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# Optimal boundary control for drag minimization in a fluid-rigid body interaction

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# Outline

## Formulation of the problem

Equations of self-propelled motion in a Navier-Stokes liquid: direct problem and control problem

## Generalized Oseen problem

Combined effect of translation and rotation

$L^2$ estimate for the velocity

## Optimal boundary control for drag reduction

State equations and cost functional

Existence of solution for the state equations

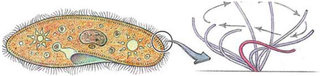
Existence of optimal controls

Characterization of the optimal controls

# Formulation of the problem

# The fluid-rigid body interaction problem

In practice, self-propelled motion can be produced by propellers (submarines), deformations (fishes), cilia (micro-organisms), etc.



**Definition**

A rigid body $\mathcal{S}$ undergoes a **self-propelled motion** in a fluid $\mathcal{F}$ if
(i) the total external force acting on $\mathcal{F}$ is identically zero,
(ii) the total net force and torque, external to $\{\mathcal{F}, \mathcal{S}\}$, acting on $\mathcal{S}$ are identically zero.

# Equations of motion of the liquid in an inertial reference frame

$$\left.\begin{aligned} \partial_t u + u \cdot \nabla u = \nabla \cdot \sigma(u, q) \\ \nabla \cdot u = 0 \end{aligned}\right\} \text{ in } \Omega(t),\ t > 0$$

$$u(y,t) = U(y,t) + u_*(y,t), \text{ at } \partial\Omega(t),\ t > 0$$

$$\lim_{|y| \to \infty} u(y,t) = 0, \text{ for all } t \in (0, \infty)$$

$$u(y,0) = u_0(y),\ y \in \Omega$$

$$\sigma(u,q) := 2\nu \mathbb{D}(u) - q\mathbb{I} = \nabla v - (\nabla v)^\top - p\mathbb{I}$$

$$\nabla \cdot \sigma(u,q) = \nu \Delta u - \nabla q$$

$$U(y,t) := \eta(t) + \theta(t) \times (y - y_C(t))$$

# Equations of motion of the solid in an inertial reference frame

$$U(y,t) = \eta(t) + \theta(t) \times (y - y_C(t))$$

$$m\frac{d\eta}{dt} = -\int_{\partial\Omega(t)} \sigma(u,q)N d\gamma + \int_{\partial\Omega(t)} (U + u_*)(u_* \cdot N) d\gamma$$

$$\frac{d(J\theta)}{dt} = \int_{\partial\Omega(t)} (y - y_C) \times \sigma(u,q)N d\gamma + \int_{\partial\Omega(t)} (y - y_C) \times (U + u_*)(u_* \cdot N) d\gamma$$

$$\eta(0) = \eta_0, \quad \theta(0) = \theta_0$$

# Equations of motion in a reference frame attached to $\mathcal{S}$

$$\left.\begin{aligned} \partial_t v + v\cdot\nabla v &= \nabla\cdot\sigma(v,p) + V\cdot\nabla v - \omega\times v \\ \nabla\cdot v &= 0 \end{aligned}\right\} \text{ in } \Omega\times(0,\infty)$$

$$v = v_* + V \text{ at } \partial\Omega\times(0,\infty)$$

$$\lim_{|x|\to\infty} v(x,t) = 0,\ \ t\in(0,\infty)$$

$$v(x,0) = v_0(x),\ \ x\in\Omega$$

$$V(x,t) = \xi(t) + \omega(t)\times x$$

$$V\cdot\nabla v = \xi\cdot\nabla v + (\omega\times x)\cdot\nabla v$$

# Equations of motion in a reference frame attached to $\mathcal{S}$

$$V(x,t) = \xi(t) + \omega(t) \times x$$

$$m\frac{d\xi}{dt} + m\omega \times \xi = -\int_\Sigma \sigma(v,p)n + \int_\Sigma (v_* + V)(v_* \cdot n)$$

$$I\frac{d\omega}{dt} + \omega \times (I\omega) = -\int_\Sigma x \times \sigma(v,p)n + \int_\Sigma x \times (v_* + V)(v_* \cdot n)$$

$$\xi(0) = \xi_0, \quad \omega(0) = \omega_0$$

$I$ is independent of time, symmetric and positive definite

# Equations of motion in a reference frame attached to $\mathcal{S}$

$$V(x,t) = \xi(t) + \omega(t) \times x$$

$$\left.\begin{aligned} \partial_t v + v \cdot \nabla v &= \nabla \cdot \sigma(v,p) + V \cdot \nabla v - \omega \times v \\ \nabla \cdot v &= 0 \end{aligned}\right\} \text{ in } \Omega \times (0,\infty)$$

$$v = v_* + V \text{ at } \partial\Omega \times (0,\infty)$$

$$\lim_{|x|\to\infty} v(x,t) = 0,\ \ t \in (0,\infty)$$

$$m\frac{d\xi}{dt} + m\omega \times \xi = -\int_\Sigma \sigma(v,p)n\ d\gamma + \int_\Sigma (v_* + V)(v_* \cdot n)\ d\gamma$$

$$I\frac{d\omega}{dt} + \omega \times (I\omega) = -\int_\Sigma x \times \sigma(v,p)n\ d\gamma + \int_\Sigma x \times (v_* + V)(v_* \cdot n)\ d\gamma$$

$$v(x,0) = v_0(x),\ \ x \in \Omega, \quad \xi(0) = \xi_0, \quad \omega(0) = \omega_0$$

# Steady states

$$\left.\begin{array}{rcl}\nabla\cdot\sigma(v,p) & = & (v-V)\cdot\nabla v+\omega\times v\\ \nabla\cdot v & = & 0\end{array}\right\}\text{ in }\Omega$$

$$v=v_*+V\text{ at }\partial\Omega$$

$$\lim_{|x|\to\infty} v(x)=0,$$

$$m\omega\times\xi=-\int_{\partial\Omega}\sigma(v,p)\cdot n\;d\gamma+\int_{\partial\Omega}(v_*\cdot n)(v_*+V)\;d\gamma$$

$$\omega\times(I\omega)=-\int_{\partial\Omega}x\times\sigma(v,p)\cdot n\;d\gamma+\int_{\partial\Omega}x\times(v_*\cdot n)(v_*+V)\;d\gamma$$

# The fluid-rigid body interaction problem

In the absence of external actions, the forward force (thrust) that makes $\mathcal{S}$ move is generated by $\mathcal{S}$. The motion is due to the interaction of the body's external surface and the fluid (velocities $v_*$).

**Self-propulsion** may be produced by

- means of drawing fluid inwards across portions of the boundary and expelling it from others, so that the net flux of momentum across the boundary is nonzero,

$$\int_{\partial\Omega} (v_* \cdot n)\,(v_* + V)\, d\gamma \neq 0 \text{ and } v_* = 0 \text{ on } \partial\Omega \setminus \Gamma;$$

  or

- moving tangentially portions of the boundary, as by belts. In this case,

$$\int_{\partial\Omega} (v_* \cdot n)\,(v_* + V)\, d\gamma = 0 \text{ because } v_* \cdot n = 0.$$

# The direct fluid-rigid body interaction problem

Notation:

$$V(x) := \xi + \omega \times x$$

$$\sigma(v,p) := \nabla v - (\nabla v)^\top - p\mathbb{I}$$

Direct problem: Given the boundary values $v_*$ at the surface of $\mathcal{S}$, prescribed relative to $\mathcal{S}$, find $(V, v, p)$ satisfying

$$-\nabla \cdot \sigma(v,p) + (v - V) \cdot \nabla v + \omega \times v = 0 \quad \text{in } \Omega$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega$$

$$v = V + v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

$$m\xi \times \omega + \int_{\partial\Omega} \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

$$(I\omega) \times \omega + \int_{\partial\Omega} x \times \left[-\sigma(v,p)n + (v_* \cdot n)(v_* + V)\right] \, d\gamma = 0$$

# The control problem - Some references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# The optimal control fluid-rigid body interaction problem

Question:

Is it possible to control the state of $\{\mathcal{S}, \mathcal{F}\}$ through a

a distribution of velocities $v_*$ at the boundary $\partial\Omega$

in order to

minimize the work needed to overcome the drag exerted by $\mathcal{F}$ on $\mathcal{S}$

$$\mathcal{J}(v_*, v, p) := \int_{\partial\Omega} v \cdot \sigma(v, p) n \, d\gamma$$

and move $\mathcal{S}$ with a target velocity $V(x) = \xi + \omega \times x$ ?

# The optimal control fluid-rigid body interaction problem

Given $V$, find $(v_*, v, p)$ that minimizes

$$\mathcal{J}(v_*, v, p) := \int_{\partial\Omega} v \cdot \sigma(v,p)n \; d\gamma = \int_{\partial\Omega} (v_* + V) \cdot \sigma(v,p)n \; d\gamma$$

and satisfies the state equations

$$-\text{div}\; \sigma(v,p) + v \cdot \nabla v - V \cdot \nabla v + \omega \times v = 0 \quad \text{in } \Omega$$

$$\text{div}\; v = 0 \quad \text{in } \Omega$$

$$v = V + v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} v(x) = 0$$

$$m\xi \times \omega + \int_{\partial\Omega} \left[ -\sigma(v,p)n + (v_* \cdot n)\,(v_* + V + \omega \times x) \right] \; d\gamma = 0$$

$$(I\omega) \times \omega + \int_{\partial\Omega} x \times \left[ -\sigma(v,p)n + (v_* \cdot n)\,(v_* + V + \omega \times x) \right] \; d\gamma = 0$$

# Generalized Oseen problem

# Classical Oseen problem

$$-\text{div } \sigma(v,p) = \xi \cdot \nabla v + f \quad \text{in } \Omega$$

$$\text{div } v = 0 \quad \text{in } \Omega$$

$$v = v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

Important case to be considered: $f = \nabla \cdot (v \otimes v)$

If $\xi \neq 0$ there is an infinite paraboloidal region within which $v$ decays like $|x|^{-1}$ and outside of which $v$ decays even faster.

# Classical Oseen problem

The non-uniform decay associated with the wake behind the body is described by the function

$$\varpi(x) := |x| \left[1 + \frac{1}{2}(|\xi||x| + \xi \cdot x)\right] = |x| \left[1 + s(x)\right]$$

$$|v(x)| \leq C \frac{1}{\varpi(x)}.$$

The wake region behind the body is

$$\mathcal{W}_\xi = \{x \in \mathbb{R}^3 : s(x) \leq 1\}.$$

# Important auxiliary problem: generalized Oseen system

$$-\text{div }\sigma(v,p) = (\xi + \omega \times x) \cdot \nabla v - \omega \times v + f \quad \text{in } \Omega$$

$$\text{div } v = 0 \quad \text{in } \Omega$$

$$v = v_* \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

In the case $\xi \neq 0$, it is also expected that $v$ decays faster outside a paraboloidal region behind the body, representative of the wake.

However, it is expected that the decay can be affected by the rotation.

# Generalized Oseen problem

The decay of $v$ and its derivatives is now described by the function

$$\varpi(x) := |x| \left[1 + \frac{1}{2}\left(\frac{|\xi \cdot \omega|}{|\omega|}|x| + \frac{(\xi \cdot \omega)}{|\omega|^2}(\omega \cdot x)\right)\right]$$

- There is no wake formation if $\xi$ and $\omega$ are orthogonal.
- If $\omega \neq 0$ and $\omega \cdot \xi \neq 0$ there is a formation of a wake $\mathcal{W}_{\xi,\omega}$, along the direction of $\omega$, whose "width" will depend on the angle between $\omega$ and $\xi$.

$$|v(x)| \leq C \frac{1}{\varpi(x)}$$

# Well-posedness of the generalized Oseen problem

**Theorem** Assume that $\partial\Omega$ is of class $C^2$, $f = \nabla \cdot F \in L^2(\Omega)$,

$$\lceil F \rceil_{2,\varpi,\Omega} := \sup_{x\in\Omega} \left[\varpi(x)^2 |F(x)|\right] < \infty$$

and $v_* \in W^{3/2,2}(\partial\Omega)$. Then, there exists a unique solution $(v, p)$ to the generalized Oseen problem with

$$\nabla v \in W^{1,2}(\Omega), \quad p \in W^{1,2}(\Omega),$$

$$\lceil v \rceil_{1,\varpi,\Omega} := \sup_{x\in\Omega} [\varpi(x)|v(x)|] < \infty$$

and

$$\|\nabla v\|_{1,2,\Omega} + \lceil v \rceil_{1,\varpi,\Omega} + \|p\|_{1,2,\Omega} \le C(\|\nabla \cdot F\|_{2,\Omega} + \lceil F \rceil_{2,\varpi,\Omega} + \|v_*\|_{3/2,2,\partial\Omega}).$$

In the above estimate, if $|\xi|, |\omega| \in [0, B]$, one can choose <u>$C$ independent of $\xi$ and $\omega$</u>.

# Consequence of the first self-propelling condition

Net force exerted by the fluid to the rigid body

$$\mathcal{N} = \int_{\partial\Omega} [\sigma(v,p) + v \otimes V - (\omega \times x) \otimes v + F]\, n \, d\gamma.$$

Important case: $F = v \otimes v$.

If $\mathcal{N} = 0$ then $v \in L^2(\Omega)$.

Let $(v,p)$ be a solution to the generalized Oseen problem of class

$$\nabla v, p \in L^2(\Omega), \quad v \in L^6(\Omega)$$

and $\Phi = \int_{\partial\Omega} n \cdot v \, d\gamma$.

Suppose $[x \mapsto (1+|x|)F(x)] \in L^2(\Omega)$.

# $L^2$-estimate of the solution to the generalized Oseen problem $(\mathcal{N} = 0)$

- Case $\omega = 0$.

$$\|v\|_{2,\Omega} \le C\big[(1+|\xi|)\left(\|\nabla v\|_{2,\Omega} + |\Phi|\right) + \|p\|_{2,\Omega} + \|F\|_{2,\Omega}\big] + C'\||x|F\|_{2,\Omega}$$

  with constants $C$, $C' > 0$ independent of $v, p, F, \xi$ and $\Phi$.

- Case $\omega \in \mathbb{R}^3 \setminus \{0\}$.

$$\|v\|_{2,\Omega} \le CK(v, q, F, \xi, \omega, \Phi) + C'\||x|F\|_{2,\Omega}$$

  with constants $C$, $C' > 0$ independent of $v, p, F, \xi, \omega$, where

$$K(v, q, F, \xi, \omega, \Phi) = \left(1 + |\omega|^{-1/4} + \frac{|\omega \cdot \xi|^{1/2}}{|\omega|} + \frac{|\omega \times \xi|}{|\omega|^2}\right) \cdot \big[(1 + |\xi| + |\omega|)\big(\|\nabla v\|_{2,\Omega} + |\Phi|\big) + \|p\|_{2,\Omega} + \|F\|_{2,\Omega}\big].$$

# Optimal boundary control for drag reduction

# State equations

$$\left.\begin{array}{rcl} \nabla \cdot \sigma(v,p) & = & (v - V) \cdot \nabla v + \omega \times v \\ \nabla \cdot v & = & 0 \end{array}\right\} \text{ in } \Omega$$

$$v = v_* + V \text{ at } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0,$$

$$m\omega \times \xi = -\int_{\partial\Omega} \sigma(v,p) \cdot n \; d\gamma + \int_{\partial\Omega} (v_* \cdot n)(v_* + V) \; d\gamma$$

$$\omega \times (I\omega) = -\int_{\partial\Omega} x \times \sigma(v,p) \cdot n \; d\gamma + \int_{\partial\Omega} x \times (v_* \cdot n)(v_* + V) \; d\gamma$$

Target velocity $V(x) = \xi + \omega \times x$ is given. The control is $v_*$.

How to satisfy the self-propelling conditions?

# Construction of correctors. Auxiliary adjoint linear systems

For $i \in \{1, 2, 3\}$, let $(v^{(i)}, q^{(i)})$ and $(V^{(i)}, Q^{(i)})$ be the solutions of the generalized Oseen problems

$$-\text{div } \sigma(v^{(i)}, q^{(i)}) + (\xi + \omega \times x) \cdot \nabla v^{(i)} - \omega \times v^{(i)} = 0 \quad \text{in } \Omega$$

$$\text{div } v^{(i)} = 0 \quad \text{in } \Omega$$

$$v^{(i)} = e_i \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} v^{(i)}(x) = 0$$

$$-\text{div } \sigma(V^{(i)}, Q^{(i)}) + (\xi + \omega \times x) \cdot \nabla V^{(i)} - \omega \times V^{(i)} = 0 \quad \text{in } \Omega$$

$$\text{div } V^{(i)} = 0 \quad \text{in } \Omega$$

$$V^{(i)} = e_i \times x \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} V^{(i)}(x) = 0$$

# The finite dimensional correctors

Recall:

$$\sigma(v,p) := \nabla v - (\nabla v)^\top - p\mathbb{I}$$

We can define velocity fields

$$g^{(i)} := \sigma(v^{(i)}, q^{(i)})n, \quad G^{(i)} := \sigma(V^{(i)}, Q^{(i)})n, \quad \text{on } \partial\Omega,$$

and the finite dimensional control spaces

- **localized boundary controls**

$\mathcal{C}_\chi := \text{span}\{\chi g^{(i)}, \chi G^{(i)}; \, i = 1,2,3\}$

- **tangential boundary controls**

$\mathcal{C}_\tau := \text{span}\{(g^{(i)} \times n) \times n, (G^{(i)} \times n) \times n \; ; \, i = 1,2,3\}$

# A linear "inverse" problem

Given $V(x) = \xi + \omega \times x$, find $v_* \in \mathcal{C}_\chi$ or $\mathcal{C}_\tau$ and $(u, q)$ satisfying

$$-\text{div}\ \sigma(u,q) - (\xi + \omega \times x) \cdot \nabla u + \omega \times u = f \quad \text{in } \Omega$$

$$\text{div}\ u = 0 \quad \text{in } \Omega$$

$$u = \vartheta + \textstyle\sum_{i=1}^{3} (\alpha_i \chi g^{(i)} + \beta_i \chi G^{(i)}) \quad \text{on } \partial\Omega \qquad \text{or}$$

$$u = \vartheta + \textstyle\sum_{i=1}^{3} (\alpha_i (g^{(i)} \times n) \times n + \beta_i (G^{(i)} \times n) \times n) \quad \text{on } \partial\Omega$$

$$\lim_{|x| \to \infty} u(x) = 0$$

$$-\int_{\partial\Omega} [\sigma(u,q)n + (\xi + \omega \times x) \cdot nu]\, d\gamma = \xi_f$$

$$-\int_{\partial\Omega} x \times [\sigma(u,q)n + (\xi + \omega \times x) \cdot nu]\, d\gamma = \omega_f .$$

where $f$, $\vartheta$, $\xi_f$ and $\omega_f$ are also given.

# Auxiliary linear systems for the inverse/control problem

and

$$-\text{div}\ \sigma(u_f, q_f) + \omega \times u_f - (\xi + \omega \times x) \cdot \nabla u_f = f$$

$$\text{div}\ u_f = 0$$

$$u_f = \vartheta \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} u_f(x) = 0$$

# Formulation as a linear algebraic system - The case of localized controls

$$A_\chi \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \delta \\ \zeta \end{pmatrix},$$

where

$$\delta_j := \xi_f \cdot e_j + \int_{\partial\Omega} \sigma(u_f, p_f)n \, d\gamma \cdot e_j \quad (j = 1, 2, 3),$$

$$\eta_j := \omega_f \cdot e_j + \int_{\partial\Omega} x \times \sigma(u_f, p_f)n \, d\gamma \cdot e_j \quad (j = 1, 2, 3),$$

and $A_\chi \in \mathbb{R}^{6\times 6}$ is defined by

$$A_{i,j} = \int_{\partial\Omega} \chi g^{(i)} \cdot g^{(j)} \, d\gamma \quad (i, j \leq 3),$$

$$A_{i,j} = \int_{\partial\Omega} \chi g^{(i)} \cdot G^{(j-3)} \, d\gamma \quad (i \leq 3, j \geq 4),$$

$$A_{i,j} = \int_{\partial\Omega} \chi G^{(i-3)} \cdot g^{(j)} \, d\gamma \quad (i \geq 4, j \leq 3),$$

$$A_{i,j} = \int_{\partial\Omega} \chi G^{(i-3)} \cdot G^{(j-3)} \, d\gamma \quad (i, j \geq 4).$$

# Formulation as a linear algebraic system - The case of localized controls

Lemma The matrix $A_\chi$ is is symmetric nonnegative. Furthermore, there exist positive constants $c_1, K$ such that if $|\xi| \leq c_1$, $|\omega| \leq c_1$, then $A$ is invertible with

$$\|A_\chi^{-1}\|_{\mathcal{L}(\mathbb{R}^6)} \leq K,$$

where $K$ is independent of $\xi, \omega$ with $|\xi|$, $|\omega| \leq c_1$.
Proof: Continuity of $A_{(\xi,\omega)}$ in $(0,0)$ and invertibility of $A_{(0,0)}$. Uses the convergence of the generalized Oseen system to the Stokes system when $(\xi,\omega) \to (0,0)$.

# The case of tangential controls

Similar formulation and results hold with the matrix $A_\tau \in \mathbb{R}^{6\times 6}$ defined by

$$A_{i,j} = -\int_{\partial\Omega} \left[(g^{(i)} \times n) \times n\right] \cdot \left[(g^{(j)} \times n) \times n\right] \, d\gamma \quad (i,j \leq 3),$$

$$A_{i,j} = -\int_{\partial\Omega} \left[(g^{(i)} \times n) \times n\right] \cdot \left[(G^{(j-3)} \times n) \times n\right] \, d\gamma \quad (i \leq 3, j \geq 4),$$

$$A_{i,j} = -\int_{\partial\Omega} \left[(G^{(i-3)} \times n) \times n\right] \cdot \left[(g^{(j)} \times n) \times n\right] \, d\gamma \quad (i \geq 4, j \leq 3),$$

$$A_{i,j} = -\int_{\partial\Omega} \left[(G^{(i-3)} \times n) \times n\right] \cdot \left[(G^{(j-3)} \times n) \times n\right] \, d\gamma \quad (i,j \geq 4).$$

# The state equations

Given $V$ and $v_*$, find $(v,p)$ and $v_*^{\mathcal{C}} \in \mathcal{C}_\tau$ or $v_*^{\mathcal{C}} \in \mathcal{C}_\chi$ such that

$$-\nabla \cdot \sigma(v,p) + (v-V)\cdot\nabla v + \omega \times v = 0 \quad \text{in } \Omega$$

$$\nabla \cdot v = 0 \quad \text{in } \Omega$$

$$v = V + v_* + v_*^{\mathcal{C}} \quad \text{on } \partial\Omega$$

$$\lim_{|x|\to\infty} v(x) = 0$$

$$m\xi \times \omega + \int_{\partial\Omega} \Big[-\sigma(v,p)n + \\ + \big((v_* + v_*^{\mathcal{C}})\cdot n\big)\,(v_* + v_*^{\mathcal{C}} + V)\Big]\; d\gamma = 0$$

$$(I\omega)\times\omega + \int_{\partial\Omega} x \times \Big[-\sigma(v,p)n + \\ + \big((v_* + v_*^{\mathcal{C}})\cdot n\big)\,(v_* + v_*^{\mathcal{C}} + V)\Big]\; d\gamma = 0$$

# Well-posedness of the state equations in the optimal control problem

Let

$$\mathcal{V}_\tau := \left\{ v_* \in W^{3/2,2}(\partial\Omega) \ ; \ v_* \cdot n = 0 \text{ on } \partial\Omega \right\}$$

and

$$\mathcal{V}_\Gamma := \left\{ v_* \in W^{3/2,2}(\partial\Omega) \ ; \ v_* = 0 \quad \text{on } \partial\Omega \setminus \Gamma \right\}.$$

**Theorem**

Let $\Omega$ be of class $C^3$. There exist constants $c_0, C_1, C_2 > 0$, which depend on $\Omega$, such that if $\xi, \omega \in \mathbb{R}^3$ and $v_* \in \mathcal{V}_\tau$ (resp. $v_* \in \mathcal{V}_\Gamma$) satisfy

$$|\xi| \leq c_0, \qquad |\omega| \leq c_0, \qquad \|v_*\|_{3/2,2,\partial\Omega} \leq c_0,$$

then the following assertions hold:

# Well-posedness of the state equations

• A solution $(v, p, v_*^{\mathcal{C}})$ of the state equations can be found within the class

$$\varpi v \in L^\infty(\Omega), \quad (v,p) \in W^{2,2}(\Omega)\times W^{1,2}(\Omega), \quad v_*^{\mathcal{C}} \in \mathcal{C}_\tau \quad \text{(resp. } v_*^{\mathcal{C}} \in \mathcal{C}_\chi\text{)}$$

along with estimates

$$\lceil v \rceil_{1,\varpi,\Omega} + \|\nabla v\|_{2,\Omega} + \|v_*^{\mathcal{C}}\|_{3/2,2,\partial\Omega} \le C_1\big(|(\xi,\omega)| + \|v_*\|_{3/2,2,\partial\Omega}\big),$$

$$\|\nabla \otimes \nabla v\|_{2,\Omega} + \|p\|_{1,2,\Omega} \le C_2\big(|(\xi,\omega)| + \|v_*\|_{3/2,2,\partial\Omega}\big).$$

Recall:

$$\varpi(x) := |x| \left[1 + \frac{1}{2}\left(\frac{|\xi\cdot\omega|}{|\omega|}|x| + \frac{(\xi\cdot\omega)}{|\omega|^2}(\omega\cdot x)\right)\right]$$

# Well-posedness of the state equations

• The energy equation is valid

$$\int_{\partial\Omega} \big(\sigma(v,p)n\big) \cdot v \, d\gamma = 2 \int_\Omega |\mathbb{D}(v)|^2 \, dx + \frac{1}{2} \int_{\partial\Omega} (v_* + v_*^{\mathcal{C}}) \cdot n |V + v_* + v_*^{\mathcal{C}}|^2 \, d\gamma.$$

• The solution of the state equations is unique (up to constants for the pressure) within the above class of functions. The pressure is singled out under the additional condition $p \in L^2(\Omega)$.

# The cost functional

$$\mathcal{J}(v_*, v, p) = \int_{\partial\Omega} v \cdot \sigma(v,p)n \; d\gamma = \int_{\partial\Omega} (v_* + V) \cdot \sigma(v,p)n \; d\gamma$$

The infimum will be taken over the set of all possible states $(v, p)$ satisfying the direct problem for $v_*$ either in

$$\mathcal{V}_\tau := \left\{ v_* \in W^{3/2,2}(\partial\Omega) \; ; \; v_* \cdot n = 0 \text{ on } \partial\Omega \right\}$$

or in

$$\mathcal{V}_\Gamma := \left\{ v_* \in W^{3/2,2}(\partial\Omega) \; ; \; v_* = 0 \quad \text{on } \partial\Omega \setminus \Gamma \right\},$$

where $\Gamma$ is a nonempty open subset of $\partial\Omega$.

# The state system and the cost functional

Using the energy equation satisfied by the state equations, we can rewrite the drag functional in the following way:

$$\mathcal{J}(v_*, v, p) = 2\|D(v)\|_{2,\Omega}^2 + \frac{1}{2}\int_{\partial\Omega}(v_* + v_*^{\mathcal{C}}) \cdot n|V + v_* + v_*^{\mathcal{C}}|^2 \, d\gamma.$$

where $v_*^{\mathcal{C}}$ are correctors of the control $v_*$ which ensure that the self-propelling conditions are satisfied

$$v_*^{\mathcal{C}} = \sum_{i=1}^{3}\left(\alpha_i \chi g^{(i)} + \beta_i \chi G^{(i)}\right) \in \mathcal{C} \text{ (localized controls)}$$

# Existence of optimal controls

The rigid body velocity $V(x) = \xi + \omega \times x$ (target velocity) is given and our aim is to minimize the drag using boundary control:

$$\min \left( 2\|D(v)\|_{2,\Omega}^2 + \frac{1}{2} \int_{\partial\Omega} (v_* + v_*^{\mathcal{C}}) \cdot n |V + v_* + v_*^{\mathcal{C}}|^2 \, d\gamma \right),$$

$$\|v_*\|_{3/2,2,\partial\Omega} \leq \kappa$$

**Theorem**

Assume that $\xi, \omega \in \mathbb{R}^3$ satisfy $|\xi| \leq c_0$, $|\omega| \leq c_0$ and $\kappa \in (0, c_0]$. Then each of the optimal control problems, with tangential controls and localized controls, admits a solution.

• Characterization of the optimal controls: adjoint state, first order optimality condition;

• In practice, to obtain solutions by an iterative method, the following strategy of resolution can be used:

→ Guess for controls, solve for states
→ Solve for adjoint state
→ Update controls
→ Repeat forward and backwards until convergence.

# The Lagrangian. Tangential case

We will introduce the Lagrangian associated with the problem, deduce the adjoint system and obtain a characterization of the optimal controls.

Let

$$\mathcal{Y} := \left\{ v \in W^{2,2}(\Omega) \cap L^{\infty}_{1,\varpi}(\Omega) \ ; \ \nabla \cdot v = 0 \text{ in } \Omega \right\},$$

$$\begin{aligned} \mathcal{U} := \Big\{ u \in L^6(\Omega) \cap D^{1,2}(\Omega) \cap D^{2,2}(\Omega) \ ; \\ V \cdot \nabla u - \omega \times u \in L^2(\Omega), \ \nabla \cdot u = 0 \text{ in } \Omega, \\ \exists \ell_u, k_u \in \mathbb{R}^3 \ u = \ell_u + k_u \times x \quad \text{on } \partial\Omega \Big\}, \end{aligned}$$

$$\mathcal{Z} := L^2(\partial\Omega).$$

# The Lagrangian. Tangential case

$$
\begin{aligned}
& \mathcal{L}_\tau(v, v_*^{\mathcal{C}}, v_*, u, \zeta) \\
:= \; & \int_\Omega |\mathbb{D}(v)|^2 \; dx - 2\int_\Omega \mathbb{D}(v) : \mathbb{D}(u) \; dx + \int_\Omega (v \cdot \nabla u) \cdot v \; dx \\
& - \int_\Omega (V \cdot \nabla u - \omega \times u) \cdot v \; dx + m(\xi \times \omega) \cdot \ell_u + ((I\omega) \times \omega) \cdot k_u \\
& - \langle v - V - v_* - v_*^{\mathcal{C}}, \zeta \rangle_{\partial\Omega}
\end{aligned}
$$

for $(v, v_*^{\mathcal{C}}, v_*, u, \zeta) \in \mathcal{Y} \times \mathcal{C}_\tau \times \mathcal{V}_\tau \times \mathcal{U} \times \mathcal{Z}$.

The adjoint system is obtained from the equations

$$D_v \mathcal{L}_\tau(\widehat{v}, \widehat{v}_*^{\mathcal{C}}, \widehat{v}_*, \widehat{u}, \widehat{\zeta}) v = 0 \quad \forall v \in \mathcal{Y},$$

$$D_{v_*^{\mathcal{C}}} \mathcal{L}_\tau(\widehat{v}, \widehat{v}_*^{\mathcal{C}}, \widehat{v}_*, \widehat{u}, \widehat{\zeta}) v_*^{\mathcal{C}} = 0 \quad \forall v_*^{\mathcal{C}} \in \mathcal{C}_\tau.$$

# The adjoint problem

Given a state $(\widehat{v}, \widehat{p})$, find $(\widehat{u}, \widehat{q}, \ell_{\widehat{u}}, k_{\widehat{u}})$

$$-\nabla \cdot \sigma(\widehat{u} - \widehat{v}, \widehat{q} - \widehat{p}) - \widehat{v} \cdot \nabla \widehat{u} - (\nabla \widehat{u})^\top \widehat{v} + V \cdot \nabla \widehat{u} - \omega \times \widehat{u} = 0 \quad \text{in } \Omega$$
$$\nabla \cdot \widehat{u} = 0 \quad \text{in } \Omega$$
$$\widehat{u} = \ell_{\widehat{u}} + k_{\widehat{u}} \times x \quad \text{on } \partial\Omega$$
$$\lim_{|x| \to \infty} \widehat{u}(x) = 0$$
$$\langle v_*^{\mathcal{C}}, \sigma(\widehat{v} - \widehat{u}, \widehat{p} - \widehat{q}) n \rangle_{\partial\Omega} = 0, \quad \forall v_*^{\mathcal{C}} \in \mathcal{C}_\tau.$$

# Optimality condition

$$\Lambda : v_* \mapsto (v, v_*^{\mathcal{C}})$$

$$\mathcal{J}(v_*) = 2\,\mathcal{L}(\Lambda(v_*), v_*, u, \zeta), \quad ((u, \zeta) \in \mathcal{U} \times \mathcal{Z}).$$

Take, in particular, the solution $\widehat{u} \in \mathcal{U}$ to the adjoint system together with $\widehat{\zeta} \in \mathcal{Z}$:

$$D_{v_*}\mathcal{J}(\widehat{v}_*)v_* = 2D_{(v,v_*^{\mathcal{C}})}\mathcal{L}(\Lambda(\widehat{v}_*), \widehat{v}_*, \widehat{u}, \widehat{\zeta})D_{v_*}\Lambda(\widehat{v}_*)v_* + 2D_{v_*}\mathcal{L}(\Lambda(\widehat{v}_*), \widehat{v}_*, \widehat{u}, \widehat{\zeta})v_*,$$

where we need to use

$$\|\widehat{z}_h - \widehat{z}\|_{2,2,\Omega} + \lceil \widehat{z}_h - \widehat{z} \rceil_{1,\varpi,\Omega} + \|\widehat{z}_{h*}^{\mathcal{C}} - \widehat{z}_*^{\mathcal{C}}\|_{3/2,2,\partial\Omega} \to 0$$

for $(\widehat{z}_h, \widehat{z}_{h*}^{\mathcal{C}}) := D_{v_*}\Lambda(\widehat{v}_*)v_* + o(h)/h$. Then

$$\frac{1}{2}D_{v_*}\mathcal{J}(\widehat{v}_*)(v_* - \widehat{v}_*) = D_{v_*}\mathcal{L}(\Lambda(\widehat{v}_*), \widehat{v}_*, \widehat{u}, \widehat{\zeta})(v_* - \widehat{v}_*) \geq 0, \ \forall v_* \in \mathcal{V}_\tau^\kappa,$$

# Optimality condition

**Theorem**

Let $\Omega$ be of class $C^3$. Suppose that $\xi, \omega \in \mathbb{R}^3$ and $v_* \in \mathcal{V}_\Gamma$ (resp. $\mathcal{V}_\tau$) satisfy

$$|\xi| \leq \kappa_0, \qquad |\omega| \leq \kappa_0, \qquad \|v_*\|_{3/2,2,\partial\Omega} \leq \kappa_0.$$

Let $\widehat{v}_*$ be a solution of the optimal control problem, $(\widehat{v}, \widehat{v}_*^{\mathcal{C}}, \widehat{p})$ the corresponding state and $(\widehat{u}, \widehat{q})$ the solution of the adjoint system. Then we have

$$\int_{\partial\Omega} \big(\sigma(\widehat{v} - \widehat{u}, \widehat{p} - \widehat{q})n\big) \cdot (v_* - \widehat{v}_*)\, d\gamma \geq 0, \quad \forall v_* \in \mathcal{V}_\tau^{\kappa_0},$$

in the case of tangential controls.

# Optimality condition

Case of localized controls:

$$\int_{\partial\Omega} \big(\sigma(\widehat{v}-\widehat{u}, \widehat{p}-\widehat{q})n\big)\cdot(v_* - \widehat{v}_*)\, d\gamma + \frac{1}{4}\int_{\partial\Omega}(v_* - \widehat{v}_*)\cdot n|V + \widehat{v}_* + \widehat{v}_*^{\mathcal{C}}|^2\, d\gamma$$
$$+ \frac{1}{2}\int_{\partial\Omega}(\widehat{v}_* + \widehat{v}_*^{\mathcal{C}})\cdot n(V + \widehat{v}_* + \widehat{v}_*^{\mathcal{C}})\cdot(v_* - \widehat{v}_*)\, d\gamma$$
$$+ \int_{\partial\Omega} ((v_* - \widehat{v}_*)\cdot n)\,(\omega \times x)\cdot \widehat{u}\; d\gamma \geq 0, \quad \forall v_* \in \mathcal{V}_\Gamma^{\kappa_0}$$

Thank you!