# Does the boundary controlled heat equation define an exactly controllable system?

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# Outline

- Motivation and background on reachability and controllability
- The 1D boundary controlled heat equation
- Concluding remarks

# Motivation and background on reachability and controllability

# Some notation

We consider control systems described by equations of the form

$$(SE) \qquad \dot{z}(t) = Az(t) + Bu(t), \qquad \text{with}$$

- $X$ (the state space) and $U$ (the input space) are complex Hilbert spaces. We have $X = \mathbb{C}^n$ and $U = \mathbb{C}^m$ for finite-dimensional control systems.

- $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ is a strongly continuous semigroup on $X$ generated by $A$. We have $\mathbb{T}_t = \mathrm{e}^{tA}$ for finite-dimensional control systems. $X_1$ is $\mathcal{D}(A)$ endowed with the graph norm and $X_{-1}$ is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space $X$.

- $B \in \mathcal{L}(U; X_{-1})$ is the control operator.

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# Admissible control operators

The solution of (SE) writes:

$$z(t) = \mathbb{T}_t z(0) + \Phi_t u,$$

where $\mathbb{T}$ is the semigroup generated by $A$ and

$$\Phi_t \in \mathcal{L}(L^2([0,\infty);U), X_{-1}), \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma)\, \mathrm{d}\sigma.$$

**Definition.** $B$ *is called an admissble control operator for* $\mathbb{T}$ *if* $\operatorname{Ran} \Phi_t \subset X$ *for one (and hence all)* $t > 0$.

**Example.** *Take* $A = -A_0$ *with* $A_0 > 0$. *For* $\alpha > 0$, *denote* $X_\alpha = \mathcal{D}(A_0^\alpha)$ *and* $X_{-\alpha}$ *is the dual of* $X_\alpha$ *with respect to the pivot space* $X$. *Then every operator* $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$ *is admissible.*

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# Controllability types

$(A, B)$ is said *exactly controllable in time* $\tau$ if $\operatorname{Ran}\Phi_\tau = X$.

$(A, B)$ is said *null controllable in time* $\tau$ if $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$. This is equivalent to the existence, for eaxh $z_0 \in X$ of $u \in L^2[0,\tau];U)$ such that the solution of

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0,$$

satisfies $z(\tau) = 0$.

$(A, B)$ is *approximatively controllable in time* $\tau$ if $\overline{\operatorname{Ran}\Phi_\tau} = X$.

**The three above concepts coincide with the usual controllability concept in the case of finite dimensional LTIs.**

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# First remarks

- In the finite dimensional case, all LTIs are controllable! Indeed, denoting by $\tilde{A}$ the restriction of $A$ to $\operatorname{Ran}\Phi_\tau$ then $(\tilde{A}, B)$, with state space $\operatorname{Ran}\Phi_\tau$ is controllable.

- Still in the finite dimensional case, $\operatorname{Ran}\Phi_\tau$ coincides with the range of the restriction of $\Phi_\tau$ to signals which can be extended to analytic functions from $\mathbb{C}$ to $U$.

- If $X$ is infinite dimensional, $A = -A_0$ with $A_0 > 0$. and $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$ is compact the $(A, B)$ is not exactly controllable.

- If $X$ is infinite dimensional and $\tilde{A}$ is the restriction of $A$ to $\operatorname{Ran}\Phi_\tau$ then $(\tilde{A}, B)$, with state space $\operatorname{Ran}\Phi_\tau$ is not, in general, exactly controllable.

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# Exact controllability and nonlinearities

Consider a nonlinear contol system

$$\dot{z} = Az + F(z) + Bu,$$

with $(A, B)$ exactly controllable in some time $\tau > 0$ and $F : X \to X$ "superlinear". Then *local exact controllability* if the nonlinear system is often easy to prove:

- Given $f \in L^2([0,\tau]; X)$ the system $\dot{z} = Az + f + Bu$ is exactly controllable. Indeed we can find an $u$ such that

$$\Phi_\tau u = -\int_0^\tau \mathbb{T}_{\tau-\sigma} f(\sigma)\,\mathrm{d}\sigma \in X.$$

  $z_f$ being the controlled trajectory we next look for a fixed point of $f \mapsto F(z_f)$.

# Null controllability and reachable space (I)

**Proposition 3.** (Fattorini, Seidman) If $(A, B)$ is null controllable in any tyime then $\operatorname{Ran}\Phi_\tau$ does not depend on $\tau > 0$.

*Proof.* Let $0 < \tau < t$, $\eta \in \operatorname{Ran}\Phi_\tau$ and let $\tilde{u} = \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_\pi \end{bmatrix}$ be a control such that $\tilde{w}(\tau, \cdot) = \eta$. Let $u$ be the control defined by

$$u(\sigma) = \begin{cases} 0 & \text{for } \sigma \in [0, t - \tau], \\ \tilde{u}(\sigma + \tau - t) & \text{for } \sigma \in [t - \tau, t]. \end{cases}$$

Then $w(t, \cdot) = \eta$, thus $\operatorname{Ran}\Phi_\tau \subset \operatorname{Ran}\Phi_t$.

Let now $\eta \in \operatorname{Ran}\Phi_t$ and $\tilde{u}(\sigma) = u(\sigma + t - \tau)$, $\quad \tilde{w}(\sigma) = w(\sigma + t - \tau, \cdot)$. Then $\eta = w(t, \cdot) = \tilde{w}(\tau, \cdot) = \mathbb{T}_\tau \tilde{w}(0, \cdot) + \Phi_\tau \tilde{u}$. Since $\operatorname{Ran}\Phi_\tau \supset \operatorname{Ran}\mathbb{T}_\tau$, we have $\eta \in \operatorname{Ran}\Phi_\tau$, thus $\operatorname{Ran}\Phi_t \subset \operatorname{Ran}\Phi_\tau$.

# Null controllability and reachable space (II)

**Proposition 4.** (Kellay, Normand and M.T., 2019)
Let $(A, B)$ be a well-posed control LTI system which is null controllable in any positive time and let $\tau,\ \alpha > 0$ and let
$\mathcal{U}_{\tau,\alpha} = \left\{u \in L^2([0,\tau];U) \ \middle|\ (t \mapsto t^{-\alpha}u(t)) \in L^2([0,\tau];U)\right\}$.
Then for every $\alpha > 0,\ \tau > 0$ we have $\Phi_\tau(\mathcal{U}_{\tau,\alpha}) = \operatorname{Ran} \Phi_\tau$.

*Proof.* Let $\eta \in \operatorname{Ran} \Phi_\tau = \operatorname{Ran} \Phi_{\frac{\tau}{2}}$ and let $\tilde{u} = \begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_\pi \end{bmatrix}$ be a control such that $\tilde{w}\left(\frac{\tau}{2}, \cdot\right) = \eta$. Let $u$ be the control defined by

$$u(\sigma) = \begin{cases} 0 & \text{for} \quad \sigma \in \left[0, \frac{\tau}{2}\right), \\ \tilde{u}\left(\sigma + \frac{\tau}{2}\right) & \text{for} \quad \sigma \in \left[\frac{\tau}{2}, \tau\right]. \end{cases}$$

Then $u \in \mathcal{U}_{\tau,\alpha}$ and $\Phi_\tau u = \eta$, thus $\operatorname{Ran} \Phi_\tau \subset \Phi_\tau(\mathcal{U}_{\tau,\alpha})$.

# The 1D boundary controlled heat equation

# Problem Statement (I)

$$(BCH)\quad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \dfrac{\partial^2 w}{\partial x^2}(t,x) & t \geqslant 0,\ \ x \in (0,\pi), \\ w(t,0) = u_0(t), \quad w\,(t,\pi) = u_\pi(t) & t \in [0,\infty), \\ w(0,x) = 0 & x \in (0,\pi)\,, \end{cases}$$

Given $\tau > 0$, define the *input to state map*

$$\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = w(\tau,\cdot) \qquad (\tau > 0,\ \ u_0,\ \ u_\pi \in L^2[0,\tau]),$$

# Problem Statement (II)

Basic question: what can we say about $\operatorname{Ran}\Phi_\tau$?

Easy answer: This is a normed space when endowed with the norm

$$\|\psi\|_{\operatorname{Ran}\Phi_\tau} = \inf\left\{u \in L^2[0,\tau];U) \;\mid\; \Phi_\tau u = \psi\right\}.$$

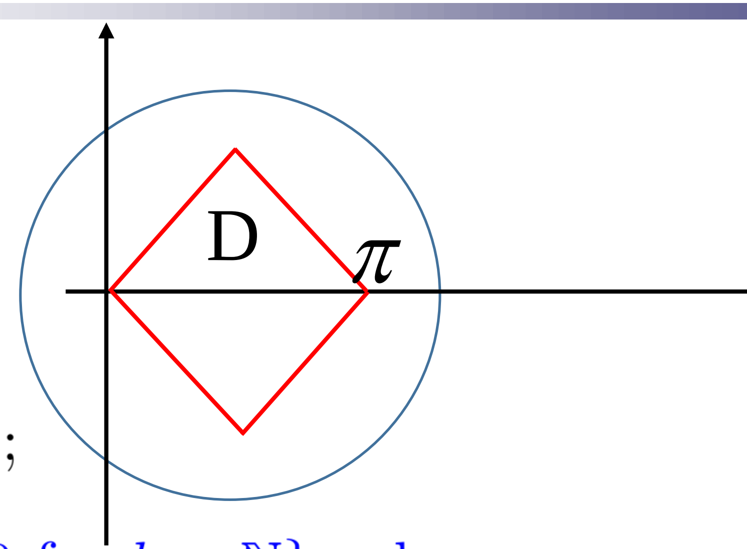
First questions:

- Is this a Banach space?
- Can we find a simple characterization of $\operatorname{Ran}\Phi_\tau$ and of $\|\cdot\|_{\operatorname{Ran}\Phi_\tau}$ ?

Applications:

- Nonlinear problems, with stronger nonlinearities ans less regularity (Laurent et Rosier, 2018).
- Time optimal control problems (Wang and Zuazua 2012, Wang, Xu and Zhang, 2015).

# Existing results




Given $\tau > 0$ it is known that:

- $\operatorname{Ran}\Phi_\tau \subset \operatorname{Hol}(D)$, where $D = \{s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x\}$;
- $\operatorname{Ran}\Phi_\tau \supset \{\psi \in \operatorname{Hol}(S) \mid \psi^{(2k)}(0) = \psi^{(2k)}(\pi) = 0 \text{ for } k \in \mathbb{N}\}$, where $S = \{s = x + iy \in \mathbb{C} \mid |y| < \pi\}$ (Fattorini and Russell, 1971);
- $\operatorname{Ran}\Phi_\tau \supset \operatorname{Hol}(B)$, where $B = \left\{s \in \mathbb{C} \mid \left|s - \frac{\pi}{2}\right| < \frac{\pi}{2}\mathrm{e}^{(2\mathrm{e})^{-1}}\right\}$ (Martin, Rosier and Rouchon, 2016);
- For every $\varepsilon > 0$ we have $\operatorname{Ran}\Phi_\tau \supset \operatorname{Hol}(D_\varepsilon)$, where $D_\varepsilon$ is an $\varepsilon$-neighbourhood of the square $D$ (Dardé and Ervedoza, 2016).
- $E^2(D) \subset \operatorname{Ran}\Phi_\tau \subset A^2(D)$ (Hartmann, Kellay and M.T. 2017)

# Hilbert spaces of analytic functions

Let $\Omega \subset \mathbb{C}$ be an open set with Lipshitz boundary.

The Hardy-Smirnov space $E^2(\Omega)$ is

$$E^2(\Omega) = \left\{ f \in Hol(\Omega) \;\middle|\; \int_{\partial\Omega} |f(\zeta)|^2 \, |d\zeta| < \infty \right\},$$

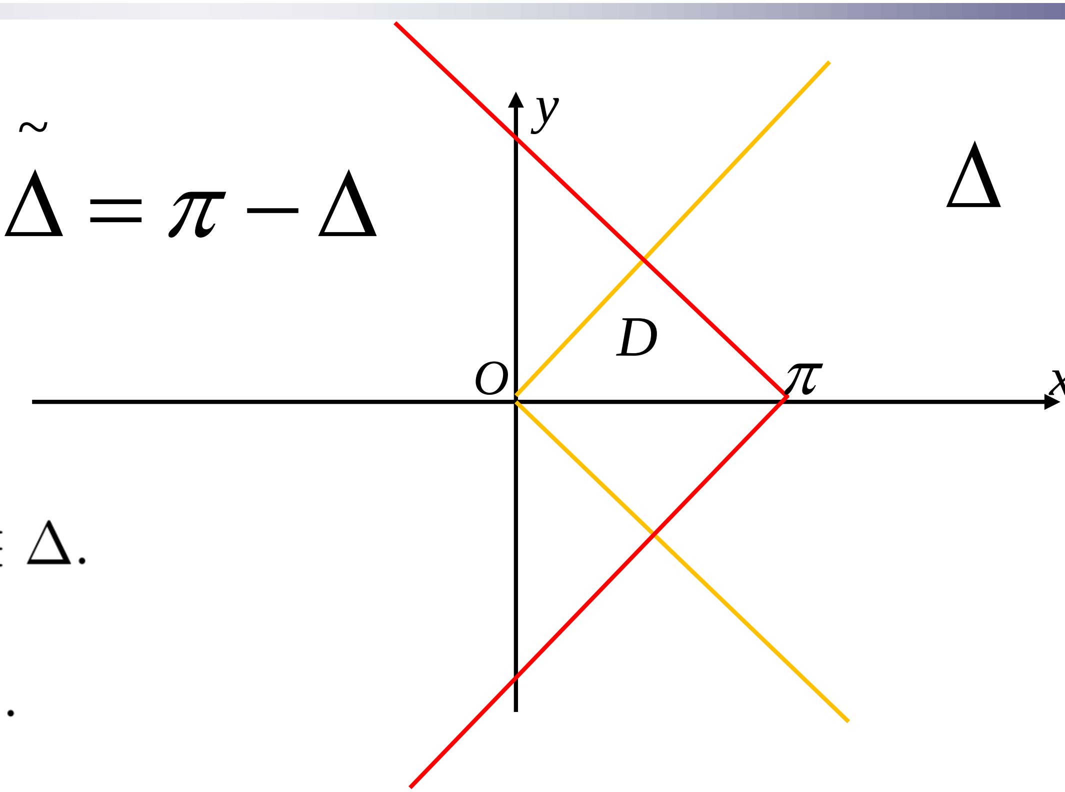
The Bergman space with weight $\omega$ is

$$A^2(\Omega, \omega) = Hol(\Omega) \cap L^2(\Omega, \omega).$$

For $\omega = 1$ we simply write $A^2(\Omega)$.

# Main results (I): notation

$$\tilde{\Delta} = \pi - \Delta$$




Let $\delta > 0$. $\omega_{0,\delta}(s) = \dfrac{e^{\frac{\mathrm{Re}\,(s^2)}{2\delta}}}{\delta}$ for $s \in \Delta$.

Let $\omega_{\pi,\delta}(\tilde{s}) = \omega_{0,\delta}(\pi - \tilde{s})$ for $\tilde{s} \in \tilde{\Delta}$.

Let $X_\delta = A^2(\Delta, \omega_{0,\delta}) + A^2(\tilde{\Delta}, \omega_{\pi,\delta})$.

$$\|\varphi\|_\delta = \inf \left\{ \|\varphi_0\|_{A^2(\Delta,\omega_{0,\delta})} + \|\varphi_\pi\|_{A^2(\tilde{\Delta},\omega_{\pi,\delta})} \;\middle|\; \begin{array}{c} \varphi_0 + \varphi_\pi = \varphi \\ \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{array} \right\}.$$

# Main results (II): statements

**Theorem 1.** (Kellay, Normand and M.T., 2019)
For every $\tau > 0$ we have $\Phi_\tau \in \mathcal{L}(L^2([0,\tau];U);X_\tau)$. Moreover, there exists $\delta^* > 0$ such that for every $\tau > 0$ and every $\delta \in (0,\delta^*)$ we have $\operatorname{Ran}\Phi_\tau = X_\delta$ for every $\tau > 0$.

**Proposition 1.** (Kellay, Normand and M.T., 2019)
We have $X_t = X_\tau$ for every $t,\ \tau > 0$.

**Corollary 1.** (Kellay, Normand and M.T., 2019)
We have $\operatorname{Ran}\Phi_\tau = X_\delta$ for every $\tau,\ \delta > 0$.

**Proposition 2.** (Kellay, Normand and M.T., 2019)
For every $\delta > 0$ we have $X_\delta = A^2(\Delta) + A^2(\tilde{\Delta})$.

**Corollary 2.** (Orsoni, 2019, Kellay, Normand and M.T., 2019)
We have $\operatorname{Ran}\Phi_\tau = A^2(\Delta) + A^2(\tilde{\Delta})$ for every $\tau,\ \delta > 0$.

# Main ingredient of the Proof of Theorem 1

**Theorem 2.** There exists $\delta^* > 0$ such that

$$\Phi_\tau(\sqrt{t}\, L^2[0,\tau]) = A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau}) := X_\tau \qquad (\tau \in (0, \delta^*)),$$

**Remark 3.** The proof is essentially contained in the previous work Hartman, Kellay, M.T. (2017), where the conclusion of Theorem 2 was seen as intermediate step. The main conlusion there was the obtained by proving the continuous embedding

$$E^2(D) \subset A^2(\Delta, \omega_{0,\tau}) + A^2(\tilde{\Delta}, \omega_{\pi,\tau}) \qquad (\tau \in (0, \delta^*)).$$

**Remark 4.** The question of the equality of $\widetilde{Hol}(\Omega_1) + \widetilde{Hol}(\Omega_2)$ with $\widetilde{Hol}(\Omega_1 \cap \Omega_2)$ (separation of singularities, Cousin problem) has been studied for various spaces, but it is open in the Bergman case. Important advancements recently obtained by Hartmann and Orsoni.

# Proof of Theorem 1

We know from Theorem 2 that $\Phi_\tau(\mathcal{U}_{\tau,\frac{1}{2}}) = X_\tau$ for $\tau \in (0, \delta^*))$. On the other hand, we know from Proposition 4 that $\Phi_\tau(\sqrt{t}\, L^2[0,\tau]) = \operatorname{Ran} \Phi_\tau$ for every $\tau > 0$. We thus have $\operatorname{Ran} \Phi_\tau = X_\tau$ for every $\tau \in (0, \delta^*)$. Finally we can apply Proposition 3 which says that $\operatorname{Ran} \Phi_\tau$ is independendent of $\tau > 0$. We obtain in this way that indeed for every $\tau > 0$ and every $\delta \in (0, \delta^*)$ we have $\operatorname{Ran} \Phi_\tau = X_\delta$.

# Proof of Proposition 1 (I)

For $t,\ \tau > 0$ we set $\Theta_{\tau,t}(s) := \mathrm{e}^{\frac{s^2(t-\tau)}{4t\tau}} + \mathrm{e}^{\frac{(\pi-s)^2(t-\tau)}{4t\tau}}$ .

**Lemma 1.** Assume $t,\ \tau,\ \varepsilon > 0$ are such that $\varepsilon < \tau$, $t = \tau + \varepsilon$ and

$$(\tau+\varepsilon)\left(\frac{\pi}{2\tau} - 1\right) < \frac{\pi}{2}.$$

Then for every $f \in A^2(\Delta; \omega_{0,t})$ and every $\tilde{f} \in A^2(\widetilde{\Delta}; \omega_{\pi,t})$ we have

$$\frac{f}{\Theta_{\tau,t}} \in A^2(\Delta; \omega_{0,\tau}), \qquad \frac{\tilde{f}}{\Theta_{\tau,t}} \in A^2(\widetilde{\Delta}; \omega_{\pi,\tau}).$$

Moreover, for every $g \in A^2(\Delta;_{0,\tau-\varepsilon})$ and every $\tilde{g} \in A^2(\widetilde{\Delta}; \omega_{\pi,\tau-\varepsilon})$ we have

$$g\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\tau}), \qquad \frac{\tilde{g}}{\Theta_{\tau,t}} \in A^2(\widetilde{\Delta}; \omega_{\pi,\tau}).$$

# Proof of Proposition 1 (II)

**Lemma 2.** Assume $t$, $\tau$, $\varepsilon > 0$ and $\Theta_{\tau,t}$ are as in Lemma 1. Moreover, assume that $\tau > 0$ is such that

$$X_\delta = X_\tau \qquad (0 < \delta \leqslant \tau).$$

(This holds, in particular, for $\tau \in (0, \delta^*)$, where $\delta^*$ is the constant in Theorem 1.). Then $X_t = X_\tau$.

*Proof.* Obviously that $X_\tau \subset X_t$. To prove that $X_t \subset X_\tau$, let $\varphi \in A^2(\Delta; \omega_{0,t})$ and $\tilde{\varphi} \in A^2(\widetilde{\Delta}; \omega_{\pi,t})$ . By Lemma 1 we have $\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} \in X_\tau = X_{\tau-\varepsilon}$. It follows that there exist $f \in A^2(\Delta; \omega_{0,\tau-\varepsilon})$ et $\tilde{f} \in A^2(\widetilde{\Delta}; \omega_{\pi,\tau-\varepsilon})$ such that $\frac{\varphi}{\Theta_{\tau,t}} + \frac{\tilde{\varphi}}{\Theta_{\tau,t}} = f + \tilde{f}$. Using again Lemma 1 it follows that

$$\varphi + \tilde{\varphi} = f\Theta_{\tau,t} + \tilde{f}\Theta_{\tau,t} \in A^2(\Delta; \omega_{0,\tau}) + A^2(\widetilde{\Delta}; \omega_{\pi,\tau}) = X_\tau,$$

which ends the proof. $\square$

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# Proof of Proposition 1 (III)

Let $\mathcal{I} = \{\tau > 0 \mid X_\delta = X_\tau \text{ for all } \delta \in (0, \tau^*)\}$. From Theorem 1 we know that $\mathcal{I} \supset (0, \delta^*)$. Let

$$\tau_0 := \min(\pi/4, \delta^*), \quad m := \min\left(2\tau_0, \frac{\pi\tau_0}{\pi - 2\tau_0}\right), \quad \varepsilon_0 := (m - \tau_0)/2.$$

We clearly have that $\tau_0 \in \mathcal{I}$ and $(\tau_0 + \varepsilon_0)\left(\frac{\pi}{2\tau_0} - 1\right) < \frac{\pi}{2}$. Since the function $t \mapsto (t + \varepsilon_0)\left(\frac{\pi}{2t} - 1\right)$ is clearly decreasing on $(0, \infty)$, it follows that

$$(t + \varepsilon_0)\left(\frac{\pi}{2t} - 1\right) < \frac{\pi}{2} \qquad\qquad (\tau > \tau_0).$$

Consider now the sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_0 = \tau_0$ and $t_{n+1} = t_n + \varepsilon_0$. Applying recursively Lemma 2 and obtain that for every $n \in \mathbb{N}$ and every $t \in [t_n, t_{n+1}]$ we have $X_t = X_{\tau_0}$, thus $\mathcal{I} = (0, \infty)$.

# Proof of Proposition 2

**Lemma 3.** Let $\psi(s) = \mathrm{e}^{\frac{s^2}{2\pi}} + \mathrm{e}^{\frac{(\pi-s)^2}{2\pi}}$. Then for every $\varphi \in A^2(\Delta)$ and every $\tilde{\varphi} \in A^2(\widetilde{\Delta})$ we have $\frac{\varphi}{\Psi} \in A^2(\Delta; \omega_{0,\pi/2})$, $\quad \frac{\tilde{\varphi}}{\Psi} \in A^2(\widetilde{\Delta}; \omega_{\pi,\pi/2})$.
Moreover, for every $f \in A^2(\Delta; \omega_{0,\pi/4})$ and $\tilde{f} \in A^2(\widetilde{\Delta}; \omega_{\pi,\pi/4})$ we have $f\Psi \in A^2(\Delta; \omega_{0,\pi/2})$, $\quad \tilde{f}\Psi \in A^2(\widetilde{\Delta}; \omega_{\pi,\pi/2})$.

*Proof of Proposition 2.* Let $\varphi \in A^2(\Delta)$ and $\tilde{\varphi} \in A^2(\widetilde{\Delta})$ and let $g = \varphi + \tilde{\varphi}$. According to Lemma 3 there exist $f \in A^2(\Delta; \omega_{0,\pi/4})$ and $\tilde{f} \in A^2(\widetilde{\Delta}; \omega_{\pi,\pi/4})$ such that $\frac{g}{\Psi} = f + \tilde{f}$. Using again Lemma 3 it follows that

$$g = f\Psi + \tilde{f}\Psi \in A^2(\Delta; \omega_{0,\pi/2}) + A^2(\widetilde{\Delta}; \omega_{\pi,\pi/2}) = X_{\frac{\pi}{2}}.$$

We have thus shown that

$$A^2(\Delta) + A^2(\widetilde{\Delta}) \subset X_{\frac{\pi}{2}}.$$

□

# Concluding remarks

# Exact controllability on the reachable space ?

The above property does not hold in the infinite dimensional case (take a heat equation on a half-line controled from the boundary).

Does it hold for systems which are null controllable in any positive time?

It "almost" holds for the 1D boundary controlled heat equation (we do not know that the heat semigroup is strongly continous on $\operatorname{Ran}\Phi_\tau$).

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# Reachability with smooth inputs

Proposition 3 . For $\tau > 0$ and $n \in \mathbb{N}$ we set

$$E_n^2(D) := \left\{ \psi \in E^2(D) \;\middle|\; \frac{\mathrm{d}^{2k}\psi}{\mathrm{d}s^{2k}} \in E^2(D) \text{ for } k = 1, \dots, n \right\} \qquad (n \geqslant 1).$$

$$W_{\mathrm{L}}^{n,2}(0,\tau) = \left\{ v \in W^{n,2}(0,\tau) \;\middle|\; v(0) = \cdots = \frac{\mathrm{d}^{n-1}v}{\mathrm{d}t^{n-1}}(0) = 0 \right\} \qquad (n \geqslant 1).$$

Then for every $\psi \in E_n^2(D)$ there exist $u_0,\ u_\pi \in W_{\mathrm{L}}^{n,2}(0,\tau)$ with $\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi$.

**Remark.** We conjecture that the following "analytic" version holds: for every $\psi \in Hol(\tilde{D})$, where $\tilde{D} \subset \mathbb{C}$ is an open set containing $\overline{D}$, there exist Gevrey type controls $u_0,\ u_\pi$, with all derivatives vanishing at $t = 0$, such that $\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \psi$.

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# Connections with the control cost

Assuming that the system $(\mathbb{T}, \Phi)$ is null controllable in some time $\tau > 0$ (this means that $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.), *the cost of null controllability in time* $\tau$ is the number $c_\tau$ defined by $c_\tau = \sup_{\|\psi\|_X \leqslant 1} \|\mathbb{T}_\tau \psi\|_{\operatorname{Ran} \Phi_\tau}$.
For our boundary controlled heat equation we set

$$d_\tau = \sup_{\|\psi\|_{W^{-1,2}(0,\pi)} \leqslant 1} \|\mathbb{T}_\tau \psi\|_\tau.$$

Proposition 4.
With the above notation we have

$$\limsup_{\tau \to 0+} \frac{c_\tau}{d_\tau} \leqslant 1.$$

# Several space dimensions

$$(BCHn) \qquad \begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \Delta w(t,x) & t \geqslant 0,\ x \in \Omega, \\ w(t,\cdot) = u, & t \in [0,\infty),\ x \in \partial\Omega \\ w(0,x) = 0 & x \in \Omega, \end{cases}$$

Given $\tau > 0$, define the *input to state map*

$$\Phi_\tau u = w(\tau,\cdot) \qquad (\tau > 0,\ u \in L^2([0,\tau];L^2(\partial\Omega)).$$

**Theorem (Strohmaier and Waters, 2020).**
For every $\tau > 0$ we have $\operatorname{Ran}\Phi\tau \supset Hol\left(\overline{\mathcal{E}(\Omega)}\right)$, where

$$\mathcal{E}(\Omega) = \{x + iy \in \mathbb{C}^n \ \mid\ x \in \Omega,\ |y| < d(x,\partial\Omega)\}\,.$$