=1}^{j=n-i}\frac{C_{(n-i)j}}{4^{n-i}}F_{ij} = \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{j=1}^{j=l}\frac{C_{lj}}{4^l}F_{ij} = \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\left(\sum_{l=j}^{\infty}\frac{C_{lj}}{4^l}\right)F_{ij}$$

It turns out that the parenthesis can be calculated and gives an exact sum for each j.

2-D case—revolution symmetry

To find the sum, consider first the generating function of the Catalan numbers C_{l1}:

$$f_1(x) = \frac{2}{1+\sqrt{1-4x}}$$

Remember that a generating function of a sequence of number is a function such that the coefficients of its power series is exactly those of the sequence of numbers.

Thus,

$$f_1(x) = C_{11} + C_{21}x + C_{31}x^2 + \ldots = \sum_{l=1}^{\infty} C_{l1}x^{l-1}$$

Therefore if we evaluate the function at $x = 1/4$ we find that

$$f_1(\frac{1}{4}) = \sum_{l=1}^{\infty} C_{l1}\frac{1}{4^{l-1}}$$

thus we find

$$\sum_{l=1}^{\infty} \frac{C_{l1}}{4^l} = \frac{1}{4}\sum_{l=1}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_1(\frac{1}{4})}{4} = \frac{1}{2}$$

2-D case—revolution symmetry

Following the previous argument, it is clear that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{4} \sum_{l=j}^{\infty} \frac{C_{lj}}{4^{l-1}} = \frac{f_j(\frac{1}{4})}{4}$$

where we define the generating function f_j as

$$f_j(x) = \sum_{l=j}^{\infty} C_{lj} x^{l-1}$$

Now since $C_{l2} = C_{l1}$ but obviously $C_{12} = 0$, it is clear that $f_2 = f_1 - C_{11} = f_1 - 1$. Thus $f_2(1/4) = 1$ and we find

$$\sum_{l=2}^{\infty} \frac{C_{l2}}{4^l} = \frac{f_2(\frac{1}{4})}{4} = \frac{1}{4}$$

2-D case—revolution symmetry

To find successive generating functions we use the properties of the Catalan's Triangle and make the following claim:

$$f_n(x) = f_{n-1}(x) - x f_{n-2}(x)$$

Based on this fact, we can now prove that

$$\sum_{l=j}^{\infty} \frac{C_{lj}}{4^l} = \frac{1}{2^j}$$

Thus we obtain

$$\begin{aligned}
|G| &\leq \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{F_{ij}}{2^j} \\
&= \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}} + \sum_{i=0}^{\infty}\sum_{j=1}^{j=\infty}\frac{\bar{\lambda}^{i+1}\alpha^i\beta^i}{i!(i+1)!}(\alpha-\beta)\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!}
\end{aligned}$$

2-D case—revolution symmetry

Summing the series

$$|G| \;\leq\; \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}\left(\sum_{j=0}^{j=\infty}\frac{\log^j\left(\frac{\alpha+\beta}{\alpha-\beta}\right)}{2^j j!}\right),$$

therefore

$$|G| \;\leq\; \frac{\sqrt{\bar{\lambda}}}{2}(\alpha-\beta)\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}e^{\log\left(\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\right)} = \frac{\sqrt{\bar{\lambda}}}{2}\sqrt{\alpha^2-\beta^2}\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}\alpha\beta}\right]}{2\sqrt{\alpha\beta}}$$

In physical variables r and ρ:

$$|G| \;\leq\; \sqrt{\bar{\lambda}}\sqrt{r\rho}\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

2-D case—revolution symmetry

Finally, going back to the original K, we find

$$|K(r,\rho)| \;\; \leq \;\; \rho\sqrt{\bar{\lambda}}\frac{\mathrm{I}_1\left[2\sqrt{\bar{\lambda}(r^2-\rho^2)}\right]}{2\sqrt{r^2-\rho^2}}$$

Thus, we have shown that the successive approximation series converges, with the solution K verifying the above bound. Uniqueness can be proved easily from the successive approximation series.

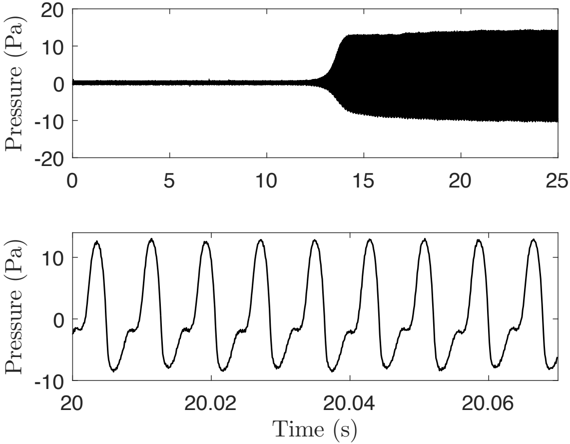
Unfortunately, this approach does not seem to be extensible for other Fourier coefficients.

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls & Symmetric disk
- Rijke Tube
- Mixed systems
- Some open problems

The Rijke Tube Experiment

Microphone signal at the onset of instability showing growth,and then saturation of the limit cycle. A zoomed-in picture shows the periodic, but nonsymetric, limit-cycle behavior.



The Rijke Tube Experiment

- Thermoacoustic instabilities are often encountered in steam and gas turbines, industrial burners, and jet and ramjet engines.
- These instabilities are undesirable and notorious difficult to model and study.
- The absence of combustion process in the Rijke tube makes the modeling and analysis more tractable.
- The Rijke tube experiment provides an accessible platform to explore and study thermoacoustic instabilities.

Nonlinear mathematical model

- The thermoacoustic oscillations can be captured using an one-dimensional model of compressible gas dynamics (Euler equations)

$$\partial_t \rho(t, x) + v(t, x)\partial_x \rho(t, x) + \rho(t, x)\partial_x v(t, x) = 0, \tag{1}$$

$$\partial_t v(t, x) + \partial_x v(t, x) + \frac{1}{\rho(t, x)}\partial_x P(t, x) = 0, \tag{2}$$

$$\partial_t P(t, x) + \gamma P(t, x)\partial_x v(t, x) + v(t, x)\partial_x P(t, x) = \overline{\gamma}\frac{1}{A}\delta(x - x_0)Q(t), \tag{3}$$

- Heat release dynamics:

$$\tau \dot{Q}(t) = -Q(t) + l_w(T_w - T_{gas})(\kappa + \kappa_v\sqrt{|v(t, x_0)|}), \tag{4}$$

- Boundary conditions:

$$P(t, 0) = P_a + U(t), \tag{5}$$

$$P(t, L) = P_a + f(v(t, L)), \tag{6}$$

Linearized mathematical model

- Assume constant steady-state solution, $(\rho,\, v\, P) = (\overline{\rho},\, \overline{v},\, \overline{P}), \forall t \in [0,\, +\infty)$, $\forall x \in [0,\, L]$, and subsonic conditions for the gas flow, i.e., $\overline{v} \approx 0$. Then,

$$\partial_t \tilde{v}(t,\, x) + \frac{1}{\overline{\rho}} \partial_x \tilde{P}(t,\, x) = 0, \tag{7}$$

$$\partial_t \tilde{P}(t,\, x) + \gamma \overline{P} \partial_x \tilde{v}(t,\, x) = \frac{\overline{\gamma}}{A} \delta(x - x_0) \tilde{Q}(t), \tag{8}$$

  and the linearized expression of the heat release dynamics

$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + f'(\overline{v})(T_w - \overline{T}_{gas}) \tilde{v}(t,\, x_0), \tag{9}$$

- Boundary conditions:

$$\tilde{P}(t,\, 0) = U(t), \tag{10}$$

$$\tilde{P}(t,\, L) = Z_L \tilde{v}(t,\, L), \tag{11}$$

Representation in characteristic coordinates

- Using the characteristic coordinates, the system (7)-(11) can be rewritten as

$$\partial_t R_1 + \lambda \partial_x R_1 = c_1 \delta(x - x_0) \tilde{Q}(t), \tag{12}$$

$$\partial_t R_2 - \lambda \partial_x R_2 = c_1 \delta(x - x_0) \tilde{Q}(t), \tag{13}$$

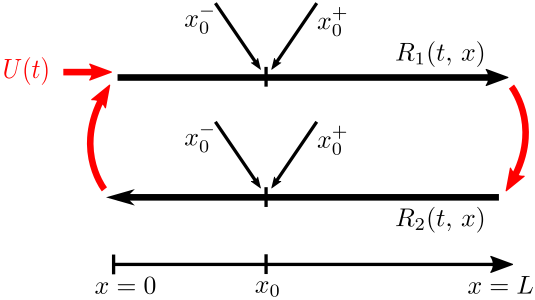
$$R_1(t, 0) = -R_2(t, 0) + 2U(t), \tag{14}$$

$$R_2(t, L) = \alpha R_1(t, L), \tag{15}$$

$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2 (R_1(t, x_0) - R_2(t, x_0)), \tag{16}$$

with λ, α, c_1, $c_2 > 0$.

Representation in characteristic coordinates



Schematic view of the jumping point at the solution of the PDE system (12)-(16).

- The following relations are satisfied:

$$R_1(t, x_0^+) = R_1(t, x_0^-) + c_1\tilde{Q}(t),$$
$$R_2(t, x_0^-) = R_2(t, x_0^+) + c_1\tilde{Q}(t).$$

Representation in characteristic coordinates

- Now, we introduce the following state variables

$$
\begin{aligned}
R_{11}(t,x) &\triangleq R_1(t,x), && \text{if } x \in [0,\, x_0] \\
R_{12}(t,x) &\triangleq R_2(t,x), && \text{if } x \in [0,\, x_0] \\
R_{21}(t,x) &\triangleq R_1(t,x), && \text{if } x \in [x_0,\, L] \\
R_{22}(t,x) &\triangleq R_2(t,x), && \text{if } x \in [x_0,\, L]
\end{aligned}
$$

and the rescaled spatial variable, so that everything evolves on the same domain:

$$
z = \begin{cases} \frac{x}{x_0} & \text{if } x \in [0,\, x_0] \\ \frac{L-x}{L-x_0} & \text{if } x \in [x_0,\, L] \end{cases}
$$

Representation in characteristic coordinates

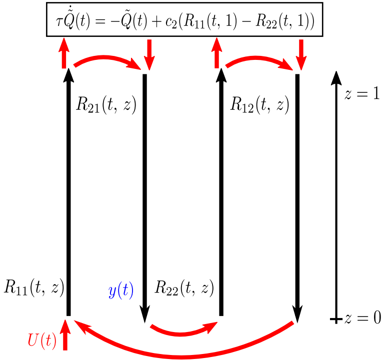
- Then, the system (12)-(16) is equivalent to

$$\partial_t R_{11}(t,\, z) + \lambda_1 \partial_z R_{11}(t,\, z) = 0, \tag{17}$$
$$\partial_t R_{12}(t,\, z) - \lambda_1 \partial_z R_{12}(t,\, z) = 0, \tag{18}$$
$$\partial_t R_{21}(t,\, z) - \lambda_2 \partial_z R_{21}(t,\, z) = 0, \tag{19}$$
$$\partial_t R_{22}(t,\, z) + \lambda_2 \partial_z R_{22}(t,\, z) = 0, \tag{20}$$

- The boundary conditions of (17)-(20) are given by

$$R_{11}(t,0) = -R_{12}(t,\, 0) + 2U(t), \tag{21}$$
$$R_{12}(t,1) = R_{22}(t,\, 1) + c_1 \tilde{Q}(t), \tag{22}$$
$$R_{21}(t,1) = R_{11}(t,\, 1) + c_1 \tilde{Q}(t), \tag{23}$$
$$R_{22}(t,0) = \alpha R_{21}(t,\, 0), \tag{24}$$
$$\tau \dot{\tilde{Q}}(t) = -\tilde{Q}(t) + c_2 (R_{11}(t,\, 1) - R_{22}(t,\, 1)). \tag{25}$$

- The boundary conditions represent two effects: reflection of the acoustic waves; and the feedback coupling between R_{21} and R_{22}, and between R_{11} and R_{12}.
- Under the right conditions the system becomes unstable due to this feedback between the states.



Backstepping-based observer design

- We design the observe as a copy of the plant (17)-(24) plus output injection terms:

$$\partial_t \hat{R}_{11}(t,\,z) + \lambda_1 \partial_z \hat{R}_{11}(t,\,z) = -p_{11}(z)\tilde{Y}(t), \tag{26}$$
$$\partial_t \hat{R}_{12}(t,\,z) - \lambda_1 \partial_z \hat{R}_{12}(t,\,z) = -p_{12}(z)\tilde{Y}(t), \tag{27}$$
$$\partial_t \hat{R}_{21}(t,\,z) - \lambda_2 \partial_z \hat{R}_{21}(t,\,z) = -p_{21}(z)\tilde{Y}(t), \tag{28}$$
$$\partial_t \hat{R}_{22}(t,\,z) + \lambda_2 \partial_z \hat{R}_{22}(t,\,z) = -p_{22}(z)\tilde{Y}(t), \tag{29}$$
$$\tau_{hr} \hat{Q}'(t) = -\hat{Q}(t) + c_2(\hat{R}_{11}(t,\,1) - \hat{R}_{22}(t,\,1)) - p_Q \tilde{Y}(t), \tag{30}$$

  with $\tilde{Y}(t) = R_{21}(t,\,0) - \hat{R}_{21}(t,\,0)$.

- The boundary conditions of (26)-(30) are given by

$$\hat{R}_{11}(t,0) = -\hat{R}_{12}(t,\,0) + 2U(t), \tag{31}$$
$$\hat{R}_{12}(t,1) = \hat{R}_{22}(t,\,1) + c_1 \hat{Q}(t), \tag{32}$$
$$\hat{R}_{21}(t,1) = \hat{R}_{11}(t,\,1) + c_1 \hat{Q}(t), \tag{33}$$
$$\hat{R}_{22}(t,0) = \alpha R_{21}(t,\,0), \tag{34}$$

- p_{11}, p_{12}, p_{21}, p_{22}, and p_Q are gains to be found.

Target system

Define the error estimation $\tilde{R}_{ij} = R_{ij} - \hat{R}_{ij}$, $i,\, j = 1,\, 2$, whose dynamics is given by

$$\partial_t \tilde{R}_{11}(t,\, z) + \lambda_1 \partial_z \tilde{R}_{11}(t,\, z) = p_{11}(z)\tilde{Y}(t), \tag{35}$$
$$\partial_t \tilde{R}_{12}(t,\, z) - \lambda_1 \partial_z \tilde{R}_{12}(t,\, z) = p_{12}(z)\tilde{Y}(t), \tag{36}$$
$$\partial_t \tilde{R}_{21}(t,\, z) - \lambda_2 \partial_z \tilde{R}_{21}(t,\, z) = p_{21}(z)\tilde{Y}(t), \tag{37}$$
$$\partial_t \tilde{R}_{22}(t,\, z) + \lambda_2 \partial_z \tilde{R}_{22}(t,\, z) = p_{22}(z)\tilde{Y}(t), \tag{38}$$
$$\tau_{hr}\tilde{Q}'(t) = -\tilde{Q}(t) + c_2(\tilde{R}_{11}(t,\, 1) - \tilde{R}_{22}(t,\, 1)) + p_Q\tilde{Y}(t), \tag{39}$$

and boundary conditions

$$\tilde{R}_{11}(t,0) = -\tilde{R}_{12}(t,\, 0), \tag{40}$$
$$\tilde{R}_{12}(t,1) = \tilde{R}_{22}(t,\, 1) + c_1\tilde{Q}(t), \tag{41}$$
$$\tilde{R}_{21}(t,1) = \tilde{R}_{11}(t,\, 1) + c_1\tilde{Q}(t), \tag{42}$$
$$\tilde{R}_{22}(t,0) = \alpha\tilde{R}_{21}(t,\, 0) - p_0\tilde{Y}. \tag{43}$$

Target system

- To design the observer output injection gains, we map (35)-(43) to the following appropriate target system:

$$\partial_t \check{R}_{11}(t,\, z) + \lambda_1 \partial_z \check{R}_{11}(t,\, z) = 0, \tag{44}$$

$$\partial_t \check{R}_{12}(t,\, z) - \lambda_1 \partial_z \check{R}_{12}(t,\, z) = 0, \tag{45}$$

$$\partial_t \check{R}_{21}(t,\, z) - \lambda_2 \partial_z \check{R}_{21}(t,\, z) = 0, \tag{46}$$

$$\partial_t \check{R}_{22}(t,\, z) + \lambda_2 \partial_z \check{R}_{22}(t,\, z) = 0, \tag{47}$$

$$\tau_{hr} \check{Q}'(t) = -(1 + c_1 c_2)\check{Q}(t) - c_2 \check{R}_{22}(t,\, 1), \tag{48}$$

with boundary conditions

$$\check{R}_{11}(t, 0) = -\check{R}_{12}(t,\, 0), \tag{49}$$

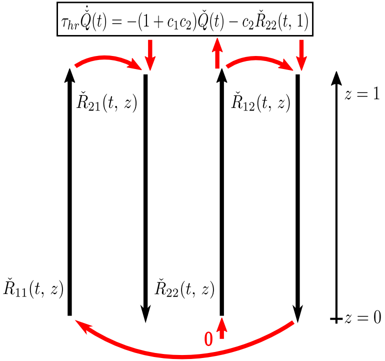
$$\check{R}_{12}(t, 1) = \check{R}_{22}(t,\, 1) + c_1 \check{Q}(t), \tag{50}$$

$$\check{R}_{21}(t, 1) = \check{R}_{11}(t,\, 1) + c_1 \check{Q}(t), \tag{51}$$

$$\check{R}_{22}(t, 0) = 0. \tag{52}$$

Target system

- The mechanism of the proof of stability of the target system is based on this scheme.
- R_{22} is identically zero for all $t \geq \lambda_2^{-1}$.
- By the cascade structure of the target system, it follows that $\check{Q} \to 0$ as $t \to \infty$.
- Finally, by computing the explicit solution of $\check{R}_{11}$, $\check{R}_{12}$ and $\check{R}_{21}$, we get that the target system is exponentially stable.



Backstepping transformation

- To map system (35)-(43) into (44)-(52), we consider the following backstepping transformation:

$$\tilde{R}_{11}(t,\, z) = \check{R}_{11}(t,\, z) - \int_0^1 P_{11}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{53}$$

$$\tilde{R}_{12}(t,\, z) = \check{R}_{12}(t,\, z) - \int_0^1 P_{12}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{54}$$

$$\tilde{R}_{21}(t,\, z) = \check{R}_{21}(t,\, z) - \int_0^z P_{21}(z,\xi)\check{R}_{21}(t,\xi)d\xi, \tag{55}$$

$$\tilde{Q}(t) = \check{Q}(t) - \int_0^1 P_Q(\xi)\check{R}_{21}(t,\xi)d\xi, \tag{56}$$

- Note that P_{21} is the kernel of a Volterra-type integral transformation, whereas P_{11} and P_{12} are the kernels of a ~~Fredholm~~-type integral transformation. P_Q is a finite dimensional kernel.

Backstepping transformation

- Differentiating (53)-(56) with respect to space and time, plugging the target system equation and integrating by parts, we obtain that (35)-(39) is mapped into (44)-(48) if and only if the kernels satisfy the following equations:

$$\lambda_2 \partial_\xi P_{11}(z,\xi) - \lambda_1 \partial_z P_{11}(z,\xi) = 0, \tag{57}$$
$$\lambda_2 \partial_\xi P_{12}(z,\xi) + \lambda_1 \partial_z P_{12}(z,\xi) = 0, \tag{58}$$
$$\partial_\xi P_{21}(z,\xi) + \partial_z P_{21}(z,\xi) = 0, \tag{59}$$
$$\tau_{hr}\lambda_2 P_Q'(\xi) = P_Q(\xi) - c_2 P_{11}(1,\xi), \tag{60}$$

and

$$P_{11}(z,1) = 0, \tag{61}$$
$$P_{12}(z,1) = 0, \tag{62}$$
$$P_Q(1) = -\frac{c_2}{\tau_{hr}\lambda_2}, \tag{63}$$
$$P_{11}(0,\xi) = -P_{12}(0,\xi), \tag{64}$$
$$P_{12}(1,\xi) = c_1 P_Q(\xi), \tag{65}$$
$$P_{21}(1,\xi) = P_{11}(1,\xi) + c_1 P_Q(\xi). \tag{66}$$

Backstepping transformation

- The observer gains are given by

$$p_{11}(z) = \lambda_2 P_{11}(z,0), \tag{67}$$
$$p_{12}(z) = \lambda_2 P_{12}(z,0), \tag{68}$$
$$p_{21}(z) = \lambda_2 P_{21}(z,0), \tag{69}$$
$$p_Q = \tau_{hr} \lambda_2 P_Q(0). \tag{70}$$

Well-posedness of the kernel equations and invertibility of the transformation

- The existence and uniqueness of the solution of the kernel equations were shown in

  - de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2018). Backstepping stabilization of a linearized ODE–PDE Rijke tube model. Automatica (pp. 98–109)

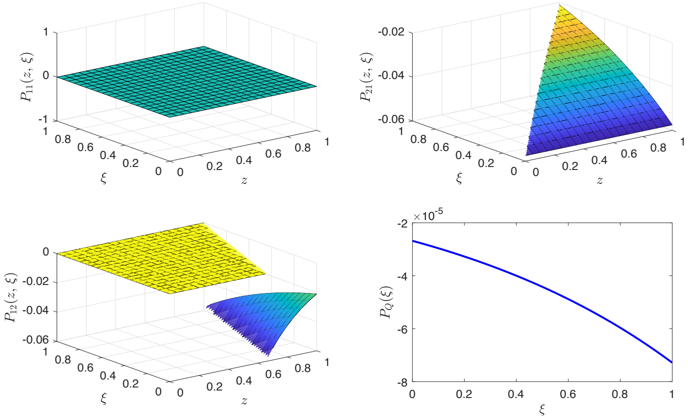
- Since these equations have a simple structure, a closed solution can be obtained by using the method of characteristics.

- In particular, the solution is piecewise-differentiable, where the number of pieces of the solution depends on the position of the heat release element.

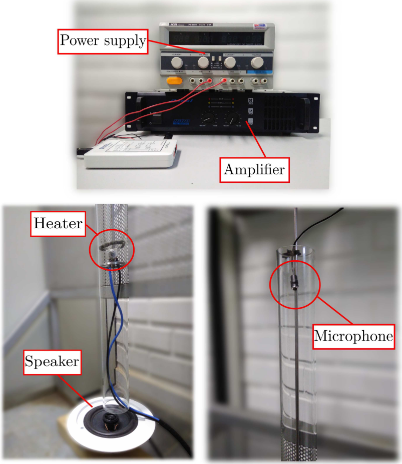
- Finally, the transformation (53)-(56) is invertible, ensuring that the target system and the observer error system have equivalent stability properties.

Numerical solution of the kernel equations

- Numerical solution of the kernel equations for the case $\lambda_1 < \lambda_2$, i.e., $x_0 > \frac{1}{2}L$ (the heater element is near from the measured boundary).

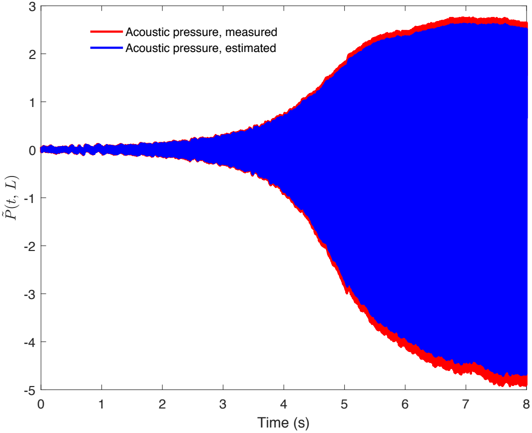


Experimental results



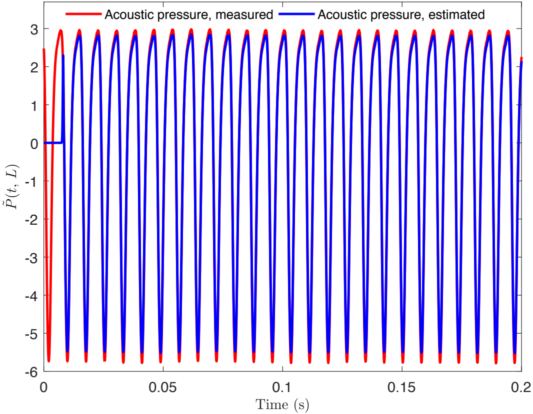
Real view of the Rijke tube experiment.

Experimental results



Time response of the measured and estimated acoustic pressure at the onset of instability.

Experimental results



Detailed view of the measured and estimated acoustic pressure.

Final remarks

- The design, requires measurements from one boundary condition and the observer gains can be computed analytically
- The resulting kernels are piecewise differentiable, with the number of pieces depending on the heat release position
- As future works, we will combine the observer design proposed in this paper with the backstepping controller that we have developed to produce real-life closed-loop experiments.
- The closed-loop experiments must be done in a real time framework because of the fast dynamics of the system and large amount of computations required to obtain the control law.

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls & Symmetric disk
- Rijke Tube
- Mixed systems
- Some open problems

Mixed hyperbolic-parabolic systems

$$\begin{aligned}
v_t(x,t) &= \varepsilon v_{xx}(x,t) + \lambda v(x,t) \\
v(0,t) &= u(0,t) \\
v(1,t) &= 0 \\
u_t(x,t) &= u_x(x,t) + \mu(x) v(x,t) + g(x) v(0,t) + \int_0^x f(x,y) v(y,t)\, dy \\
u(1,t) &= U(t) \\
& \quad (x,t) \in [0,1] \times [0,\infty)
\end{aligned}$$

Strategy: two possible transformations

Target system & transformation number 1

$$
\begin{aligned}
\eta_t(x,t) &= \varepsilon\eta_{xx}(x,t) - c\eta(x,t) \\
\eta(1,t) &= 0 \\
\eta(0,t) &= \omega(0,t) \\
\omega_t(x,t) &= \omega_x(x,t) + \mu(x)\eta(x,t) \\
\omega(1,t) &= 0
\end{aligned}
$$

Transformation:

$$
\begin{aligned}
\eta(x,t) &= v(x,t) - \int_x^1 p(x,y)v(y,t)dy \\
\omega(x,t) &= u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^1 l(x,y)v(y,t)dy
\end{aligned}
$$

Control: $U(t) = \int_0^1 k(1,y)u(y,t)dy + \int_0^1 l(1,y)v(y,t)dy$

Kernel equations number 1

For $0 \le y \le x \le 1$:

$$\begin{aligned} k_x(x,y) &= -k_y(x,y) \\ k(x,0) &= \varepsilon l_y(x,0) - g(x) + \int_0^x k(x,y) g(y)\, dy \end{aligned}$$

For $0 \le x, y \le 1$:

$$\begin{aligned} l_x(x,y) &= \varepsilon l_{yy}(x,y) + \lambda l(x,y) - h(x-y) \left[k(x,y)\mu(y) + f(x,y) - \int_y^x k(x,s) f(s,y) ds \right] \\ l(x,0) &= 0, \quad l(x,1) = 0, \quad l(0,y) = p(0,y) \end{aligned}$$

For $0 \le x \le y \le 1$:

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= \frac{\lambda + c}{\varepsilon} p(x,y) \\ p(x,1) &= 0 \\ p(x,x) &= \frac{\lambda + c}{2\varepsilon}(x-1) \end{aligned}$$

Target system & transformation number 2

Target system 1 enough for stability but it requires very large c, thus large controls.

Thus a second cleaner system is better:

$$
\begin{aligned}
\eta_t(x,t) &= \varepsilon\eta_{xx}(x,t) - c\eta(x,t) \\
\eta(1,t) &= 0 \\
\eta(0,t) &= \omega(0,t) \\
\omega_t(x,t) &= \omega_x(x,t) \\
\omega(1,t) &= 0
\end{aligned}
$$

Now arbitrary c works! The transformation is the same:

$$
\begin{aligned}
\eta(x,t) &= v(x,t) - \int_x^1 p(x,y)v(y,t)dy \\
\omega(x,t) &= u(x,t) - \int_0^x k(x,y)u(y,t)dy - \int_0^1 l(x,y)v(y,t)dy
\end{aligned}
$$

Control: $U(t) = \int_0^1 k(1,y)u(y,t)dy + \int_0^1 l(1,y)v(y,t)dy$

Kernel equations number 2

Price to pay: kernel equations are more involved

For $0 \le y \le x \le 1$:

$$
\begin{aligned}
k_x(x,y) &= -k_y(x,y) \\
k(x,0) &= \varepsilon l_y(x,0) - g(x) + \int_0^x k(x,y) g(y)\, dy
\end{aligned}
$$

For $0 \le x, y \le 1$:

$$
\begin{aligned}
l_x(x,y) &= \varepsilon l_{yy}(x,y) + \lambda l(x,y) - h(x-y) \left[k(x,y)\mu(y) + f(x,y) - \int_y^x k(x,s) f(s,y) ds \right] \\
&\quad -\delta(y-x)\mu(y) \\
l(x,0) &= 0, \quad l(x,1) = 0, \quad l(0,y) = p(0,y)
\end{aligned}
$$

For $0 \le x \le y \le 1$:

$$
\begin{aligned}
p_{xx}(x,y) - p_{yy}(x,y) &= \frac{\lambda + c}{\varepsilon} p(x,y) \\
p(x,1) &= 0 \\
p(x,x) &= \frac{\lambda + c}{2\varepsilon}(x-1)
\end{aligned}
$$

Outline

- Foundations of backstepping: basic design for a 1D parabolic equation
- Coupled hyperbolic systems
- Coupled parabolic systems
- Extension to n-balls & Symmetric disk
- Rijke Tube
- Mixed systems
- Some open problems

Some open problems

- Underactuated coupled hyperbolic and parabolic systems.

- Robustness properties of backstepping controllers.

- Non-strict-feedback terms (terms that are not “spatially-causal”).

- Reaction-diffusion equation in the n-ball with non-constant diffusion.

Design on the disk with $\lambda(r,\theta)$

$$u_t = \frac{\varepsilon}{r}(ru_r)_r + \frac{\varepsilon}{r^2}u_{\theta\theta} + \lambda(r,\theta)u,$$

It is not possible to use spherical harmonics (they are no longer eigenfunctions that decouple the problem).

Pose a physical-space transformation:

$$w = u - \int_0^r \int_{-\pi}^{\pi} K(r,\rho,\theta,\psi)u(\rho,\psi)d\psi d\rho,$$

to transform the u equation into the target system

$$w_t = \frac{\varepsilon}{r}(rw_r)_r + \frac{\varepsilon}{r^2}w_{\theta\theta},$$

Design on the disk with $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2}-\frac{K_{\theta\theta}}{r^2}+\frac{K_{\psi\psi}}{\rho^2}=\frac{\lambda(\rho,\psi)}{\varepsilon}K$$

with BC

$$\begin{aligned}
K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi)\\
K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi)\\
K(r,0,\theta,\psi) &= 0,\\
\int_{-\pi}^{\pi} K(r,r,\theta,\psi)u(r,\psi)d\psi &= -\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho u(r,\theta),
\end{aligned}$$

this last boundary condition can be verified if

$$\lim_{\rho\to r} K(r,\rho,\theta,\psi) = -\delta(\theta-\psi)\int_0^r \frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho.$$

Design on the disk with $\lambda(r,\theta)$

The kernel verifies the ultrahyperbolic equation

$$K_{rr}+\frac{K_r}{r}-K_{\rho\rho}+\frac{K_\rho}{\rho}-\frac{K}{\rho^2}-\frac{K_{\theta\theta}}{r^2}+\frac{K_{\psi\psi}}{\rho^2}=\frac{\lambda(\rho,\psi)}{\varepsilon}K$$

with BC

$$\begin{aligned}
K(r,\rho,0,\psi) &= K(r,\rho,\pi,\psi)\\
K(r,\rho,\theta,0) &= K(r,\rho,\theta,\pi)\\
K(r,0,\theta,\psi) &= 0,\\
\int_{-\pi}^{\pi}K(r,r,\theta,\psi)u(r,\psi)d\psi &= -\int_0^r\frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho u(r,\theta),
\end{aligned}$$

this last boundary condition can be verified if

$$\lim_{\rho\to r}K(r,\rho,\theta,\psi) = -\delta(\theta-\psi)\int_0^r\frac{\lambda(\rho,\theta)}{2\varepsilon}d\rho.$$

We don't know how to solve it, only know there is a solution for constant λ!

$$K(r,\rho,\theta,\psi)=-\rho\frac{\lambda}{2\pi\varepsilon}\frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda}{\varepsilon}(r^2-\rho^2)}}\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\theta-\psi)}$$

Gracias! Questions?

Some references:

- A. Smyshlyaev and M. Krstic, “Closed form boundary state feedbacks for a class of partial integro-differential equations,” IEEE Transactions on Automatic Control, 2004.

- Long Hu, Florent Di Meglio, Rafael Vazquez, Miroslav Krstic, ”Control of Homodirectional and General Heterodirectional Linear Coupled Hyperbolic PDEs,” IEEE Transactions on Automatic Control, 2016.

- R. Vazquez and M. Krstic, ”Boundary Control of Coupled Reaction-Advection-Diffusion Systems with Spatially-Varying Coefficients,” IEEE Transactions on Automatic Control, 2017.

- Long Hu, Rafael Vazquez, Florent Di Meglio, Miroslav Krstic, ”Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems,” under review in SICON, 2017. Preprint in ArXiv.

- R. Vazquez and M. Krstic, ”Explicit output-feedback boundary control of reaction-diffusion PDEs on arbitrary-dimensional balls,” ESAIM:COCV, 2016.

- R. Vazquez and M. Krstic, ”Boundary control of a singular reaction-diffusion equation on a disk,” CPDE 2016 (2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations), Bertinoro, Italy, 2016.

- de Andrade, G. A., Vazquez, R., and Pagano, D. J. (2018). Backstepping stabilization of a linearized ODEPDE Rijke tube model. Automatica (pp. 98109)

- R. Vazquez, J. Zhang, M. Krstic, J. Qi, ”Output Feedback Control of Radially-Dependent Reaction-Diffusion PDEs on Balls of Arbitrary Dimensions,” IFAC WC 2020

- G. Andrade, R. Vazquez and D. Pagano, ”Backstepping-based linear boundary observer for estimation of thermoacoustic instabilities in the Rijke tube with experimental validation,” IEEE Transactions on Automatic Control, 2020

- Guangwei Chen, Rafael Vazquez, Zhitao Liu and Hongye Su, ”Backstepping control of mixed hyperbolic-parabolic PDE system with multiple coupling terms,” to be submitted to 2021 CDC.

Contact: rvazquez1@us.es

- **In these times of "academic distancing":** open webinar in the topic of DPS control.
- **Format:** bi-weekly, 1-hour seminar, given by volunteers willing to present their results. Open to everyone!
- **Time:** Tuesdays 17-18 (Central European Time), 8-9 am (PDT), 11 am-12 pm (EDT). Dear Asian colleagues: sorry :(
- **Mailing list:** Send an email to Jean Auriol(jean.auriol@centralesupelec.fr) if you want to be included.
- **To give a talk:** Send an email to Rafael Vazquez(rvazque1@us.es) if you want to volunteer.
- **Website**: http://aero.us.es/DPSOnlineSeminar/Seminar.html