Resource Allocation in an Open Ecosystem

Control in Times of Crisis. Online seminar.

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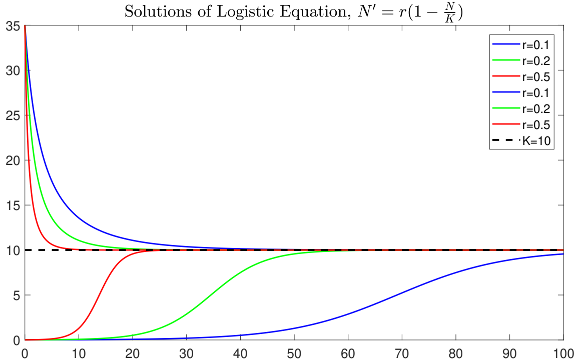
Content

Motivation

In the usual logistic model, the intrinsic growth rate r and carrying capacity K seem to be not related.

$$N' = rN\left(1 - \frac{N}{K}\right) \tag{1}$$



One way to understand better if r and K should be related is separating the per-capita population change as linear density dependence births and deaths,

$$\frac{N'}{N} = (b_0 - \beta N) - (d_0 + \delta N)$$

Rearranging

$$N' = (b_0 - d_0)N\left(1 - \frac{N}{\frac{b_0 - d_0}{\beta + \delta}}\right) \tag{2}$$

Thus,

$$K = \frac{r}{\beta + \delta}$$

Another approach is including mechanisms (motivated by Loreau's book). Start off with a resource limited system:

$$\begin{aligned} N' &= \beta\alpha R N \\ R' &= -\alpha R N \end{aligned} \tag{3}$$

Then $(N+\beta R)' = 0$ and $N + \beta R = N_0 + \beta R_0$ which lead to one DE,

$$N' = \alpha(N_0 + \beta R_0) N \left(1 - \frac{N}{N_0 + \beta R_0}\right) \tag{4}$$

Thus,

$$K = \frac{r}{\alpha}$$

- We are interested in such relationships and in the ideas of resources having their own dynamics in the PDEs setting to allow for habitat heterogeneity in space and time.

Motivation

- The effect of heterogeneous habitat on a diffusing population is crucial to understand population dynamics.
- Loreau in his book [6] identified an increasing gap between community ecology and ecosystem ecology.
- One aspect of the problem is to understand how the spatial and temporal arrangement of favorable and unfavorable resource affects the population growth.

Next, we review some mathematical work that have been done in spatial ecology.

The effect of spatial heterogeneity in population dynamics, Cantrell and Cosner (1991)

They were motivated by Skellman [7] to study the steady state of a population that can diffuse through the habitat via random walks with linear or logistic growth rate and spatially dependent coefficients.

$$u_t = d\Delta u + m(x)u - c(x)u^2 \quad \text{in } \Omega \times (0, \infty) \tag{5}$$

subject to necessary boundary and initial conditions.

- $u(x,t)$ represents the population density, $m(x)$ is the intrinsic growth rate and $c(x)$ is the carrying capacity
- They examined a number of one dimensional cases where the local growth rate $m(x)$ is a piecewise constant function which has positive effect on part of the habitat and negative effect everywhere else to model the growth and spread of a population in steady state.

Dispersal and spatial heterogeneity: single species, DeAngelis et al. (2016)

DeAngelis et al. analyzed the properties of the positive steady state u_d for

$$\begin{aligned} d\Delta u + r(x)u\left[1 - \frac{u}{K(x)}\right] &= 0 \quad u > 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

They focused on properties of u_d in terms of the intrinsic growth rate $r(x)$ and the carrying capacity $K(x)$. In particular, when the following inequality holds:

$$\int_\Omega u_d > \int_\Omega K.$$

In other words, when is the total population strictly greater that the total carrying capacity?

Optimal Resource Allocation for a Parabolic Population Model, Bintz and Lenhart (2020)

They considered an optimal resource allocation for a population with diffusion and logistic growth and a resource function $m(x,t)$ depending on space and time rather than only space. Their goal is to find

$$m^* \in V = \left\{ m \in L^\infty : \quad 0 \leq m(x,t) \leq M, \quad \int_0^T \int_\Omega m(x,t)\, dxdt = \delta \right\}$$

such that

$$\sup_{m \in V} J(m) = J(m^*)$$

where

$$J(m) = A \int_\Omega u(x,T)\, dx + \int_0^T \int_\Omega [u - (Bm^2)]\, dx$$

subject to

$$\begin{aligned} u_t - \mu \Delta u &= u(m-u), \qquad (x,t) \in \Omega \times (0,T) \\ \frac{\partial u}{\partial n} &= 0 \qquad (x,t) \in \partial\Omega \times (0,T). \end{aligned} \tag{7}$$

- Part of their results showed that in the presence of inhospitable boundary, the resources should be allocated further away from the hostile boundary. However, with hospitable boundary, more resources are allocated.

Aims of our Resource Allocation Model

Motivated by Loreau considering that the resources are important to understand population ecology and all the aforementioned reviewed papers, our goals are to

- Formulate a reaction-diffusion population model to study the effect of resource allocation in an open ecosystem with resources having their own dynamics in space and time.
- Develop an optimal control problem of our resource allocation model to maximize the abundance of a single species while minimizing the cost of the inflow resource allocation.

Formulation of Optimal Control Problems with PDE

We outline an approach of optimal control for parabolic PDE systems [5]:

1. Choose solution space. For most parabolic PDE problems with control is

$$w \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty(Q) \quad \text{and} \quad w_t \in L^2([0,T]; H_0^1(\Omega)^*)$$

2. Let U be the set of all admissible Lebesgue measurable controls with corresponding lower and upper bounds and consider the following problem

$$Aw = f(w,u) \qquad \text{in } Q \tag{8}$$

   with corresponding initial and boundary conditions.

3. The goal is to find $u^* \in U$ such that

$$J(u^*) = \sup_u J(u)$$

   where

$$J(u) = \int_Q g(x,t,w(x,t),u(x,t))\, dxdt.$$

   subject to problem (8).

One needs to show that for any given $u \in U$, there exist an associated state solution $w \in L^2([0,T]; H_0^1(\Omega)) \cap L^\infty(Q)$ that depends on u. In this case, we write $w(u)$ showing that w depends on u.

4. Require *a priori* estimates of states in the solution space to support convergence of a maximizing sequence and finally prove the existence of an optimal control $u^* \in U$
5. Derive the necessary conditions for our optimal control problem with PDE by finding corresponding adjoint and sensitivity PDEs
6. Use the adjoint and sensitivity PDEs to find an optimal control characterization.
7. One must show later that the optimal control is unique when time is sufficiently small.

We will illustrate steps 5 and 6 later on with our resource allocation model.

Our Model

Consider the following resource allocation model with diffusing population for an open ecosystem:

$$
\begin{aligned}
N_t - d\Delta N &= \alpha_1 NR - \mu N && \text{on } Q \\
R_t &= I(x,t) - \alpha_2 NR + p\mu N - qR && \text{on } Q \\
N(x,0) &= N_0(x) && \text{on } \Omega \\
R(x,0) &= R_0(x) && \text{on } \Omega \\
N(x,t) &= 0 && \text{on } \partial\Omega \times [0,T],
\end{aligned}
\tag{9}
$$

where $Q = \Omega \times (0,T)$ is the domain.

- $N = N(x,t)$ is measuring the population density.
- $R = R(x,t)$ represents the amount of resources.
- d represents the diffusion rate, μ is the natural death rate of the population and α_1 is the conversion from resources into population growth.

$$
\begin{aligned}
N_t - d\Delta N &= \alpha_1 NR - \mu N && \text{on } Q \\
R_t &= I(x,t) - \alpha_2 NR + p\mu N - qR && \text{on } Q \\
N(x,0) &= N_0(x) && \text{on } \Omega \\
R(x,0) &= R_0(x) && \text{on } \Omega \\
N(x,t) &= 0 && \text{on } \partial\Omega \times [0,T],
\end{aligned} \tag{10}
$$

- $I(x,t)$ is the input amount of resources (resources/time)
- α_2 is the rate of loss of resources due to the population, $p\mu N$ is the contribution to the resources from the death of the population and q is the resource decay rate.
- All the parameters are assumed to be non-negative and the function $I \in L^\infty(Q)$.

In the next two slides we analyze a simpler case of our model that's connected to previous works.

A Generalized Carrying Capacity

Consider the special case when $\alpha_1 = \alpha_2$, $p = 1$ and $q = 0$, then our system in (9) reduces to

$$N_t + R_t = I + d\Delta N \tag{11}$$

and integrating over the time interval $(0, t)$ leads to

$$R(x,t) = N_0(x) - N(x,t) + R_0(x) + \int_0^t I(x,s)ds + d\int_0^t \Delta N(x,s)ds. \tag{12}$$

Substituting equation (12) for R into the PDE equation for N in (9) and simplifying:

$$\begin{aligned} N_t - d\Delta N &= \alpha_1 N\left(N_0(x) - N + R_0(x) + \int_0^t I(x,s)ds + d\int_0^t \Delta N(x,s)ds\right) - \mu N \\ &= r(K)N\left(1 - \frac{N}{K(x,t)}\right). \end{aligned} \tag{13}$$

Thus, our PDE system can be rewritten as

$$N_t - d\Delta N = r(K)N\left(1 - \frac{N}{K(x,t)}\right). \tag{14}$$

where

$$K(x,t) = N_0(x) - \frac{\mu}{\alpha_1} + R_0(x) + \int_0^t I(x,s)ds + d\int_0^t \Delta N(x,s)ds$$

is the generalized carrying capacity and

$$r(K) = \alpha_1 K(x,t)$$

would be the generalized intrinsic growth rate.

1. The special case of our model reduces to study a single parabolic PDE for the population density. The equation (14) has been widely studied when the carrying capacity depends only on space by Cosner and Cantrell [2] and DeAngelis et al. [3, 4].

Weak Formulation Problem

Definition:

Assume that d, μ, α_1, α_2, p and q are non-negative constants with $I \in L^\infty(Q)$. The functions

1. $N \in L^2((0,T), H_0^1(\Omega)) \cap L^\infty(Q)$,
2. $N_t \in L^2((0,T), H_0^1(\Omega)^*)$
3. $R \in L^2(Q)$

are said to be weak solutions of problem (9) if for any test function $\Phi \in L^2((0,T), H_0^1(\Omega)) \cap L^\infty(Q)$

$$\begin{aligned} \int_0^T \langle N_t, \Phi\rangle dt + d\int_0^T \int_\Omega \nabla N \cdot \nabla \Phi \, dxdt &= \int_0^T \int_\Omega \alpha_1 N(x,t)R(x,t)\Phi(x,t) \\ &\quad - \mu N(x,t)\Phi(x,t)\, dxdt \\ R(x,t) - R(x,0) &= \int_0^t I(x,s) - \alpha_2 N(x,s)R(x,s) \\ &\quad + p\mu N(x,s) - qR(x,s) ds \end{aligned} \tag{15}$$

with initial and boundary conditions

$$\begin{aligned} N(x,0) &= N_0(x) && \text{on } \Omega \\ R(x,0) &= R_0(x) && \text{on } \Omega \\ N(x,t) &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned} \tag{16}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H_0^1(\Omega)^*$ and $H_0^1(\Omega)$.

A priori Estimates

Lemma 1: Let $N_0(x)$, $R_0(x)$ be positive and bounded on Ω. Then, any weak solutions N, R of Problem (15) must be non-negative on Q.
Lemma 2: Let N be a weak solution of problem (15) with $N_0(x) \geq 0$, then there exists a constant $B > 0$ depending only on given coefficients and final time T such that

$$\|N\|_{L^2(0,T;H_0^1(\Omega))} \leq B \|N_0\|_{L^2(\Omega)} .$$

Lemma 3: Let N be a weak solution of problem (15)-(16) with $N_0(x) \geq 0$, then there exist a constant $B_1 > 0$ depending on given data such that

$$\|N_t\|_{L^2(0,T;H_0^1(\Omega)^*)} \leq B_1.$$

Optimal Control Problem Formulation

$U = \left\{ I \in L^\infty(Q) |\ 0 \leq I(x,t) \leq M_1, \text{ a.e.} \right\}$ is the set of admissible controls. Then, we want to find $I^* \in U$ such that

$$\sup_{I \in U} J(I) = J(I^*) \quad \text{where } J(I) = \int_0^T \int_\Omega C_1 N - (C_3 + C_2 I) I \; dxdt$$

subject to

$$\begin{aligned} N_t - d\Delta N &= \alpha_1 NR - \mu N && \text{on } Q \\ R_t &= I(x,t) - \alpha_2 NR + p\mu N - qR && \text{on } Q \\ N(x,0) &= N_0(x) && \text{on } \Omega \\ R(x,0) &= R_0(x) && \text{on } \Omega \\ N(x,t) &= 0 && \text{on } \partial\Omega \times [0,T], \end{aligned} \tag{17}$$

- The objective functional $J(I)$ maximizes the total abundance of the population when the cost related to input amount of resources is minimized.

Sketch of Necessary Conditions

1. Define $I^\epsilon = I + \epsilon k$ for $\epsilon > 0$ and any variation function $k \in U$ with $I^\epsilon \in U$. We will differentiate the map $I \mapsto N$ and $I \mapsto R$.
2. Let $(N^\epsilon, R^\epsilon) = (N(I^\epsilon), R(I^\epsilon))$ be the corresponding state solutions for I^ϵ and $(N, R) = (N(I), R(I))$ be the corresponding state solutions for I.
3. Find the sensitivity of the N-state w.r.t. control

$$\lim_{\epsilon \to 0^+} \frac{N^\epsilon - N}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{N(I + \epsilon k) - N(I)}{\epsilon} = \Psi_1(x, t)$$

4. Find the sensitivity of the R-state w.r.t. control

$$\lim_{\epsilon \to 0^+} \frac{R^\epsilon - R}{\epsilon} = \lim_{\epsilon \to 0^+} \frac{R(I + \epsilon k) - R(I)}{\epsilon} = \Psi_2(x, t)$$

5 Form the PDE-difference quotient between N^ϵ and N

$$\left(\frac{N^\epsilon - N}{\epsilon}\right)_t - d\Delta\left(\frac{N^\epsilon - N}{\epsilon}\right) = \alpha_1 \frac{N^\epsilon R^\epsilon - NR}{\epsilon} - \mu\frac{N^\epsilon - N}{\epsilon}$$
$$\frac{N^\epsilon - N}{\epsilon} = 0 \qquad \text{on } \partial\Omega \times (0,T)$$
$$\frac{N^\epsilon(x,0) - N(x,0)}{\epsilon} = 0.$$

6 Let $\epsilon \to 0$ and completed corresponding convergence results. Thus, the sensitivity PDE problem for Ψ_1 is given by

$$(\Psi_1)_t - d\Delta\Psi_1 = \alpha_1(\Psi_1 R + \Psi_2 N) - \mu\Psi_1$$
$$\Psi_1 = 0 \qquad \text{on } \partial\Omega \times (0,T)$$
$$\Psi_1(x,0) = 0.$$

7 Form the ODE-difference quotient between R^ϵ and R

$$\left(\frac{R^\epsilon - R}{\epsilon}\right)_t = k - \alpha_2 \frac{N^\epsilon R^\epsilon - NR}{\epsilon} + p\mu \frac{N^\epsilon - N}{\epsilon} - q\frac{R^\epsilon - R}{\epsilon}$$
$$\frac{R^\epsilon(x,0) - R(x,0)}{\epsilon} = 0.$$

8 Let $\epsilon \to 0$ and completed corresponding convergence results. Thus, the sensitivity ODE problem for Ψ_2 is given by

$$(\Psi_2)_t = k - \alpha_2(\Psi_1 R + \Psi_2 N) + p\mu\Psi_1 - q\Psi_2$$
$$\Psi_2(x,0) = 0.$$

9 Write the sensitivity PDE system for Ψ_1 and Ψ_2 as

$$L\Psi = \begin{pmatrix} L_1\Psi_1 \\ L_2\Psi_2 \end{pmatrix} + M\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad \text{on } Q$$
$$\Psi_1(x,0) = 0 \qquad \text{on } \Omega$$
$$\Psi_2(x,0) = 0 \qquad \text{on } \Omega$$
$$\Psi_1(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

with

$$M = \begin{pmatrix} \mu - \alpha_1 R & -\alpha_1 N \\ \alpha_2 R - p\mu & q + \alpha_2 N \end{pmatrix}$$

10 Find the adjoint operator L^* by formally multiplying λ_1 times $L_1\Psi_1$ and integrate over our domain Q. Do the same now with λ_2 and $L_2\Psi_2$. Thus, the adjoint PDE system in vector form is:

$$L^*\lambda = \begin{pmatrix} L_1^*\lambda_1 \\ L_2^*\lambda_2 \end{pmatrix} + M^T \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} \qquad \text{on } Q$$

$$\lambda_1(x,T) = 0 \qquad \text{on } \Omega$$

$$\lambda_2(x,T) = 0 \qquad \text{on } \Omega$$

$$\lambda_1(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T)$$

where

$$L_1^*\lambda_1 = -(\lambda_1)_t - d\Delta\lambda_1$$
$$L_2^*\lambda_2 = -(\lambda_2)_t$$

Observe that

$$\frac{\partial \text{ integrand of } J}{\partial N} = C_1 \quad \text{and} \quad \frac{\partial \text{ integrand of } J}{\partial R} = 0.$$

11 Form the difference quotient of our objective functional where I^* is an optimal control

$$\frac{J(I^\epsilon) - J(I^*)}{\epsilon} = \int_0^T \int_\Omega C_1 \frac{N^\epsilon - N}{\epsilon} - C_3 \frac{I^\epsilon - I^*}{\epsilon} - C_2 \frac{(I^\epsilon)^2 - (I^*)^2}{\epsilon} \, dxdt.$$

12 Passing the limit as $\epsilon \to 0^+$,

$$\begin{aligned}
0 &\geq \int_0^T \int_\Omega C_1 \Psi_1 - k(C_3 + 2C_2 I^*) \, dxdt \\
&= \int_0^T \int_\Omega \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} \begin{pmatrix} C_1 \\ 0 \end{pmatrix} - k(C_3 + 2C_2 I^*) \, dxdt \\
&= \int_0^T \int_\Omega \begin{pmatrix} \Psi_1 & \Psi_2 \end{pmatrix} L^* \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} - k(C_3 + 2C_2 I^*) \, dxdt \\
&= \int_0^T \int_\Omega \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} L \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} - k(C_3 + 2C_2 I^*) \, dxdt \\
&= \int_0^T \int_\Omega \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 \\ k \end{pmatrix} - k(C_3 + 2C_2 I^*) \, dxdt \\
&= \int_0^T \int_\Omega k(\lambda_2 - C_3 - 2C_2 I^*) \, dxdt.
\end{aligned}$$

In order to differentiate $J(I)$ with respect to I.

13 Assuming $C_2 \neq 0$ and taking into account the bounds for $I^* \in U$ with inequality in step (12) we obtain that the optimal control characterization is

$$I^*(x,t) = \min\left\{M_1, \max\left\{0, \frac{\lambda_2(x,t) - C_3}{2C_2}\right\}\right\}$$

for any $(x,t) \in Q$ and any positive constant C_2 and C_3.

Theorem 1: There exists an optimal control I^* and corresponding states (N^*, R^*) such that

$$J(I^*) = \sup_{I \in U} J(I).$$

Theorem 2: If $C_2 \neq 0$ and T is sufficiently small, then the optimal control of problem (17) is unique.

Numerical simulations

We want to investigate two numerical scenarios to illustrate our resource allocation model in (9). For all these scenarios $\mu = 0$, and $q = 0$ with $Q = [0,1] \times [0,1]$. These assumptions on the parameters mean that natural death in the population and resource losses do not occur.

$$
\begin{aligned}
N_t - dN_{xx} &= \alpha_1 NR && \text{on } Q \\
R_t &= I(x,t) - \alpha_2 NR && \text{on } Q \\
N(x,0) &= N_0(x) && \text{on } [0,1] \\
R(x,0) &= R_0(x) && \text{on } [0,1] \\
N(x,t) &= 0 && \text{on } \{0,1\} \times [0,1],
\end{aligned}
\tag{18}
$$

Numerical Cases Without Control

- The two numerical scenarios about to be presented illustrate the dynamics of our resource system for a fixed amount of input resource that depends only on space.
- We perform numerical simulations for a population diffusing at different levels when

(1) there are no added resources $I(x) = 0$ and $\alpha_1 = \alpha_2$

(2) there are added resources $I(x) \neq 0$ and $\alpha_1 \neq \alpha_2$

Initial conditions without added resources, $I(x) = 0$

This initial conditions assume that the resources have no added resource.

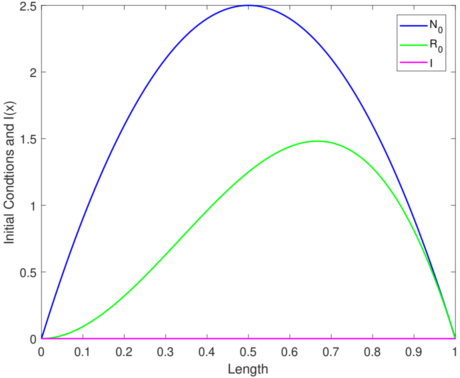


Figure: Set of initial conditions without added resources for Fig. 2 without added resources

Scenario 1 with IC in Fig. 1

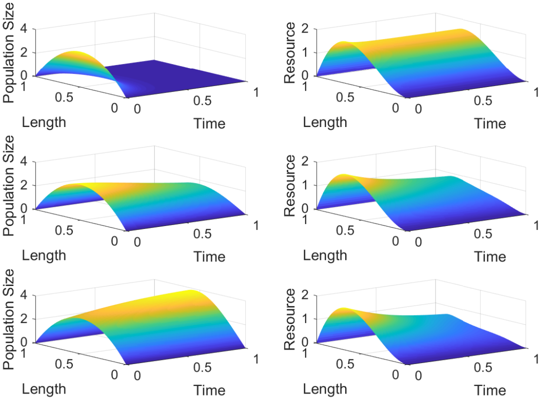


Figure: Three levels of diffusion rate subject to initial conditions in Fig. 1 with $\alpha_1 = \alpha_2 = 0.4$. The top, middle and bottom scenarios correspond to high, medium and low diffusion rates for $d = 1$, $d = 0.1$ and $d = 0.01$ respectively.

Initial conditions with added resources

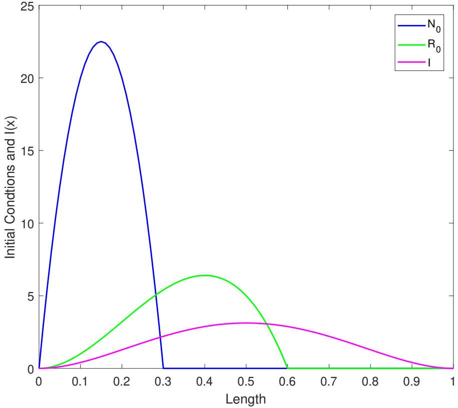


Figure: Set of initial conditions with only space dependent added resources for Fig. 4

Scenario 2 with IC in Fig. 3

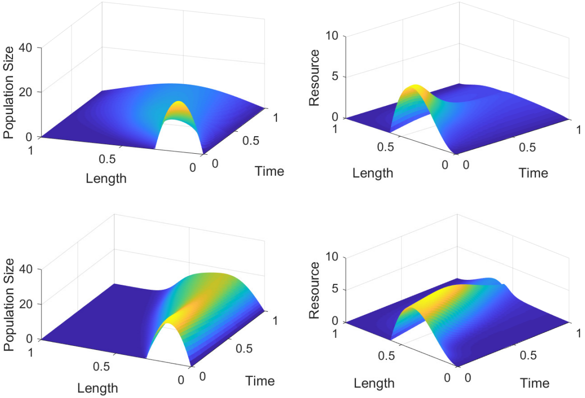


Figure: Two levels of diffusion rate subject to initial conditions in Fig. 3 with $\alpha_1 = 1$ and $\alpha_2 = 0.6$. The top and bottom scenarios correspond to medium and low diffusion rates for $d = 0.1$ and $d = 0.01$ respectively.

Conclusions

- All *a priori* estimates as well as theorem 1 and 2 have been proven.
- The previous scenarios were just done to start the process of understanding our model.
- We have performed some numerical simulations using optimal control as well.
- We will consider different sets of initial conditions with more parameter choices. If we allow natural death in the population and decay in the resources.
- In the future, we would like to fit our model to some real world examples.

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Thank You for Listening.
Any Questions?