Local null controllability of a class of non-Newtonian incompressible viscous fluids

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Control in Times of Crisis Webinar

October 27, 2020

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Outline

The model

Let us consider $d \in \{2,3\}$, an open subset Ω of $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and let us write $Q :=]0,T[\times \Omega$, as well as $\Sigma := [0,T] \times \partial\Omega$. We consider a fluid for with a velocity field $y = y(t,x) \in \mathbb{R}^d$ and pressure $p = p(t,x)$ governed by

$$\begin{cases} \frac{Dy}{Dt} - \nabla \cdot \mathcal{T}(y,p) = \chi_\omega v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,\cdot) = y_0 & \text{in } \Omega, \end{cases}$$

where:

- The function $v = v(t,x) \in \mathbb{R}^d$ is a control;
- We represent by χ_ω the indicator function of an open subset ω of Ω;
- The material derivative $\frac{Dy}{Dt}$ is equal to $\partial_t y + (y \cdot \nabla)\, y$;
- The stress tensor is $\mathcal{T}(y,p) = -pI + \nu(\nabla y)\nabla y$.

Shear-thickening power-law fluid

The constitutive law for the shear tensor here is

$$\tau = \nu(\nabla y)\nabla y,$$

and we actually assume

$$\nu(\nabla y) := \nu_0 + \nu_1 \left|\nabla y\right|^r ,$$

where $r \in \{1, 2\}$ or $r \geqslant 3$. Moreover, we remark that we assume no-slip boundary conditions.

A derivation from the turbulence viewpoint

Some turbulence models are derived by averaging the Navier-Stokes equations. Namely, if u solves the Navier-Stokes equations, we write

$$u = y + u',$$

where $y = \overline{u}$ is some average of u (around each point). Taking the average on the PDE solved by u,

$$y_t - \nu_0 \Delta y + \overline{(u \cdot \nabla)\, u} + \nabla p = v,$$

where we have written $p = \overline{\pi}$ and $v = \overline{f}$. We assume

$$\overline{(u \cdot \nabla)\, u} = (y \cdot \nabla)\, y - \nabla \cdot R.$$

With the further hypothesis $R = \nu_1 |\nabla y|^r \nabla y$, we retrieve the current model. For $r = 1$, this is the Smagorinsky turbulence model.

A few properties of the model

We consider the class of shear-dependent non-Newtonian fluids known as the power-law ones. The shear stress is a given power of the spatial gradient of the velocity field.

- The pioneers in the study of these models were O. A. Ladyzhenskaya and J.-L. Lions. Well-posedness results can be found in [Ladyzhenskaya, 1968, Lions, 1969].
- For the values of r we are focusing on, these fluids are shear-thickening (dilatant).
- A stylized fact about this class of fluids is that the particular form of their shear tensor has a stabilizing effect.

Motivations to consider this model

It enjoys wide applicability in real world problems, namely:

- Behavior of colloid, suspensions, and a variety of polymeric liquids in chemical engineering;
- Ice mechanics and glaciology;
- Blood-rheology;
- Geology.

See [Málek et al., 1995] and the references therein for particular instances of the use of power-law models in these problems.

Motivations to establish a local null controlability result

- It is usual to prove global controllability by means of extending a local null controllability argument;
- In the case of the Navier-Stokes equations ($\nu_1 = 0$) with the no-slip boundary condition, the global null-controllability is an open problem as of today. It may be that, for positive ν_1, the stabilizing effect renders the current problem easier than the $\nu_1 = 0$ case;
- For spatial dimension greater than one, even for quasilinear parabolic equations, there seems to be no work in the literature considering gradient dependent diffusion coefficients.

The distributed local null controllability result

Theorem

Given $T > 0$, *there exists* $\eta > 0$ *such that, if* $\|y_0\|_{H^5(\Omega)^N} < \eta$, *then we can find a control* $v \in L^2\left(]0,T[\times\omega\right)^N$ *for which the corresponding velocity field* y *satisfies*

$$y(T,x) = 0$$

for almost every $x \in \Omega$.

An outline of the proof

1. We take advantage of a variational characterization of the control and the estimates following from it to obtain a higher regularity for the control. This in turn allows us to deduce enough regularity for the state variable to make the approach work.
2. We set up functional spaces Y and Z and a mapping $H : Y \to Z$ such that a solution (y, p, v) of the equation $H(y, p, v) = f$, for suitable $f \in Z$, consists of a control v and the corresponding states (y, p).
3. We verify that H is well-defined, that it is of class C^1, and moreover that its derivative at the origin is onto.
4. We conclude via an appropriate inversion theorem.

Local null controllability with boundary controls

Let us suppose $y_0 \in H_0^5(\Omega)^N \cap V$. We extend y_0 to zero outside of Ω. Using standard geometrical arguments, we can extend Ω to $\widehat{\Omega}$ in such a way that

$$\partial\Omega \backslash \gamma \subseteq \partial\widehat{\Omega},$$

where $\gamma \subseteq \partial\Omega$ is the control region. We build a distributed control on $\widehat{\Omega} \backslash \overline{\Omega}$. The restriction of the corresponding controlled state to $\Gamma := [0, T] \times \gamma$ provides a suitable boundary control.

A numerical algorithm

Algorithm 1: A quasi-Newton algorithm

Result: A control v and the corresponding velocity field y.

Initialize with the error variable ϵ, the tolerance ϵ_0, $n = 0$, and an initial guess (y^0, p^0, v^0).

while $\epsilon > \epsilon_0$ **do**

We let f^n be the solution of

$$DH(0,0,0) \cdot f^n = H(y^n, p^n, v^n) - (0, y_0);$$

We set $(y^{n+1}, p^{n+1}, v^{n+1}) := (y^n, p^n, v^n) - f^n$;

We update ϵ;

We update $n \leftarrow n + 1$;

end

Local convergence

Theorem

There exists a small enough constant $\eta > 0$ *and appropriate Banach spaces* Y *and* Z *such that, if* $\|y_0\|_{H^5(\Omega)^N} < \eta$, *then it is possible to find* $\kappa \in]0,1[$ *with the following property: the relations* $(y^0, p^0, v^0) \in Y$ *and*

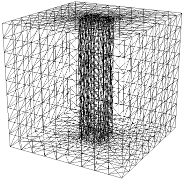
$$\|(y^0, p^0, v^0) - (y, p, v)\|_Y < \kappa,$$

imply the existence of $\theta \in]0,1[$ *for which*

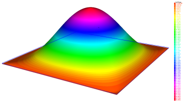
$$\|(y^{n+1}, p^{n+1}, v^{n+1}) - (y, p, v)\|_Y \leqslant \theta \|(y^n, p^n, v^n) - (y, p, v)\|_Y,$$

for all $n \geqslant 0$. *In particular,* $(y^n, p^n, v^n) \to (y, p, v)$ *in* Y.

An experiment

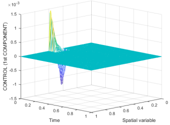


(a) Triangulation of Q.

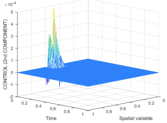


(b) ψ_0, where $y_0 = \nabla \times \psi_0$.

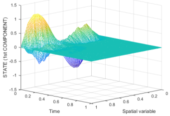
The following are the time evolutions of the cuts $x_1 = 0.5$ for v and the corresponding y.



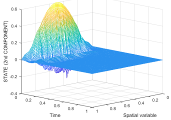
(a) v_1



(b) v_2



(c) y_1



(d) y_2

The linearized system

We consider the null controllability of the forced Stokes problem:

$$\begin{cases} Ly + \nabla p = \chi_\omega v + f & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{1}$$

The adjoint state (φ, π) satisfies

$$\begin{cases} L^* \varphi + \nabla \pi = g & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases} \tag{2}$$

Notations: $L := \partial_t - \nu_0 \Delta$ and $L^* := \partial_t + \nu_0 \Delta$.

Observability

There is a constant $C > 0$ such that the solution φ of (2) corresponding to $g \in L^2(Q)^N$ and $\varphi^T \in H$ satisfies

$$\|\varphi(0)\|^2 + \int_Q \left[\rho_0^{-2}|\varphi|^2 + \rho_1^{-2}|\nabla\varphi|^2 + \rho_2^{-2}\left(|\varphi_t|^2 + |\Delta\varphi|^2\right)\right] d(t,x) \leqslant C\left(\int_Q \rho_3^{-2}|g|^2 d(t,x) + \int_0^T \int_{\omega_1} \rho_4^{-2}|\varphi|^2 dx\ dt\right),$$

for suitable weights ρ_0, ρ_1, ρ_2, ρ_3 and ρ_4 which blow up as $t \uparrow T$.

Variational characterization of a control

- We define
  $P_0 := \left\{ (w,\sigma) \in C^2(\overline{Q})^{N+1} : \nabla \cdot w \equiv 0,\ w|_\Sigma \equiv 0,\ \int_\Omega \sigma \ dx = 0 \right\}.$
- We take $\chi \in C_c^\infty(\omega)$, with $0 \leqslant \chi \leqslant 1$, $\chi|_{\omega_1} \equiv 1$, and we consider on P_0 the continuous bilinear form

$$b((w,\sigma),(\widetilde{w},\widetilde{\sigma})) := \int_Q \left\{ \rho_3^{-2} (L^* w + \nabla\sigma) \cdot (L^* \widetilde{w} + \nabla \widetilde{\sigma}) + \chi \rho_4^{-2} w \cdot \widetilde{w} \right\} d(t,x).$$

- For $P := \overline{P}_0^b$, we also consider the linear form

$$\Lambda : (w,\sigma) \in P \longmapsto \int_\Omega y_0 \cdot w(0) dx + \int_Q f \cdot w \ d(t,x) \in \mathbb{R}.$$

- The Riesz representation theorem guarantees the existence of a unique $(\varphi,\pi) \in P$ for which

$$\Lambda(w,\sigma) = b((w,\sigma),(\varphi,\pi)) \qquad \text{(for all } (w,\sigma) \in P).$$

Variational characterization of the control

Then, we set

$$y := \rho_3^{-2}\left(L^*\varphi + \nabla\pi\right),\; z := \rho_4^{-2}\varphi,\; v := -\chi z.$$

We see that v is a control, with corresponding velocity field y. Moreover,

$$\int_Q \rho_3^2|y|^2 d(t,x) + \int_0^T \int_\omega \rho_4^2|v|^2 dx\; dt \leqslant C\kappa_0(y_0, f).$$

First estimates for the state

For suitable weights ρ_6, ρ_7, we have

$$\sup_{[0,T]}\left(\int_\Omega \rho_6^2|y|^2dx\right)+\int_Q \rho_6^2|\nabla y|^2 d(t,x)\leqslant C\kappa_0(y_0,f),$$

If $y_0\in H_0^1(\Omega)^N$, then

$$\int_Q \rho_7^2\left(|y_t|^2+|\Delta y|^2\right)d(t,x)+\sup_{[0,T]}\left(\int_\Omega \rho_7^2|\nabla y|^2dx\right)\leqslant C\kappa_1(y_0,f).$$

A bootstrapping argument provides regularity for the control

We can find an appropriate weight ζ for which

$$\zeta v \in L^2(0, T; H^2(\omega) \cap H_0^1(\omega)) \cap C([0, T]\,; V), \;\; (\zeta v)_t \in L^2(]0, T[\times \omega)^N,$$

with the estimate

$$\int_0^T \int_\omega \left[|(\zeta v)_t|^2 + |\zeta \Delta v|^2 \right] dx\, dt + \sup_{[0,T]} \|\zeta v\|_V^2 \leqslant C\kappa_0(y_0, f).$$

A bootstrapping argument provides higher regularity for the control

Furthermore, if $y_0 \in H_0^1(\Omega)^N$, then for some $\widehat{\zeta}$ we have the memberships

$$(\widehat{\zeta} v_t)_t \in L^2(]0,T[\times\omega)^N,\ \widehat{\zeta} v_t \in L^2\left(0,T;\left[H^2(\omega)\cap H_0^1(\omega)\right]^N\right),$$

$$\widehat{\zeta} v \in L^2\left(0,T;\left[H^4(\omega)\cap H_0^1(\omega)\right]^N\right),$$

and the following inequality holds

$$\int_0^T\int_\omega\left[\left|(\widehat{\zeta} v_t)_t\right|^2+\left|\widehat{\zeta}\Delta v_t\right|^2+\left|\widehat{\zeta} D^4 v\right|^2\right]dx\ dt \leqslant C\kappa_1(y_0,f).$$

Idea of the proof

Let us write $u := \zeta z$ and $\widetilde{\pi} := \mu\pi$, (for a suitable weight μ). Then, (u, π) solves the Stokes equation

$$\begin{cases} L^* u + \nabla\widetilde{\pi} = h & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T) = 0 & \text{in } \Omega, \end{cases}$$

where

$$h := -\frac{d\mu}{dt}\varphi + \mu_1 y \in L^2(Q)^N.$$

By standard regularity results for solutions of the Stokes system, we can infer the stated regularity for $\zeta v = -\chi u$. For higher regularity, we carry out similar arguments to $L^*(\widehat{\zeta} z_t)$ and $L^*(\widehat{\zeta}\Delta z)$.

Higher-order estimates for the state

Supposing $y_0 \in H^2(\Omega)^N \cap V$, there are weights ρ_8 and ρ_9 such that, for $\rho_8 f_t \in L^2(Q)^N$, we have the following estimates:

$$\sup_{[0,T]} \left(\int_\Omega \rho_8^2 |y_t|^2 dx \right) + \int_Q \rho_8^2 |\nabla y_t|^2 d(t,x) \leqslant C\kappa_2(y_0, f).$$

If furthermore $y_0 \in H^3(\Omega)^N$, $f(0) \in H_0^1(\Omega)^N$, then

$$\int_Q \rho_9^2 \left(|y_{tt}|^2 + |\Delta y_t|^2 \right) d(t,x) + \sup_{[0,T]} \left[\int_Q \rho_9^2 \left(|\nabla y_t|^2 + |\Delta y|^2 \right) dx \right] \leqslant C\kappa_3(y_0, f).$$

Higher-order estimates for the state

Let us assume that $y_0 \in H^4(\Omega)^N \cap V$. For adequate weights ρ_{10} and ρ_{11}, we also suppose

$$\rho_9 \Delta f \in L^2(Q)^N,\ \rho_{10} f_{tt} \in L^2(Q)^N,\ \rho_{10} f_t \in L^2\left(0, T; H_0^1(\Omega)^N\right),$$

$$f(0) \in \left[H^2(\Omega) \cap H_0^1(\Omega)\right]^N \text{ and } f_t(0) \in L^2(\Omega).$$

Then, the following estimate holds

$$\sup_{[0,T]} \left[\int_\Omega \rho_{10}^2 \left(|y_{tt}|^2 + |\Delta y_t|^2\right) dx + \|\rho_9 y\|^2_{H^3(\Omega)^N}\right]$$
$$+ \int_Q \left(\rho_{10}^2 |\nabla y_{tt}|^2 + \rho_{10}^2 |D^3 y_t|^2 + \rho_9^2 |D^4 y|^2\right) d(t,x) \leqslant C\kappa_4(y_0, f).$$

If $y_0 \in H^5(\Omega)^N$, $f(0) \in H^3(\Omega)^N$ and $f_t(0) \in H_0^1(\Omega)^N$, then

$$\sup_{[0,T]} \left(\int_\Omega \rho_{11}^2 |\nabla y_{tt}|^2 dx\right) + \int_Q \rho_{11}^2 \left(|y_{ttt}|^2 + |\Delta y_{tt}|^2\right) d(t,x)$$
$$\leqslant C\kappa_5(y_0, f).$$

A local inversion to the right theorem

Theorem

Let us consider a continuous mapping $H : Y \to Z$ with $H(0) = 0$. We assume that there are $\delta, \eta', M > 0$ and a continuous linear mapping $\Lambda : Y \to Z$ with the following properties:

(i) *For all $e \in Y$, we have*

$$\|e\|_Y \leqslant M \|\Lambda(e)\|_Z;$$

(ii) *The constants δ and M satisfy $\delta < M^{-1}$;*

(iii) *Whenever $e_1, e_2 \in B_Y(0; \eta')$, the inequality*

$$\|H(e_1) - H(e_2) - \Lambda(e_1 - e_2)\|_Z \leqslant \delta \|e_1 - e_2\|_Y$$

holds.

Then, for each $k \in B_Z(0; \eta)$, the equation $H(e) = k$ has a solution $e \in B_Y(0; \eta')$, where $\eta := \left(M^{-1} - \delta\right) \eta'$.

The functional spaces

Let us set

$$X_1 := L^2(Q;\rho_3^2)^N \text{ and } X_2 := L^2(]0,T[\times\omega;\rho_4^2)^N.$$

We define

$$\begin{aligned}Y := \Big\{ &(y,p,v) \in X_1 \times L^2(Q) \times X_2 : y_t \in L^2(Q)^N,\ \nabla y \in L^2(Q)^{N\times N},\\ &(\zeta v)_t,\ \zeta\Delta v,\ (\widehat{\zeta} v_t)_t,\ \widehat{\zeta}\Delta v_t \in L^2(\omega_T)^N,\ \widehat{\zeta} D^4 v \in L^2(\omega_T)^{N^4},\\ &\text{for } f := Ly + \nabla p - \chi_\omega v,\ \rho_0 f,\ \rho_8 f_t,\ \rho_9 \Delta f,\ \rho_{10} f_{tt} \in L^2(Q)^N,\\ &\rho_{10} f_t \in L^2\left(0,T;H_0^1(\Omega)^N\right),\ f(0) \in \left[H^3(\Omega)\cap H_0^1(\Omega)\right]^N,\\ &f_t(0) \in H_0^1(\Omega)^N,\ y|_\Sigma \equiv 0,\ \nabla\cdot y \equiv 0,\ y(0) \in \left[H^5(\Omega)\cap V\right]^N,\\ &\int_\Omega p\, dx = 0 \Big\},\end{aligned}$$

for $\omega_T :=]0,T[\times\omega$.

The functional spaces

We endow Y with the norm

$$
\begin{aligned}
\|(y,p,v)\|_Y^2 &:= \int_Q \rho_3^2 |y|^2 d(t,x) \\
&+ \int_Q \left(\rho_0^2|f|^2 + \rho_8^2|f_t|^2 + \rho_9^2|\Delta f|^2 + \rho_{10}^2|\nabla f_t|^2 + \rho_{10}^2|f_{tt}|^2\right) d(t,x) \\
&+ \int_{\omega_T} \left(\rho_4^2\,|v|^2 + |(\zeta v)_t|^2 + |\zeta \Delta v|^2 + \left|(\widehat{\zeta} v_t)_t\right|^2 + \left|\widehat{\zeta}\Delta v_t\right|^2\right) d(t,x) \\
&+ \int_{\omega_T} \left|D^4\left(\widehat{\zeta} v\right)\right|^2 d(t,x) + \|f(0)\|^2_{[H^3(\Omega)\cap H_0^1(\Omega)]^N} + \|f_t(0)\|^2_{H_0^1(\Omega)^N} \\
&+ \|y(0)\|^2_{H^5(\Omega)^N}.
\end{aligned}
$$

In this way, Y is a Banach space.

The functional spaces

We put

$$F := \Big\{ f \in L^2(Q)^N : \rho_0 f,\ \rho_8 f_t,\ \rho_9 \Delta f,\ \rho_{10} f_{tt} \in L^2(Q)^N,\\
\rho_{10} f_t \in L^2\left(0, T; H_0^1(\Omega)^N\right), f(0) \in \left[H^3(\Omega) \cap H_0^1(\Omega)\right]^N,\\
f_t(0) \in H_0^1(\Omega)^N \Big\},$$

with norm

$$\|f\|_F^2 := \int_Q \left(\rho_0^2 |f|^2 + \rho_8^2 |f_t|^2 + \rho_9^2 |\Delta f|^2 + \rho_{10}^2 |\nabla f_t|^2 + \rho_{10}^2 |f_{tt}|^2\right) d(t,x)\\
+ \|f(0)\|^2_{\left[H^3(\Omega)\cap H_0^1(\Omega)\right]^N} + \|f_t(0)\|^2_{H_0^1(\Omega)^N}.$$

Finally, we set

$$Z := F \times \left[H^5(\Omega) \cap V\right]^N, \tag{3}$$

The space Z with the natural product topology is also a Banach space.

Step 1: the mapping H is well-defined

We define the mapping $H : Y \to Z$ by

$$H(y,p,v) := \left(\frac{Dy}{Dt} - \nabla \cdot \mathcal{T}(y,p) - \chi_\omega v, y(0) \right).$$

We must check that H is well-defined. We write

$$H = (h_1 + h_2 + h_3, y(0)),$$

where

$$\begin{cases} h_1 := y_t - \nu_0 \Delta y + \nabla p - \chi_\omega v, \\ h_2 := -\nabla \cdot \left[\left(\nu(\nabla y) - \nu_0 \right) \nabla y \right], \\ h_3 := (y \cdot \nabla)\, y. \end{cases}$$

It follows easily that $\|h_1\|_F < \infty$. We must show $\|h_2\|_F < \infty$.

Step 1: the mapping H is well-defined

We have

$$
\begin{aligned}
|\Delta h_2(y,p,v)| \leqslant C\big[&(r+1)r|r-1||\nabla y|^{r-2}|D^2 y|^3 \\
&+(r+1)r|\nabla y|^{r-1}|D^2 y||D^3 y| + (r+1)|\nabla y|^r|D^4 y|\big] \\
&= C(D_{1,1} + D_{1,2} + D_{1,3}).
\end{aligned}
$$

The relation $r = 1$ implies $D_{11} \equiv 0$.

Step 1: the mapping H is well-defined

If $r \geqslant 2$, then

$$\begin{aligned}\int_Q \rho_9^2 D_{1,1}^2 d(t,x) &\leqslant C(r)\int_0^T \rho_9^2 \int_\Omega |\nabla y|^{2(r-2)}|D^2 y|^6 dx\ dt\\ &\leqslant C(r)\int_0^T \rho_9^2 \|D^3 y\|^{2(r-2)}\|D^2 y\|_{L^6(\Omega)}^6 dt\\ &\leqslant C(r)\int_0^T \rho_9^2 \|D^3 y\|^{2(r-2)}\|y\|_{H^3(\Omega)}^6 dt\\ &\leqslant C(r)\int_0^T \rho_9^2 \|D^3 y\|^{2(r+2)} dt\\ &\leqslant C(r)\left(\sup_{[0,T]}\|\rho_9 D^3 y\|\right)^{2(r+2)}\int_Q \rho_9^{-2(r+1)} d(t,x) < \infty.\end{aligned}$$

We used the immersions $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

Step 1: the mapping H is well-defined

We can estimate the remaining terms $D_{1,2}$ and $D_{1,3}$ likewise, yielding

$$\|\rho_9 \Delta h_2\|_{L^2(Q)^N} < \infty.$$

In a similar way, we can prove that the remaining terms constituting the $F-$norm of h_2 are finite, whence the well-definiteness of H follows.

Step 2: The mapping H is of class C^1

Lemma

The mapping H is of class $C^1(Y,Z)$ and, for each $(\overline{y},\overline{p},\overline{v}) \in Y$, its derivative $DH(\overline{y},\overline{p},\overline{v}) \in \mathcal{L}(Y,Z)$ is given by

$$DH(\overline{y},\overline{p},\overline{v}) \cdot (y,p,v) = \left(\Lambda_1(\overline{y},\overline{p},\overline{v}) \cdot (y,p,v),\ y(0)\right), \tag{4}$$

where we have written

$$\begin{cases} \Lambda_1(\overline{y},\overline{p},\overline{v}) \cdot (y,p,v) := y_t - \nu_0 \Delta y + \nabla p - \chi_\omega v \\ -r\nu_1 \nabla \cdot \left[\chi_{\overline{y}} |\nabla \overline{y}|^{r-2} \nabla \overline{y} : \nabla y \, \nabla \overline{y} + |\nabla \overline{y}|^r \nabla y\right] + (y \cdot \nabla)\, \overline{y} + (\overline{y} \cdot \nabla)\, y, \\ \nabla \overline{y} : \nabla y := \operatorname{tr}\left(\nabla \overline{y}^{\intercal} \nabla y\right), \\ \chi_{\overline{y}} \text{ is the indicator function of the set } \{\nabla \overline{y} \neq 0\}. \end{cases}$$

Step 2: The mapping H is of class C^1

We analyze the differentiability around the zero trajectory for simplicity of notations. We write $H = (H_1, H_2)$. We have only to analyze H_1. Given $(y, p, v), (\overline{y}, \overline{p}, \overline{v}) \in Y$, we note that

$$H_1(y, p, v) - H_2(\overline{y}, \overline{p}, \overline{v}) - \Lambda_1 \cdot (y - \overline{y}, p - \overline{p}, v - \overline{v}) = -\nu_1 D_1 + D_2,$$

where

$$D_1 := \nabla \cdot \left(|\nabla \overline{y}|^r \nabla \overline{y} - |\nabla y|^r \nabla y \right),$$

$$D_2 := (\overline{y} \cdot \nabla)\overline{y} - (y \cdot \nabla)y.$$

Let us take two positive real numbers, ϵ and δ, and we suppose $\|(y, p, v)\|_Y \leqslant \delta$, $\|(\overline{y}, \overline{p}, \overline{v})\|_Y \leqslant \delta$. We show that we can take $\delta = \delta(\epsilon)$ such that

$$\|H_1(y, p, v) - H_2(\overline{y}, \overline{p}, \overline{v}) - \Lambda_1 \cdot (y - \overline{y}, p - \overline{p}, v - \overline{v})\|_F \leqslant \epsilon \|(y - \overline{y}, p - \overline{p}, v - \overline{v})\|_Y.$$

Step 2: The mapping H is of class C^1

It is enough to show that

$$\nu_1\|D_1\|_F + \|D_2\|_F \leqslant \epsilon\|(y-\overline{y}, p-\overline{p}, v-\overline{v})\|_Y. \tag{5}$$

We observe that

$$\begin{aligned}
|\Delta D_1| \leqslant\ & C(r+1)r\Big[|r-1|\,\big||\nabla\overline{y}|^{r-2} - |\nabla y|^{r-2}\big|\,|D^2\overline{y}|^3 \\
& + |r-1||\nabla y|^{r-2}\left(|D^2\overline{y}|^2 + |D^2 y|^2\right)|\nabla\overline{y} - \nabla y| \\
& + |r-1|\left(|\nabla\overline{y}|^{r-2} + |\nabla y|^{r-2}\right)|\nabla\overline{y} - \nabla y||D^2\overline{y}||D^3\overline{y}| \\
& + |\nabla y|^{r-1}|D^2\overline{y} - D^2 y||D^3\overline{y}| + |\nabla y|^{r-1}|D^2 y||D^3(\overline{y} - y)| \\
& + |\nabla\overline{y}|^{r-1}|\nabla(\overline{y} - y)||D^4\overline{y}| + |\nabla y|^{r}|D^4(\overline{y} - y)|\Big] \\
=\ & C(r+1)r\left(D_{1,1} + \cdots + D_{1,7}\right).
\end{aligned}$$

Step 2: The mapping H is of class C^1

If $r \in \{1,2\}$, then $D_{1,1} \equiv 0$. For $r \geqslant 3$,

$$\int_Q \rho_9^2 D_{1,1}^2 d(t,x) \leqslant C(r) \int_0^T \rho_9^2 \|D^3(\overline{y}-y)\|^2 \|D^3\overline{y}\|^{2(r-3)} \|D^2\overline{y}\|_{L^6(\Omega)}^6 dt$$
$$+C(r) \int_0^T \rho_9^2 \|D^3(\overline{y}-y)\|^2 \|D^3 y\|^{2(r-3)} \|D^2\overline{y}\|_{L^6(\Omega)}^6 dt$$
$$\leqslant C(r) \int_0^T \rho_9^2 \|D^3(\overline{y}-y)\|^2 \left(\|D^3\overline{y}\|^{2(r-3)} + \|D^3 y\|^{2(r-3)} \right) \|D^3\overline{y}\|^6 dt$$
$$= C(r) \int_0^T \rho_9^{-2r} \|\rho_9 D^3(\overline{y}-y)\|^2 \|\rho_9 D^3\overline{y}\|^{2(r-3)} \|\rho_9 D^3\overline{y}\|^6 dt$$
$$+C(r) \int_0^T \rho_9^{-2r} \|\rho_9 D^3(\overline{y}-y)\|^2 \|\rho_9 D^3 y\|^{2(r-3)} \|\rho_9 D^3\overline{y}\|^6 dt$$
$$\leqslant C(r)\delta^{2r} \|(y-\overline{y}, p-\overline{p}, v-\overline{v})\|_Y^2.$$

We analyze the remaining terms similarly.

Conclusion of the proof

- The condition on $DH(0,0,0)$ follows from the null controllability of the forced Stokes problem.
- The mapping $H : Y \to Z$ is continuous and strictly differentiable at $(0,0,0) \in Y$.

Thus, there exist $\eta, \eta' > 0$ such that the equation

$$H(y,p,v) = (h, y_0)$$

has a solution $(y,p,v) \in B_Y(0;\eta')$ for each $(h,y_0) \in B_Z(0;\eta)$. In particular, given $y_0 \in H^5(\Omega)^N \cap V$, with $\|y_0\|_{H^5(\Omega)^N} < \eta$, there is $(y,p,v) \in Y$ satisfying

$$H(y,p,v) = (0, y_0).$$

It is straightforward to infer that v is the control we desired.

On more general initial datum

A useful idea is to apply some smoothing effect. Thus, we should carry out estimates to show that, for sufficiently small $\|y_0\|_H$, there exists $t_0 \in]0, T/2]$ such that the uncontrolled state satisfies

$$\|y(t_0, \cdot)\|_{H^5(\Omega)^N} < \eta.$$

For this particular model, these estimates are not straightforward.

On the restrictions on the exponent

It seems that we can get rid of the restrictions on r by means of a regularization argument.

On large-time global null controllability

Once the result for more general initial datum, we can apply well-known dissipation results, see [Málek et al., 1995], to conclude that large-time global null controllability holds.

On small-time global null controllability

- Proving this might shed light on the corresponding problem for the Navier-Stokes equations.
- If instead of no-slip boundary conditions we use Navier boundary conditions, a method used in the literature for Navier-Stokes and Boussinesq equations might be extensible to the current model.

Thank you!

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