Some results about the stability and the controllability of the KdV equation

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Outline

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Motivation

We are interested to systems of type

$$\dot{Z}(t) = f(Z(t), U(t)), \qquad Z(0) = Z_0,$$

where $Z(t) \in \mathcal{H}$ is the state of the system at time t, $U(t) \in \mathcal{U}$ is the control, Z_0 is the initial state and $f : \mathcal{H} \times \mathcal{U} \to \mathcal{H}$.

Controllability

Stabilization by feedback (or closed loop)

Motivation

$$\dot{Z}(t) = f(Z(t), U(t)), \qquad Z(0) = Z_0$$

Controllability

The possibility to find a time $T > 0$ and a control $U(t)$ allowing to bring the state Z from a given initial state Z_0 to a final state Z_f at time T.

$\hookrightarrow$ *Exact controllability, null controllability, approximate controllability, global, local, at finite time,...*

Stabilization by feedback (or closed loop)

Motivation

$$\dot{Z}(t) = f(Z(t), U(t)), \qquad Z(0) = Z_0$$

Controllability

Stabilization by feedback (or closed loop)

Bring the state Z closed to a state Z_f by taking U in the form $U(t) = K(Z(t))$, i.e.

$$\|Z(t) - Z_f\| \to 0 \qquad \text{when } t \to +\infty.$$

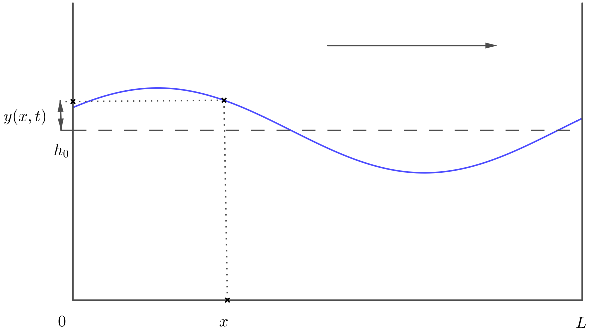
$\hookrightarrow$ *Exponential stability, polynomial stability,...*

Motivation

Korteweg-de Vries equation 1895 (Russell 1834, Boussinesq 1877, Bona-Winther 1983)

$$y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) = 0$$

Model water waves propagating along a shallow canal.



The stability problem

The goal is to study the stability of the following non-linear KdV equation with a boundary feedback term on a bounded domain

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) & \\ \qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\, t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = \alpha y_x(0,t), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{1}$$

In the above equations:

- $y(x,t)$: amplitude of the water wave at position x at time t;
- $L > 0$ is the length of the spacial domain;
- α is a real constant parameter;
- $y_0 \in L^2(0,L)$.

Known results with boundary feedback

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) & \\ \qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t>0, \\ y(0,t) = y(L,t) = 0, & t>0, \\ y_x(L,t) = \alpha y_x(0,t), & t>0, \\ y(x,0) = y_0(x), & x \in (0,L). \end{cases}$$

Stability result [Zhang 1994 ($L = 1$), Perla Menzala, Vasconcellos, Zuazua 2002]

For $L \notin \mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}},\ k,l \in \mathbb{N}^* \right\}$ and $|\alpha| < 1$, local exponential stability result (i.e. for small initial data).

Remark

If $L = 2\pi$, there exists a solution ($y(x,t) = 1 - \cos x$) of the linearized system around 0 which has a constant energy.

Known results with internal feedback

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) & \\ \qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases}$$

where a is a nonnegative function in $L^\infty(0,L)$ such that $a(x) \geq a_0 > 0$ a.e. in an open nonempty subset ω of $(0,L)$.

Stability result [Perla Menzala, Vasconcellos, Zuazua 2002, Pazoto 2005]

For any $L > 0$, local exponential stability result (i.e. for small initial data) and semi-global stability result (i.e. for any initial data in a given ball).

The controllability problem with one control

The second question is to study the controllability of the following non-linear KdV equation with a boundary control on a bounded domain

$$
\begin{cases}
y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) & \\
\qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t > 0, \\
y(0,t) = y(L,t) = 0, & t > 0, \\
y_x(L,t) = u(t), & t > 0, \\
y(x,0) = y_0(x), & x \in (0,L),
\end{cases}
\tag{2}
$$

In the above equations:

- $y(x,t)$: amplitude of the water wave at position x at time t;
- $L > 0$ is the length of the spacial domain;
- $u(t)$ is the control in $L^2(0,T)$;
- $y_0 \in L^2(0,L)$.

Known results with one control

Theorem (Rosier 1997)

- KdV equation linearized around 0 is exactly controllable in $L^2(0,L)$ iff $L \notin \{2\pi\sqrt{\frac{k^2+l^2+kl}{3}},\ k,l \in \mathbb{N}^*\}$.
- If $L \notin \{2\pi\sqrt{\frac{k^2+l^2+kl}{3}},\ k,l \in \mathbb{N}^*\}$, then KdV equation is locally exactly controllable.

Theorem ([Coron, Crépeau 2004], [Cerpa 2007], [Cerpa, Crépeau 2009])

For all $L > 0$, KdV equation is locally exactly controllable at a time T large enough.

Outline

The first problem at a glance

The first main goal is to study the stability of the following non-linear KdV equation with a boundary feedback delayed term

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) & \\ \qquad\qquad + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t>0, \\ y(0,t) = y(L,t) = 0, & t>0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0,t-h), & t>0, \\ y_x(0,t) = z_0(t), & t \in (-h,0), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{3}$$

- $y(x,t)$: amplitude of the water wave at position x at time t;
- $h>0$ is the delay;
- $L>0$ is the length of the spacial domain;
- α and $\beta \neq 0$ are real constant parameters;
- $y_0 \in L^2(0,L)$ and $z_0 \in L^2(-h,0)$.

Energy

Let us choose the following definition of the energy of system (3):

$$E(t) = \int_0^L y^2(x,t)dx + |\beta|\, h \int_0^1 y_x^2(0, t - h\rho)d\rho.$$

Moreover, we will assume, that the parameters α and β satisfy the following limitation:

$$|\alpha| + |\beta| < 1.$$

Goal

- Long-time behavior of the energy $E(t)$
- Exponential stability: $E(t) \leq Ce^{-\nu t}E(0)$, $\forall t > 0$
- Robustness with respect to the delay

Previous results for equations with delay

Consider, for instance, the wave equation with boundary feedback delay:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & x \in (0,L),\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u_x(L,t) = -\alpha u_t(L,t) - \beta u_t(L,t-h), & t > 0, \\ u_t(L,t) = z_0(t), & t \in (-h,0), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & x \in (0,L). \end{cases}$$

Assumption

$$0 \leq \beta < \alpha$$

If it is not the case, it can be shown that instabilities may appear:

- Datko 1988, Datko, Lagnese, Polis 1986 with $\alpha = 0$
- Nicaise, Pignotti 2006 in the more general case for the wave equation (see also Nicaise, V. 2010).

Strategy for the well-posedness

Ideas [Rosier 1997]

- Well-posedness result of the linear equation, with a priori estimates and regularity of the solutions,
- KdV linear equation with a right hand side,
- Well-posedness result of the nonlinear equation by a fixed point argument.

The linear KdV equation

We begin by proving the well-posedness of the KdV equation linearized around 0, that writes

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) = 0, & x \in (0,L),\ t>0, \\ y(0,t) = y(L,t) = 0, & t>0, \\ y_x(L,t) = \alpha y_x(0,t) + \beta y_x(0,t-h), & t>0, \\ y_x(0,t) = z_0(t), & t \in (-h,0), \\ y(x,0) = y_0(x), & x \in (0,L). \end{cases} \tag{4}$$

Taking into consideration of the delay

Following Nicaise and Pignotti 2006, we set

$$z(\rho,t)=y_x(0,t-\rho h)$$

for any $\rho\in(0,1)$ and $t>0$. Then z satisfies the transport equation

$$\begin{cases} hz_t(\rho,t)+z_\rho(\rho,t)=0, & \rho\in(0,1),\ t>0,\\ z(0,t)=y_x(0,t), & t>0,\\ z(\rho,0)=z_0(-\rho h), & \rho\in(0,1). \end{cases}$$

Consequently, (4) can be written as

$$\begin{cases} y_t(x,t)+y_{xxx}(x,t)+y_x(x,t)=0, & x\in(0,L),\ t>0,\\ hz_t(\rho,t)+z_\rho(\rho,t)=0, & \rho\in(0,1),\ t>0,\\ y(0,t)=y(L,t)=0, & t>0,\\ z(0,t)=y_x(0,t), & t>0,\\ y_x(L,t)=\alpha y_x(0,t)+\beta z(1,t), & t>0,\\ z(\rho,0)=z_0(-\rho h), & \rho\in(0,1),\\ y(x,0)=y_0(x), & x\in(0,L). \end{cases}$$

First order system

We introduce the Hilbert space $H = L^2(0,L) \times L^2(0,1)$ equipped with the inner product

$$\left\langle \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^L y\tilde{y}\,dx + |\beta|\,h \int_0^1 z\tilde{z}\,d\rho.$$

We denote by $\|\cdot\|_H$ the associated norm and this new norm is equivalent to the usual norm on H.

We then rewrite (4) as a first order system:

$$\begin{cases} U_t(t) = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0 \in H, \end{cases} \quad \text{where } U = \begin{pmatrix} y \\ z \end{pmatrix}, U_0 = \begin{pmatrix} y_0 \\ z_0(-h\ \cdot) \end{pmatrix},$$

and where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \begin{pmatrix} -\partial_{xxx} - \partial_x & 0 \\ 0 & -\frac{1}{h}\partial_\rho \end{pmatrix}, \text{ with domain}$$

$$\mathcal{D}(\mathcal{A}) = \{(y,z) \in H^3(0,L) \times H^1(0,1) \,|\, y(0) = y(L) = 0,$$
$$z(0) = y_x(0), y_x(L) = \alpha y_x(0) + \beta z(1)\}.$$

Well-posedness result

We define the space

$$\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$$

endowed with the norm

$$\|y\|_{\mathcal{B}} = \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y\|_{H^1(0,L)}^2 \, dt \right)^{1/2} .$$

To prove the well-posedness result of the non-linear KdV equation, we exactly follow Coron-Crépeau 2004 (see also Cerpa 2014).

Proposition

Assume $|\alpha| + |\beta| < 1$*. There exist* $r > 0$ *and* $C > 0$ *such that for every* $(y_0, z_0(-h\cdot)) \in H$ *such that*

$$\|(y_0, z_0(-h\cdot))\|_H \leq r,$$

there exists a unique solution of (3) *which satisfies*

$$\|y\|_{\mathcal{B}} \leq C \|(y_0, z_0(-h\cdot))\|_H .$$

Decay of the energy

Proposition

Let $|\alpha| + |\beta| < 1$. *Then, for any regular solution of* (3) *the energy* E *is non-increasing and satisfies*

$$E'(t) = \left(\alpha^2 - 1 + |\beta|\right) y_x^2(0,t) + \left(\beta^2 - |\beta|\right) y_x^2(0,t-h) + 2\alpha\beta y_x(0,t) y_x(0,t-h) \leq 0.$$

Why E is non-increasing ?

Differentiating E and using the system, we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &= -2\int_0^L y(x,t)(y_{xxx}+y_x+yy_x)(x,t)dx \\
&\quad -2\,|\beta|\int_0^1 y_x(0,t-h\rho)\partial_\rho y_x(0,t-h\rho)d\rho \\
&= y_x^2(L,t)-y_x^2(0,t)-|\beta|\,y_x^2(0,t-h)+|\beta|\,y_x^2(0,t) \\
&= \left(\alpha^2-1+|\beta|\right)y_x^2(0,t)+\left(\beta^2-|\beta|\right)y_x^2(0,t-h) \\
&\quad +2\alpha\beta y_x(0,t)y_x(0,t-h) \\
&= (MX(t),X(t))\,,
\end{aligned}$$

where

$$X(t)=\begin{bmatrix} y_x(0,t) \\ y_x(0,t-h)\end{bmatrix} \quad \text{and} \quad M=\begin{bmatrix}\alpha^2-1+|\beta| & \alpha\beta \\ \alpha\beta & \beta^2-|\beta|\end{bmatrix}.$$

Lyapunov functionnal

We choose now the following Lyapunov functionnal

$$V(t) = E(t) + \mu_1 V_1(t) + \mu_2 V_2(t),$$

where μ_1 and $\mu_2 \in (0,1)$ are positive constants that will be fixed small enough later on, V_1 is defined by

$$V_1(t) = \int_0^L x y^2(x,t)\,dx,$$

and V_2 is defined by

$$V_2(t) = h \int_0^1 (1-\rho) y_x^2(0, t-h\rho)\,d\rho.$$

It is clear that the two energies E and V are equivalent, in the sense that

$$E(t) \leq V(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{|\beta|}\right\}\right) E(t).$$

First stability result [Baudouin, Crépeau, V. 2019]

Theorem

Assume $|\alpha| + |\beta| < 1$ *and assume that the length* L *fulfills*

$$L < \pi\sqrt{3}.$$

Then, there exist $r > 0$ *sufficiently small, such that for every* $(y_0, z_0) \in L^2(0,L) \times L^2(-h,0)$ *satisfying*

$$\|(y_0, z_0)\|_{L^2(0,L)\times L^2(-h,0)} \leq r,$$

the energy E *of system* (3) *decays exponentially:*

$$E(t) \leq \kappa E(0) e^{-2\gamma t}, \qquad t > 0,$$

where for $\mu_1 > 0$ *and* $\mu_2 \in (0,1)$ *sufficiently small*

$$\gamma \leq \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2} r\pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2(\mu_2 + |\beta|)h} \right\}.$$

Idea of the proof

Let y be a regular solution of (3). For any $\gamma > 0$, we have

$$\begin{aligned}
\frac{d}{dt}V(t) + 2\gamma V(t) \quad \leq \quad & \left(\widetilde{M}X(t), X(t)\right) - 3\mu_1 \int_0^L y_x^2(x,t)dx \\
& + (2\gamma h(\mu_2 + |\beta|) - \mu_2) \int_0^1 y_x^2(0, t - h\rho)d\rho \\
& + (2\gamma(1 + L\mu_1) + \mu_1) \int_0^L y^2(x,t)dx \\
& + \frac{2}{3}\mu_1 \int_0^L y^3(x,t)dx,
\end{aligned}$$

where

$$X(t) = \begin{bmatrix} y_x(0,t) \\ y_x(0,t-h) \end{bmatrix}$$

and

$$\widetilde{M} = \begin{bmatrix} (1 + L\mu_1)\alpha^2 - 1 + |\beta| + \mu_2 & \alpha\beta(1 + L\mu_1) \\ \alpha\beta(1 + L\mu_1) & (1 + L\mu_1)\beta^2 - |\beta| \end{bmatrix}.$$

Idea of the proof

Thus we have,

$$\widetilde{M} = M + \mu_1 L \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where M is defined as previously. As M is definite negative, we easily prove that for $\mu_1 > 0$ and $\mu_2 > 0$ sufficiently small the matrix $\widetilde{M}$ is definite negative, by continuity of the applications Trace and Determinant.

Finally, for μ_1 and μ_2 sufficiently small, using Poincaré inequality, we obtain that

$$\frac{d}{dt}V(t) + 2\gamma V(t) \leq \left(\frac{L^2\left(2\gamma\left(1 + L\mu_1\right) + \mu_1\right)}{\pi^2} - 3\mu_1 \right) \int_0^L y_x^2(x,t)dx$$
$$+ \left(2\gamma h(\mu_2 + |\beta|) - \mu_2\right) \int_0^1 y_x^2(0, t - h\rho)d\rho$$
$$+ \frac{2}{3}\mu_1 \int_0^L y^3(x,t)dx.$$

Idea of the proof

Moreover, using Cauchy-Schwarz inequality and since $H^1(0,L)$ embeds in $L^\infty(0,L)$, we have:

$$\begin{aligned}
\int_0^L y^3(x,t)dx &\leq \|y\|^2_{L^\infty(0,L)} \int_0^L y(x,t)dx \\
&\leq L\sqrt{L}\,\|y\|^2_{H^1(0,L)}\,\|y\|_{L^2(0,L)} \\
&\leq L^{3/2}\,\|(y_0,z_0(-h\cdot))\|_H\,\|y\|^2_{H^1(0,L)} \\
&\leq L^{3/2}\,r\,\|y\|^2_{H^1(0,L)}\,.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
&\frac{d}{dt}V(t) + 2\gamma V(t) \\
&\leq \left(\frac{L^2\left(2\gamma\left(1+L\mu_1\right)+\mu_1\right)}{\pi^2} - 3\mu_1 + \frac{2L^{3/2}r\mu_1}{3}\right)\int_0^L y_x^2(x,t)dx \\
&\qquad + \left(2\gamma h(\mu_2+|\beta|)-\mu_2\right)\int_0^1 y_x^2(0,t-h\rho)d\rho.
\end{aligned}$$

End of the proof

It is then sufficient to choose r small enough such that
$r < \dfrac{3(3\pi^2 - L^2)}{2L^{3/2}\pi^2}$ (which is possible due to $L < \sqrt{3}\pi$) and $\gamma > 0$
such that

$$\gamma \leq \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{3/2} r\pi^2)\mu_1}{6L^2(1 + L\mu_1)}, \frac{\mu_2}{2(\mu_2 + |\beta|)h} \right\}.$$

to have

$$\frac{d}{dt} V(t) + 2\gamma V(t) \leq 0,$$

which is equivalent to $V(t) \leq V(0)e^{-2\gamma t}$ for any $t > 0$. Using the equivalence between E and V, we obtain that

$$E(t) \leq \left(1 + \max\left\{L\mu_1, \frac{\mu_2}{|\beta|}\right\}\right) E(0)e^{-2\gamma t}, \qquad t > 0.$$

By density of $\mathcal{D}(\mathcal{A})$ in H, the results extend to arbitrary $(y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)$.

Observability result of the linear equation

Theorem

Assume that $|\alpha| + |\beta| < 1$ *is satisfied. Let* $L \in (0, +\infty) \setminus \mathcal{N}$, *where*

$$\mathcal{N} = \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}},\ k, l \in \mathbb{N}^* \right\}$$

and $T > h$. *Then there exists* $C > 0$ *such that for all* $(y_0, z_0(-h\cdot)) \in H$, *we have*

$$\int_0^L y_0^2(x)dx + |\beta|\, h \int_0^1 z_0^2(-h\rho)d\rho \leq C \int_0^T \left(y_x^2(0,t) + z^2(1,t)\right) dt$$

where $(y, z) = S(.)(y_0, z_0(-h\cdot))$.

Ideas of the proof

- Contradiction argument (as in Rosier 1997);
- Generalized Aubin-Lions theorem (Simon 1987).

Stability result for the linear KdV equation

Theorem

Assume that

$$L \in (0, +\infty) \setminus \mathcal{N} \qquad \textit{and} \qquad |\alpha| + |\beta| < 1.$$

Then, for every $(y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)$, *the energy of the* *linear* system (4) *decays exponentially.*

Ideas of the proof

Combinate

- the observability inequality,
- the decay of the energy,
- the fact that the system is invariant by translation in time.

Remark: The value of the decay rate can not be estimated in this approach.

Exponential decay of small amplitude solutions of the non-linear KdV equation [Baudouin, Crépeau, V. 2019]

Theorem

Assume that

$$L \in (0, +\infty) \setminus \mathcal{N} \quad \text{and} \quad |\alpha| + |\beta| < 1.$$

Then, there exists $r > 0$ *such that for* $(y_0, z_0) \in L^2(0, L) \times L^2(-h, 0)$ *st*

$$\|(y_0, z_0)\|_{L^2(0,L)\times L^2(-h,0)} \leq r,$$

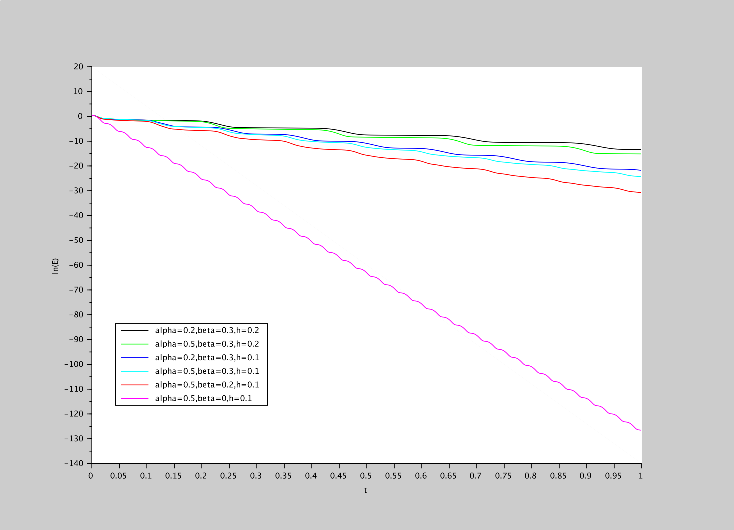
the energy of the non-linear system (3) *decays exponentially.*

Idea of the proof: follows Cerpa 2014

- To decompose the solution as the solution of the linear system and the solution of the linear system with trivial initial data and right hand side;
- Use the exponential stability of the linear system.

Numerical simulations: $t \mapsto \ln(E(t))$ for different values of α, β and h (adaptation of [Colin, Gisclon, 2001])

$T = 1$, $L = 1$ and $y_0(x) = 1 - \cos(2\pi x)$ and $z_0(\rho) = 0.1\sin(-2\pi\rho h)$



Outline

First case: $\operatorname{supp} b \subset \operatorname{supp} a$

$$
\begin{cases}
y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + a(x)y(x,t) & \\
\qquad + b(x)y(x,t-h) + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t > 0, \\
y(0,t) = y(L,t) = y_x(L,t) = 0, & t > 0, \\
y(x,0) = y_0(x), & x \in (0,L), \\
y(x,t) = z_0(x,t), & x \in \omega,\ t \in (-h,0),
\end{cases}
\tag{5}
$$

where $a = a(x)$ and $b = b(x)$ are nonnegative functions belonging to $L^\infty(0,L)$. We will also assume that $\operatorname{supp} b = \omega$, $b(x) \geq b_0 > 0$ a.e. in an open, nonempty subset ω of $(0,L)$. We first assume that

$$
\exists c_0 > 0, \qquad b(x) + c_0 \leq a(x), \qquad \text{a.e. in } \omega.
$$

We define the energy as

$$
E(t) = \int_0^L y^2(x,t)dx + h \int_\omega \int_0^1 \xi(x) y^2(x, t-h\rho) d\rho dx,
$$

where $\xi \in L^\infty(0,L)$ is chosen such that $\operatorname{supp} \xi = \operatorname{supp} b = \omega$ and

$$
b(x) + c_0 \leq \xi(x) \leq 2a(x) - b(x) - c_0, \qquad x \in \omega.
$$

First case: $\operatorname{supp} b \subset \operatorname{supp} a$

$$\begin{aligned}
\frac{d}{dt}E(t) &= -y_x^2(0,t) - 2\int_0^L a(x)y^2(x,t)dx \\
&\quad -2\int_0^L b(x)y(x,t)y(x,t-h)dx + \int_\omega \xi(x)y^2(x,t)dx \\
&\quad - \int_\omega \xi(x)y^2(x,t-h)dx \\
&\leq -y_x^2(0,t) + \int_\omega \left(-2a(x) + b(x) + \xi(x)\right) y^2(x,t)dx \\
&\quad -2\int_{(0,L)\setminus\omega} a(x)y^2(x,t)dx + \int_\omega \left(b(x) - \xi(x)\right) y^2(x,t-h)dx.
\end{aligned}$$

Results [V. 2020]:

- Local exponential stability result with a Lyapunov approach for $L < \pi\sqrt{3}$,
- Local exponential stability result for any $L > 0$,
- Semi-global stability result for any $L > 0$.

Second case: $\operatorname{supp} b \not\subset \operatorname{supp} a$

In this case, the derivative of the energy E satisfies

$$\begin{aligned}
\frac{d}{dt}E(t) &= -y_x^2(0,t) - 2\int_{\operatorname{supp} a} a(x)y^2(x,t)dx \\
&\quad -2\int_\omega b(x)y(x,t)y(x,t-h)dx \\
&\quad + \int_\omega \xi(x)y^2(x,t)dx - \int_\omega \xi(x)y^2(x,t-h)dx \\
&\leq -y_x^2(0,t) - 2\int_{\operatorname{supp} a} a(x)y^2(x,t)dx + \int_\omega (b(x)+\xi(x))y^2(x,t)dx \\
&\quad + \int_\omega (b(x)-\xi(x))y^2(x,t-h)dx,
\end{aligned}$$

and so the energy is not decreasing in general due to the term $b(x)+\xi(x) > 0$ on ω.

Second case: $\operatorname{supp} b \not\subset \operatorname{supp} a$

Following Nicaise Pignotti 2014, we consider the next auxiliary problem, which is "close" to (5) but whose the energy is decreasing:

$$\begin{cases} \widetilde{y}_t(x,t) + \widetilde{y}_{xxx}(x,t) + \widetilde{y}_x(x,t) + \widetilde{y}(x,t)\widetilde{y}_x(x,t) + a(x)\widetilde{y}(x,t) + b(x)\widetilde{y}(x,t-h) \\ \qquad + \xi b(x)\widetilde{y}(x,t) = 0, \\ \widetilde{y}(0,t) = \widetilde{y}(L,t) = \widetilde{y}_x(L,t) = 0, \\ \widetilde{y}(x,0) = y_0(x), \\ \widetilde{y}(x,t) = z_0(x,t), \end{cases} \tag{6}$$

where ξ is a positive constant. Then the derivative of the energy E defined by

$$E(t) = \int_0^L \widetilde{y}^2(x,t)dx + h\xi \int_\omega \int_0^1 b(x)\widetilde{y}^2(x,t-h\rho)d\rho dx,$$

with $\xi > 1$ satisfies

$$\begin{aligned} \frac{d}{dt}E(t) &= -\widetilde{y}_x^2(0,t) - 2\int_{\operatorname{supp} a} a(x)\widetilde{y}^2(x,t)dx - 2\int_\omega b(x)\widetilde{y}(x,t)\widetilde{y}(x,t-h)dx \\ &\quad -2\xi\int_\omega b(x)\widetilde{y}^2(x,t)dx + \xi\int_\omega b(x)\widetilde{y}^2(x,t)dx - \xi\int_\omega b(x)\widetilde{y}^2(x,t-h)dx \\ &\leq -\widetilde{y}_x^2(0,t) - 2\int_{\operatorname{supp} a} a(x)\widetilde{y}^2(x,t)dx + \int_\omega (b(x) - \xi b(x))\widetilde{y}^2(x,t)dx \\ &\quad + \int_\omega (b(x) - \xi b(x))\widetilde{y}^2(x,t-h)dx \leq 0. \end{aligned}$$

Second case: $\operatorname{supp} b \not\subset \operatorname{supp} a$

We would like to use the classical perturbation result of Pazy:

Theorem (Pazy)

Let X be a Banach space and let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X satisfying $\|T(t)\| \leq Me^{\omega t}$. If B is a bounded linear operator on X, then $A+B$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on X satisfying $\|S(t)\| \leq Me^{(\omega+M\|B\|)t}$.

Strategy:

1. Exponential stability for (6) linearized around 0 by the Lyapunov approach for all $L < \sqrt{3}\pi$;
2. Exponential stability for (5) linearized around 0 using the perturbation theorem of Pazy for all $L < \sqrt{3}\pi$ and for $\|b\|_{L^\infty(0,L)}$ small enough ($-\alpha + \sqrt{\beta}\xi \|b\|_{L^\infty(0,L)} < 0$);
3. Local exponential stability for the nonlinear system (5) for all $L < \sqrt{3}\pi$ and for $\|b\|_{L^\infty(0,L)}$ small enough using the same proof as previously.

Result [V. 2020]

Theorem

Let $L < \sqrt{3}\pi$ and $\xi > 1$. Then there exist $\delta > 0$ (depending on ξ, L, h) and $r > 0$ sufficiently small such that if

$$\|b\|_{L^\infty(0,L)} \leq \delta,$$

for every $(y_0, z_0) \in \mathcal{H}$ satisfying

$$\|(y_0, z_0)\|_{\mathcal{H}} \leq r,$$

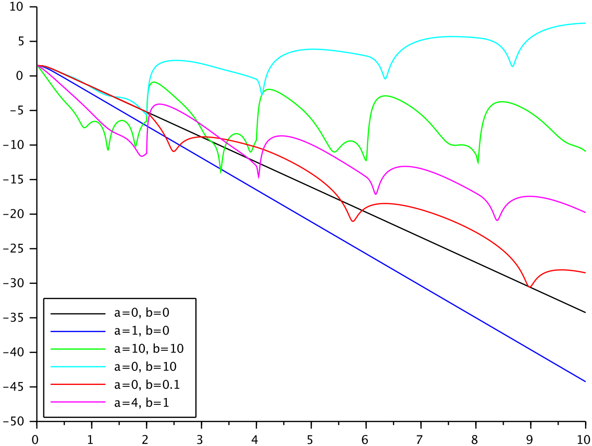
the energy decays exponentially.

Remarks:

- we can take $a = 0$,
- if h is large, the choice of b is such that $\|b\|_{L^\infty(0,L)}$ is small.

Numerical simulations: $t \mapsto \ln(E(t))$ for different values of a and b

$T = 10$, $L = 3$, $h = 2$ and $y_0(x) = 1 - \cos(2\pi x)$ and
$z_0(x,\rho) = (1 - \cos(2\pi x))\cos(2\pi\rho h)$ with $\operatorname{supp} a = \operatorname{supp} b = (0, L/5)$



Some open problems

- Time-varying delay;
- Improve some previous results, for instance remove $L < \sqrt{3}\pi$, with an appropriate Lyapunov functional;
- Time-delay on the nonlinear term "$y(x, t-h)y_x(x,t)$";

Outline

Known results with one control

$$\begin{cases} y_t(x,t) + y_{xxx}(x,t) + y_x(x,t) + y(x,t)y_x(x,t) = 0, & x \in (0,L),\ t > 0, \\ y(0,t) = y(L,t) = 0, & t > 0, \\ y_x(L,t) = u(t), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases}$$

Theorem (Rosier 1997)

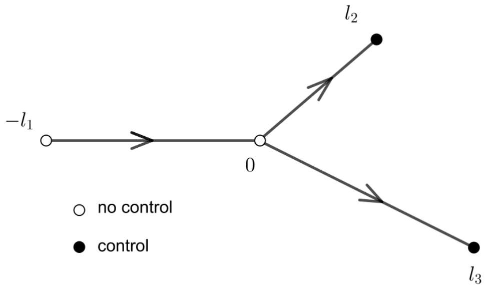
- KdV equation linearized around 0 is exactly controllable in $L^2(0,L)$ iff $L \notin \{2\pi\sqrt{\frac{k^2+l^2+kl}{3}},\ k,l \in \mathbb{N}^*\}$.
- If $L \notin \{2\pi\sqrt{\frac{k^2+l^2+kl}{3}},\ k,l \in \mathbb{N}^*\}$, then KdV equation is locally exactly controllable.

Theorem ([Coron, Crépeau 2004], [Cerpa 2007], [Cerpa, Crépeau 2009])

For all $L > 0$, KdV equation is locally exactly controllable at a time T large enough.

A tree-shaped network

We consider a tree-shaped network $\mathcal{R}$ of $(N+1)$ edges e_i, of lengths $l_i > 0$, $i \in \{1, .., N+1\}$, connected at one vertex that we assume to be 0 for all the edges.
e_1 is parametrized on the interval $I_1 := (-l_1, 0)$ and the N other edges e_i are parametrized on the interval $I_i := (0, l_i)$.



The nonlinear KdV equation on a tree

$$\begin{cases} (y_{i,t} + y_{i,x} + y_{i,xxx} + y_i y_{i,x})(x,t) = 0, \ i \in \{1, \cdots, N+1\}, \ x \in I_i, \ t > 0, \\ y_1(-l_1, t) = 0, & t > 0, \\ \left.\begin{array}{l} y_i(l_i, t) = 0, \\ y_{i,x}(l_i, t) = h_i(t), \\ y_1(0,t) = \alpha_i y_i(0,t), \end{array}\right\} & \forall i \in \{2, \cdots, N+1\}, \ t > 0, \\ y_{1,x}(0,t) = \displaystyle\sum_{i=2}^{N+1} \beta_i y_{i,x}(0,t), & t > 0, \\ y_{1,xx}(0,t) = \displaystyle\sum_{i=2}^{N+1} \frac{1}{\alpha_i} y_{i,xx}(0,t), & t > 0, \\ y_i(x,0) = y_{i0}(x), & \forall i \in \{1, \cdots, N+1\}, \ x \in I_i, \end{cases}$$

- $y_i(x,t)$: amplitude of the water wave on e_i at $x \in I_i$ at t,
- $h_i = h_i(t)$ is the control on the edge e_i belonging to $L^2(0,T)$,
- α_i and β_i are positive constants, $y_{i0} \in L^2(I_i)$,
- the transmission conditions at the central node 0 are inspired by [Mugnolo, Noja, Seifert 2018] and [Cavalcante 2018].

Notation

Goal: exact controllability

For any $T > 0$, $l_i > 0$, $y_0 \in L^2(\mathcal{R})$ and $y_T \in L^2(\mathcal{R})$, is it possible to find N Neumann boundary controls $h_i \in L^2(0, T)$ such that the solution y on the tree shaped network of $N + 1$ edges satisfies $y(\cdot, 0) = y_0$ and $y(\cdot, T) = y_T$?

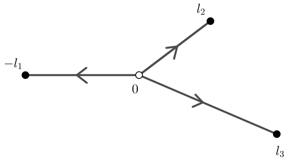
Known results about the controllability of the KdV equation on a network

Known results: star-shaped network

- Ammari, Crépeau 2018: $N+1$ controls for N edges,
- Cerpa, Crépeau, Moreno 2020: N controls for N edges.

Main differences

- the sense of the propagation of the water wave on the first edge,
- the transmission conditions at the central node,
- the fact that we improve the previous results having one control less.



Well-posedness result

Proposition

Let $T>0$, $l_i>0$ *and assume*

$$\sum_{i=2}^{N+1}\frac{1}{\alpha_i^2}\leq 1 \quad \textit{and} \quad \sum_{i=2}^{N+1}\beta_i^2\leq 1.$$

Then, there exist $r>0$ *and* $C>0$ *such that for every* $y_0\in L^2(\mathcal{R})$ *and* $h_i\in L^2(0,T)$ *verifying*

$$\|y_0\|_{L^2(\mathcal{R})}+\sum_{i=2}^{N+1}\|h_i\|_{L^2(0,T)}\leq r,$$

there exists a unique $y\in\mathcal{B}=C([0,T],L^2(\mathcal{R}))\cap L^2(0,T,H_0^1(\mathcal{R}))$ *which satisfies*

$$\|y\|_{\mathcal{B}}\leq C\left(\|y_0\|_{L^2(\mathcal{R})}+\sum_{i=2}^{N+1}\|h_i\|_{L^2(0,T)}\right).$$

The proof of the well-posedness result

Ideas of the proof

- Linear equation with no control, then with regular initial data and controls,
- Linear equation with less regularity on the data using density and multiplier arguments,
- Linear equation with a source term,
- Nonlinear equation by fixed point argument.

Controllability result [Cerpa, Crépeau, V. 2020]

Theorem

Let $\alpha_i = \sqrt{N}$, $\beta_i = \dfrac{1}{\sqrt{N}}$ *and* $l_i > 0$ *satisfying*

$$L := \max_{i=1,\cdots,N+1} l_i < \sqrt{3}\pi. \tag{7}$$

There exists $T_{\min} > 0$ *such that the system is locally exactly controllable in any time* $T > T_{\min}$*: there exists* $r > 0$ *sufficiently small such that for any* $y_0 \in L^2(\mathcal{R})$ *and* $y_T \in L^2(\mathcal{R})$ *with*

$$\|y_0\|_{L^2(\mathcal{R})} < r \quad \text{and} \quad \|y_T\|_{L^2(\mathcal{R})} < r,$$

there exist N *Neumann boundary controls* $h_i \in L^2(0,T)$ *such that* y *satisfies* $y(\cdot,0) = y_0$ *and* $y(\cdot,T) = y_T$ *for* $T > T_{\min}$.

Remarks

Remark

This results can be extended to more general weights α_i and β_i.

Ideas of the proof

- Linearize the system around a stationnary solution (here 0),
- Show the exact controllability result of the linear KdV equation (by linearity, we can take $y_0 = 0$) with an observability inequality of the linear backward adjoint system obtained by the multiplier method,
- Apply a fixed point theorem to have the local exact controllability result of the nonlinear equation.

Comments

Remarks

- Drawback of this method: we do not obtain sharp conditions on the lengths l_i and on the time of control $T_{\min}$.
- Advantage: we get an explicit constant of observability.

Comment

A same type of result can be obtained for a general tree with $N + 1$ external vertices, we get the controllability result with only N Neumann controls.

Open questions

- Observability inequality with a contradiction argument, case of critical lengths.
- Is it possible to reduce the number of controls at the external vertices and still having a control result ?
- Network with a circuit.

Thank you for your attention !