A constructive algorithm for building mixing coupling potentials

Application to bilinear control

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Control in Time of Crisis, 24/11/2020

Outline

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Control systems and controllability along a trajectory

Dynamical system: $u' = f(u, \mathsf{p})$

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$\bar{u}_0 \bullet$

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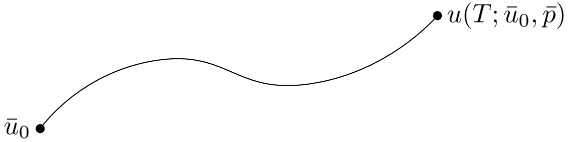

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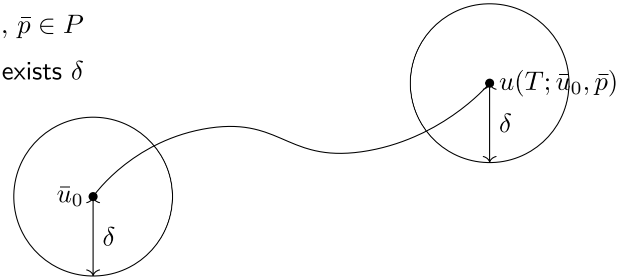

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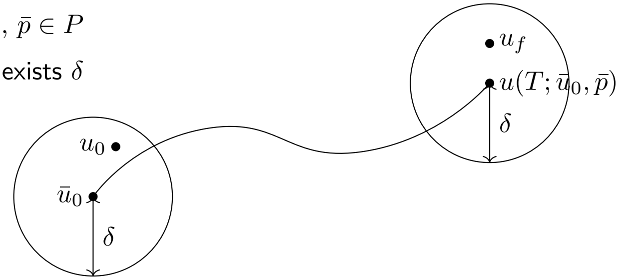

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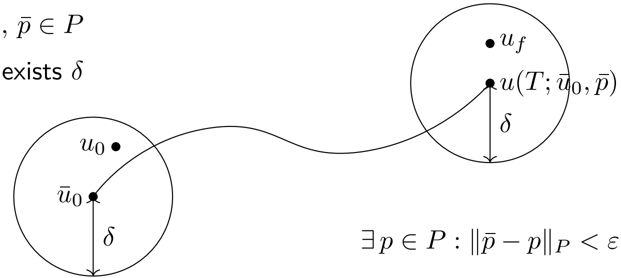

$\exists\, p \in P : \|\bar{p} - p\|_P < \varepsilon$

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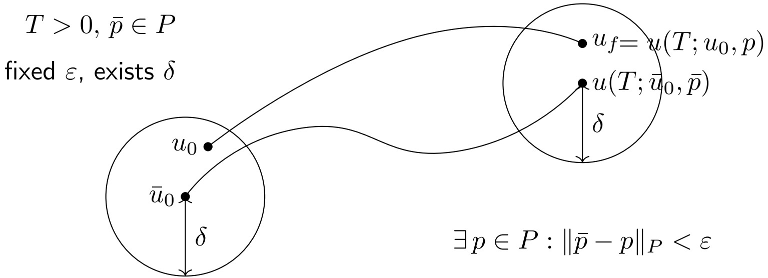

Different kinds of control systems

Boundary control:

$$\begin{cases} u' = Au + Bu \\ u = \mathbf{p}|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

Locally distributed control:

$$\begin{cases} u' = Au + Bu + \mathbf{p}\mathbb{1}_{\omega} \\ u = g|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

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Theorem (Ball, Marsden, Slemrod 1982)

Let X be a Banach space with dim(X)=$+\infty$. Let A generate a C^0-semigroup of bounded linear operators on X and $B : X \to X$ be a bounded linear operator. Let $u_0 \in X$ be fixed, and let $u(t; p, u_0)$ denote the unique solution of ($\ast$) for $p \in L^1_{loc}([0, +\infty), \mathbb{R})$. The set of states accessible from u_0 defined by

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^r_{loc}([0, +\infty), \mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

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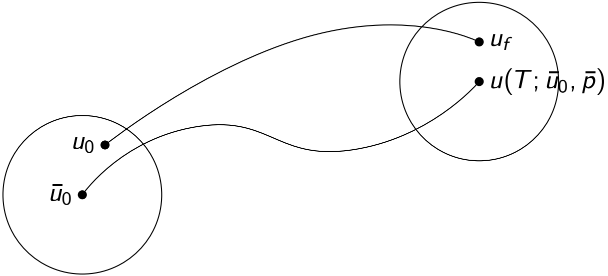

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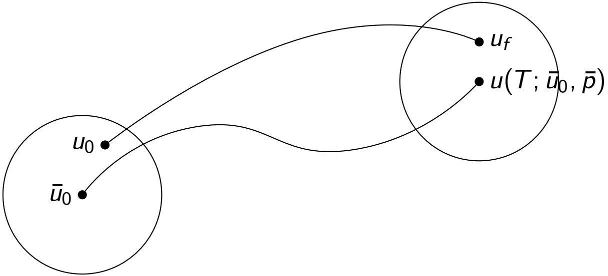

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Controllability to a trajectory

$(X, \langle \cdot, \cdot \rangle)$ separable Hilbert space.

$\boldsymbol{A} : D(\boldsymbol{A}) \subset X \to X$ densely defined linear operator:

$$
\begin{aligned}
&(a) \quad \boldsymbol{A} \text{ is self-adjoint },\\
&(b) \quad \exists\, \sigma > 0 \,:\, \langle \boldsymbol{A}x, x\rangle \geq -\sigma \|x\|^2, \forall x \in D(\boldsymbol{A}),\\
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Bilinear control problem:

$$\begin{cases} u'(t) + \boldsymbol{A}u(t) + p(t)\boldsymbol{B}u(t) = 0,\\ u(0) = u_0. \end{cases} \qquad (\bigstar)$$

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Trajectories: **eigensolutions** $\boldsymbol{\psi_j} = e^{-\boldsymbol{\lambda_j} t}\boldsymbol{\varphi_j}$: solutions of (★) for $p = 0$ and $u_0 = \boldsymbol{\varphi_j}$, where $\boldsymbol{A\varphi_j} = \boldsymbol{\lambda_j\varphi_j}$, for all $j \in \mathbb{N}^*$.

Controllability to a trajectory

Definition

*Let $T > 0$ and let A satisfy (☕). The pair $\{A, B\}$ is called **j-null controllable in time T** if there exists a constant $N(T) > 0$ such that for every $y_0 \in X$ one can find a control $p \in L^2(0,T)$ satisfying $\|p\|_{L^2(0,T)} \leq N(T)\|y_0\|$, and for which $y(T) = 0$, where $y(\cdot)$ is the solution of*

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0,T] \\ y(0) = y_0. \end{cases}$$

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Let $\{A, B\}$ be a j-null controllable pair. Then, system (★) is superexponentially stabilizable to ψ_j, for any $j \in \mathbb{N}^*$:

$$\|u(t) - \psi_j(t)\| \leq M e^{-\rho e^{\omega t}} \qquad \forall t \geq 0,$$

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Theorem (of Controllability. Alabau-Boussouira, Cannarsa, U.)

Let $\{A, B\}$ be a j-null controllable pair and $N(T) \leq e^{C/T}$ for T small. Then, for any $T > 0$, system (★) is exactly controllable to ψ_j, for any $j \in \mathbb{N}^*$:

$$u(T) = \psi_j(T).$$

Sufficient conditions for j-null controllability

Theorem (Alabau-Boussouira, Cannarsa, U.)

Let $\boldsymbol{A} : D(\boldsymbol{A}) \subset X \to X$ *satisfy* (☕) *and such that* $\exists\, \alpha > 0$ *for which its eigenvalues fulfill the gap condition*

$$\sqrt{\boldsymbol{\lambda_{k+1}} - \boldsymbol{\lambda_1}} - \sqrt{\boldsymbol{\lambda_k} - \boldsymbol{\lambda_1}} \geq \alpha, \quad \forall\, k \in \mathbb{N}^*. \tag{✈}$$

Let $\boldsymbol{B} : X \to X$ *be a bounded linear operator such that*

$$\begin{aligned} &i) \quad \langle \boldsymbol{B}\boldsymbol{\varphi_j}, \boldsymbol{\varphi_k} \rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*, \\ &ii) \quad \exists\, \tau > 0 \ : \ \sum_{k \in \mathbb{N}^*} \frac{e^{-2\boldsymbol{\lambda_k}\tau}}{|\langle \boldsymbol{B}\boldsymbol{\varphi_j}, \boldsymbol{\varphi_k} \rangle|^2} < +\infty. \end{aligned}$$

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Then, the pair $\{A, B\}$ *is* j*-null controllable in any time* $T > 0$ *with control cost* $N(T)$ *that satisfies* $N(T) \leq e^{C/T}$*, for* T *small.*

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Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control

Karine Beauchard[a,*,1], Camille Laurent[b]

Theorem 1. *Let $T > 0$ and $\mu \in H^3((0,1),\mathbb{R})$ be such that*

$$\exists c > 0 \quad \textit{such that} \quad \frac{c}{k^3} \leqslant \left|\langle \mu\varphi_1, \varphi_k\rangle\right|, \quad \forall k \in \mathbb{N}^*. \tag{5}$$

There exists $\delta > 0$ and a C^1 map

$$\Gamma : \mathcal{V}_T \to L^2\big((0,T),\mathbb{R}\big),$$

where

$$\mathcal{V}_T := \big\{\psi_f \in \mathcal{S} \cap H^3_{(0)}\big((0,1),\mathbb{C}\big);\ \big\|\psi_f - \psi_1(T)\big\|_{H^3} < \delta\big\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1) with initial condition

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$\Longrightarrow$ provide more examples of potentials μ with **(NVFC)** property,

$\Longrightarrow$ extend existing controllability results to other boundary conditions

Constructive algorithm for building mixing coupling potentials

Consider the $1D$ Laplacian operator (with some boundary conditions) and let (φ_k, λ_k) be the eigenfunctions and eigenvalues.

Theorem

For any function $\mu \in H^4(0,1)$ and for any $k \geqslant 2$ the following relation holds

$$(\lambda_k - \lambda_1)^2 \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \int_0^1 T_k(\mu)(x)\varphi_1(x)\varphi_k(x)dx + B_{G,k}(\mu).$$

where $T_k : H^4(0,1) \mapsto L^2(0,1)$ are the linear operators defined by

$$T_k(\mu) = -\left[\mu^{(4)} + \alpha_k \mu''\right], \quad \forall\, \mu \in H^4(0,1), \quad \forall\, k \in \mathbb{N}^*,$$

where $\alpha_k := 2(\lambda_1 + \lambda_k)$, and $B_{G,k} : H^4(0,1) \mapsto \mathbb{R}$ are boundary linear operators.

Constructive algorithm for building mixing coupling potentials

Theorem

For any function $\mu \in H^{4n}(0,1)$, for any $n \in \mathbb{N}^*$ and any $k \geqslant 2$, we have the following inductive formulas:

$$(\lambda_k - \lambda_1)^{2n} \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \int_0^1 T_k^n(\mu)(x)\varphi_1(x)\varphi_k(x)dx + \sum_{p=0}^{n-1} (\lambda_k - \lambda_1)^{2p} \left[B_{G,k}(T_k^{n-p-1}(\mu)) \right],$$

where $T_k^n(\mu) = \underbrace{(T_k \circ \cdots \circ T_k)}_{n}(\mu)$ *and* $T_k^0(\mu) = Id$,

$$T_k^p(\mu) = (-1)^p \sum_{l=0}^{p} C_p^l \alpha_k^l \mu^{(4p-2l)}, \quad \forall\, p \in \mathbb{N}^*,$$

where the notation C_p^l stands for the binomial coefficient $C_p^l = \binom{p}{l} = \frac{p!}{l!(p-l)!}$, for all $0 \leq l \leq p$.

Constructive algorithm for building mixing coupling potentials

$$R_r(\alpha) := \sum_{j=\lceil \frac{r}{2} \rceil}^{r} (-1)^j \alpha^j C_j^{r-j}, \quad \forall\, \alpha \in \mathbb{R}.$$

Constructive algorithm for building mixing coupling potentials

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Corollary

Let $\mu \in \mathcal{P}_q(\mathbb{R})$ and n be such that $2n > q$. Then, for any $k \geqslant 2$

$$(\lambda_k - \lambda_1)^{2n} \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = \sum_{r=0}^{n-1} B_{G,k}(\mu^{(2r)}) \frac{(\lambda_k - \lambda_1)^{2(n-1)}}{\alpha_k^r} R_r\left(\left(\frac{\alpha_k}{\lambda_k - \lambda_1}\right)^2\right)$$

Moreover, $\left(\dfrac{\alpha_k}{\lambda_k - \lambda_1}\right)^2 > 4, \quad \forall\, k \geqslant 2.$

Constructive algorithm for building mixing coupling potentials

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Moreover, $\left(\dfrac{\alpha_k}{\lambda_k - \lambda_1}\right)^2 > 4, \quad \forall\, k \geqslant 2.$

Lemma

$$R_{2m}(\alpha) > 0, \qquad R_{2m+1}(\alpha) < 0, \qquad \forall\, \alpha > 4, \forall\, m \geq 0.$$

A constructive algorithm: Dirichlet-Robin boundary conditions

$$\begin{cases} -\varphi_k''(x) = \lambda_k \varphi_k(x), & x \in (0,1), \\ \varphi_k(0) = 0, \quad \varphi_k(1) + \varphi_k'(1) = 0. \end{cases}$$

Eigenvalues and eigenfunctions are given by

$$\lambda_k = r_k^2, \qquad \varphi_k(x) = \eta_k \sin(r_k x),$$

where r_k are the positive solutions of

$$\sin r_k + r_k \cos r_k = 0,$$

and $\eta_k > 0$ are normalization constants.

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Proposition

For any $\mu \in H^{4n}(0,1)$, for any $n \in \mathbb{N}^*$, and any $k \geqslant 2$, we have

$$(\lambda_k - \lambda_1)^{2n} \int_0^1 \mu(x)\varphi_1(x)\varphi_k(x)dx = (-1)^n \sum_{l=0}^{n} C_n^l \alpha_k^l \int_0^1 \mu^{(4n-2l)}(x)\varphi_1(x)\varphi_k(x)dx + \eta_1\eta_k r_1 r_k D_{k,n}(\mu),$$

where $D_{k,n}(\cdot)$ is a boundary linear operator defined on $H^{4n}(0,1)$.

A constructive algorithm: Dirichlet-Robin boundary conditions

We introduce

$$
\begin{aligned}
a_r &:= \mu^{(2r+3)}(1) + 2\mu^{(2r+2)}(1), && r = 0, 1, \dots, l-1 \\
b_r &:= \mu^{(2r+1)}(1), && r = 0, 1, \dots, l-1 \ (\, r = 0, 1, \dots, l \text{ if } q \text{ is odd}), \\
c_r &:= 2\mu^{(2r+1)}(0), && r = 0, 1, \dots, l-1 \ (\, r = 0, 1, \dots, l \text{ if } q \text{ is odd}).
\end{aligned} \tag{1}
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REMARK: $\forall\, ((a_r), (b_r))$, $\exists\, !\mu \in \mathcal{P}_q^*(\mathbb{R})$ s.t. (1) holds:

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\begin{cases}
\mu^{(2r)}(1) = \dfrac{1}{2}\left(a_{r-1} - b_r\right), & \forall\ r \in \{1, \dots, l-1\}, \\
\mu^{(2r+1)}(1) = b_r, & \forall\ r \in \{0, \dots, l-1\}.
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Viceversa, $\forall \mu \in \mathcal{P}_q^*(\mathbb{R})$, $(\mu^{(k)}(1))_{1 \le k \le q}$ uniquely determine $((a_r), (b_r))$.

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Viceversa, $\forall \mu \in \mathcal{P}_q^*(\mathbb{R})$, $(\mu^{(k)}(1))_{1 \le k \le q}$ uniquely determine $((a_r), (b_r))$.

REMARK: the potential μ is completely determined by the knowledge of $\mu^{(k)}(1)$ for all $k \in \{0, \ldots, q\}$, thanks to the (finite) Taylor expansion of μ at 1:

$$\mu(x) = K + \sum_{k=1}^{q} \frac{\mu^{(k)}(1)}{k!}(x-1)^k = K + \tilde{\mu}(x).$$

A constructive algorithm: Dirichlet-Robin boundary conditions

Corollary

For any $\mu \in \mathcal{P}_q(\mathbb{R})$, *choosing* n *such that* $2n > q$, *it holds that for all* $i \geq 1$

$$A_{i,n} := -D_{2i,n} = \sum_{r=0}^{n-1} \left[\frac{1}{\sqrt{\lambda_1 + 1}} g_r(\lambda_{2i}) + c_r \right] \frac{(\lambda_{2i} - \lambda_1)^{2(n-1)}}{\alpha_{2i}^r} R_r\left(\tau_{2i}\right),$$

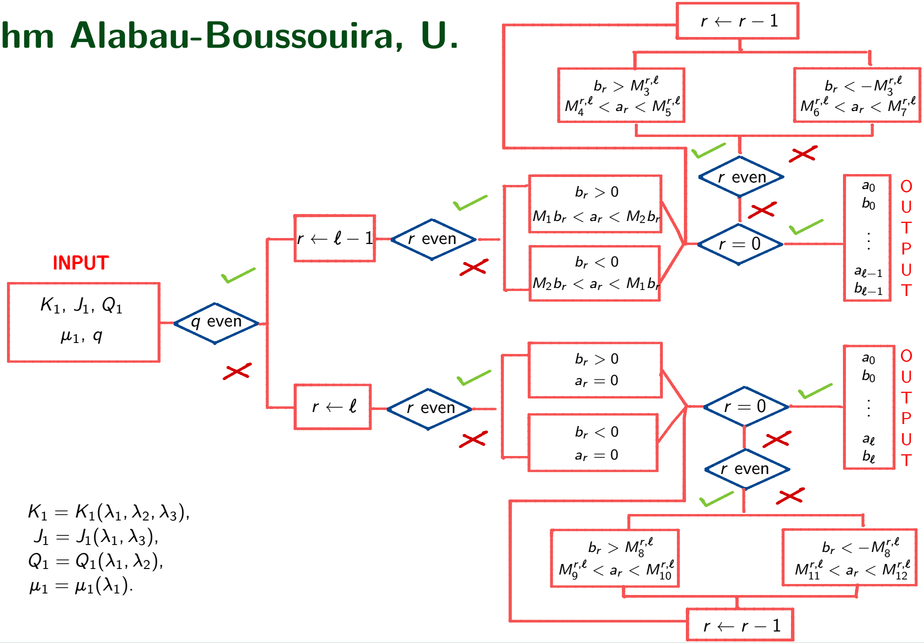
$$B_{i,n} := D_{2i+1,n} = \sum_{r=0}^{n-1} \left[\frac{1}{\sqrt{\lambda_1 + 1}} g_r(\lambda_{2i+1}) - c_r \right] \frac{(\lambda_{2i+1} - \lambda_1)^{2(n-1)}}{\alpha_{2i+1}^r} R_r\left(\tau_{2i+1}\right),$$

where

$$g_r(x) := \frac{1}{\sqrt{x+1}} \left(a_r + (x + \lambda_1 + 2) b_r\right), \quad \forall\, x \in (\lambda_1, +\infty).$$

$$\langle \mu\varphi_1, \varphi_k \rangle \neq 0, \quad \forall\, k \geq 2 \iff D_{k,n} \neq 0, \quad \forall\, k \geq 2 \iff A_{i,n}, B_{i,n} \neq 0, \quad \forall\, i \geq 1$$

Algorithm Alabau-Boussouira, U.



$$K_1 = K_1(\lambda_1, \lambda_2, \lambda_3),$$
$$J_1 = J_1(\lambda_1, \lambda_3),$$
$$Q_1 = Q_1(\lambda_1, \lambda_2),$$
$$\mu_1 = \mu_1(\lambda_1).$$

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- output of the algorithm: $((a_r),(b_r))$, $0 \leq r \leq l-1$ for q even ($0 \leq r \leq l$ for q odd),

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- determine $\mu^{(k)}(1)$, $k = 1,\dots,q$ and hence define $\tilde{\mu} \in \mathcal{P}_q^*(\mathbb{R})$ by the finite Taylor expansion

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Our final polynomial $\mu \in \mathcal{P}_q(\mathbb{R})$ that satisfies **(NVFC)** is

$$\mu(x) := \mu^{(0)}(1) + \tilde{\mu}(x) = \sum_{k=0}^{q} \frac{\mu^{(k)}(1)}{k!}(x-1)^k.$$

Outline

Application: Schrödinger equation, mixed boundary conditions

Consider the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\partial_x^2 u(t,x) - p(t)\mu(x)u(t,x), & (t,x) \in (0,T)\times(0,1) \\ u(t,0)=0,\ u(t,1)+\partial_x u(t,1)=0, & \end{cases}$$

which describes the motion of a quantum particle in a box (perfectly reflecting wall at $x=0$ and a nonstandard wall at $x=1$) subject to an electric field, whose magnitude in given by the control $p(\cdot)$.

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which describes the motion of a quantum particle in a box (perfectly reflecting wall at $x=0$ and a nonstandard wall at $x=1$) subject to an electric field, whose magnitude in given by the control $p(\cdot)$. Robin boundary condition: $\varphi(1)+L\varphi'(1)=0$, L measures the time delay in scattering (difference between moment of incidence and reflection).

- Fülöp, T., Cheon, T., Tsutsui, I.. Classical aspects of quantum walls in one dimension. Physical Review A, 66(5), 052102. (2002)
- Belchev, B., Walton, M. A.. On Robin boundary conditions and the Morse potential in quantum mechanics. Journal of Physics A: Mathematical and Theoretical, 43(8), 085301. (2010)
- Allwright, G., Jacobs, D. M.. Robin boundary conditions are generic in quantum mechanics. arXiv preprint arXiv:1610.09581 (2016)

Application: Schrödinger equation, mixed boundary conditions

Theorem

Let $T > 0$ *and* $\mu \in H^2(0,1;\mathbb{R})$ *be such that* $\exists\, C > 0$ *such that* $|\langle \mu\varphi_1, \varphi_k\rangle| \geq \frac{C}{\lambda_k}$, $\forall\, k \in \mathbb{N}^*$.

Then, there exists $\delta > 0$ *and a* C^1*-map*

$$\Gamma : \mathcal{R}_T \to L^2(0,T)$$

where

$$\mathcal{R}_T := \left\{ u_f \in \mathcal{S} \cap H^2_{(0)}(0,1;\mathbb{C}) \ : ||u_f - \psi_1(T)||_{H^2} < \delta \right\},$$

such that $\Gamma(\psi_1(T)) = 0$ *and for any* $\psi_f \in \mathcal{R}_T$*, the solution of the Schrödinger equation with Dirichlet-Robin boundary conditions, initial condition* φ_1 *and control* $p = \Gamma(\psi_f)$ *satisfies* $u(T) = \psi_f$.

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Outline

Examples of admissible potentials for D.R. boundary conditions

The following polynomials have been computed through A.U. algorithm:

- $\mu(x) = 1 + 16(x-1) + 12(x-1)^2 - 4(x-1)^3 - \frac{3}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$
- $\mu(x) = 1 + 52(x-1) + 41(x-1)^2 - 16(x-1)^3 - \frac{13}{4}(x-1)^4 + (x-1)^5 + \frac{1}{6}(x-1)^6$
- $\mu(x) = \frac{1}{10} + 240(x-1) + 74(x-1)^2 - 36(x-1)^3 - \frac{11}{2}(x-1)^4 + \frac{6}{5}(x-1)^5 + \frac{1}{15}(x-1)^6 - \frac{1}{70}(x-1)^7$

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THANK YOU! GRACIAS! MERCI! GRAZIE!