Controlling elliptic PDE, and applications to inverse problems

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Outline

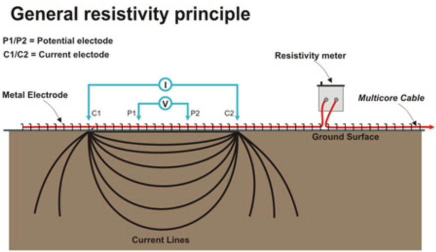
1. Calderón problem

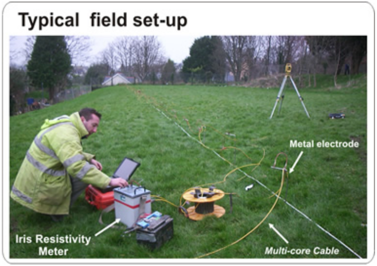
2. Controlling solutions

3. Fractional equations

Calderón problem

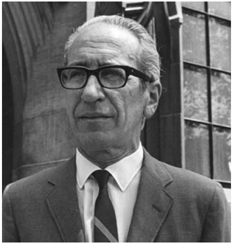
Electrical Resistivity Imaging in geophysics (1920's) [image: TerraDat]





A.P. Calderón (1980):

- mathematical formulation
- solution of the linearized problem
- exponential solutions



Calderón problem

Conductivity equation

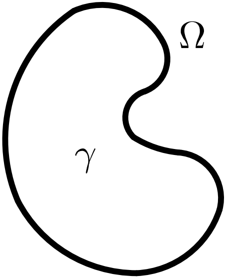
$$\begin{cases} \mathrm{div}(\gamma(x)\nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $\gamma \in L^\infty(\Omega)$ positive scalar function (electrical conductivity).

Boundary measurements given by the *Dirichlet-to-Neumann map*[1]

$$\Lambda_\gamma : f \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega}.$$

Inverse problem: given Λ_γ, determine γ.



[1] as a map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$

Calderón problem

Model case of inverse boundary problems for elliptic equations (Schrödinger, Maxwell, elasticity, Navier-Stokes).

Related to:

- optical / acoustic / hybrid imaging
- inverse scattering
- geometric problems (boundary rigidity)
- invisibility cloaking

Calderón problem

Uniqueness results:

$n \geq 3$	$\gamma \in C^2$	Sylvester-Uhlmann 1987
	$\gamma \in W^{1,\infty}$	Haberman-Tataru, Caro-Rogers 2016
	$\gamma \in W^{1,n}$	Haberman 2016, n=3,4
$n = 2$	$\gamma \in C^2$	Nachman 1996
	$\gamma \in L^\infty$	Astala-Päivärinta 2006

Connections to *Carleman estimates* and *unique continuation* (u vanishes in a ball $\implies u \equiv 0$).

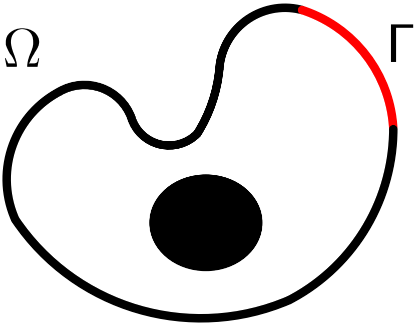
Calderón problem

Techniques: solutions $u \approx e^{\rho \cdot x}$, $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, and

$n \geq 3$	$\gamma \in C^2$	L^2 Carleman estimates
	$\gamma \in W^{1,\infty}$	Bourgain spaces / averaging
	$\gamma \in W^{1,n}$	L^p Carleman estimates, n=3,4
$n = 2$	$\gamma \in C^2$	$\overline{\partial}$-scattering theory
	$\gamma \in L^\infty$	quasiconformal methods

Local data problem

Prescribe voltages on Γ, measure currents on Γ:



Measure $\Lambda_\gamma f|_\Gamma$ for any f with $\mathrm{supp}(f) \subset \Gamma$.
Partial results if $n \geq 3$ [Kenig-S 2013, Kenig et al 2007, Isakov 2007].

Open questions

1. (Local data, $n \geq 3$) If $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \partial\Omega$, solve the Calderón problem with measurements on Γ.

2. (Anisotropic problem, $n \geq 3$) Determine a C^∞ matrix A up to gauge from measurements for $\mathrm{div}(A(x)\nabla u) = 0$.

3. (Counterexamples, $n \geq 3$) Can one find $\gamma_1, \gamma_2 \in C^\alpha$ with $0 < \alpha < 1$ so that

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \quad \text{but} \quad \gamma_1 \neq \gamma_2?$$

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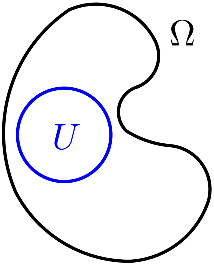
Runge approximation

Runge's theorem (for $\overline{\partial} u = 0$):

analytic functions in simply connected $U \subset \mathbb{C}$ can be approximated by complex polynomials.

General Runge property (for $Pu = 0$):

any solution in U, where $U \subset \Omega \subset \mathbb{R}^n$, can be approximated using solutions in Ω.



Valid e.g. for $Pu := \mathrm{div}(\gamma \nabla u)$ if $\gamma \in W^{1,\infty}(\Omega)$ is positive.
Reduces by duality to *unique continuation* [Lax/Malgrange 1956].

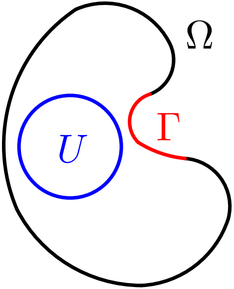
Approximate controllability

Theorem

Let $U \subset \Omega \subset \mathbb{R}^n$ be Lipschitz with $\Omega \setminus \overline{U}$ connected, and $\Gamma \subset \partial\Omega$ open. Given a solution[2] $v \in H^1(U)$ *and* $\varepsilon > 0$, *there is* $f \in C_c^\infty(\Gamma)$ *so that*

$$\|u_\varepsilon - v\|_{L^2(U)} \leq \varepsilon$$

where $Pu_\varepsilon = 0$ *in* Ω *with* $u_\varepsilon|_{\partial\Omega} = f$.



One can think of f as a *boundary control*, which makes u_ε approximate the profile v within U.

[2] Only solutions in U can be approximated!

Application: localized potentials

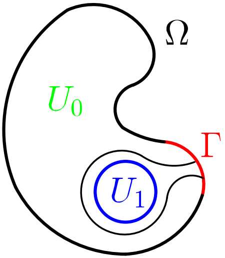
Theorem

If $U_0, U_1 \subset \Omega$ are open sets so that

$\overline{U}_0 \cap \overline{U}_1 = \emptyset, \quad \Omega \setminus (\overline{U}_0 \cup \overline{U}_1)$ connected and meets Γ,

then $\exists\ u_j \in H^1(\Omega)$, $Pu_j = 0$, with

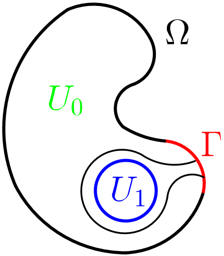
$$u_j|_{U_0} \approx 0, \quad u_j|_{U_1} \approx j,$$
$$\mathrm{supp}(u_j|_{\partial\Omega}) \subset \Gamma.$$



Proof. Apply Runge approximation to piecewise constant solutions $w_j \in H^1(U_0 \cup U_1)$ with $w_j|_{U_0} = 0$, $w_j|_{U_1} = j$. □

Runge approximation

Produce solutions with $u|_{U_0} \approx 0$ and $u|_{U_1} \gg 1$ (region of interest), but with *very little control outside $U_0 \cup U_1$*. Useful in the Calderón problem for



- piecewise analytic conductivities [Kohn-Vogelius 1985]
- local data if γ is known near $\partial\Omega$ [Ammari-Uhlmann 2004]
- detecting shapes of obstacles (γ known near $\partial\Omega$), e.g.
  - singular solutions [Isakov 1988]
  - probe method [Ikehata 1998]
  - monotonicity tests [Harrach 2008, ..., Harrach-Pohjola-S 2019]

Runge approximation

Four recent applications in inverse problems:

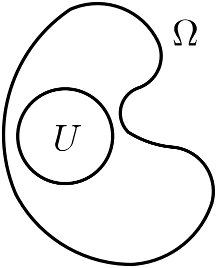
1. Quantitative Runge approximation
2. Anisotropic Calderón problem / Poisson embedding
3. Monotonicity methods for non-positive equations
4. Inverse problems for fractional equations

1. Quantitative Runge approximation

Given a solution $v \in H^1(U)$, find a solution $u_\varepsilon \in H^1(\Omega)$ satisfying

$$\|u_\varepsilon|_U - v\|_{L^2(U)} \leq \varepsilon$$

such that $\|u_\varepsilon\|_{H^1(\Omega)}$ is as small as possible (cost of approximation)?



Theorem (Rüland-S 2018) The cost of approximation satisfies

$$\|u_\varepsilon\|_{H^1(\Omega)} \lesssim \begin{cases} 1 & \text{if } v \text{ is a solution in } \Omega, \\ \varepsilon^{-\mu} & \text{if } v \text{ is a solution near } \overline{U}, \\ e^{C\varepsilon^{-\mu}} & \text{if } v \text{ is a general solution in } U. \end{cases}$$

Proved using duality and quantitative unique continuation.
Optimality: [Koch-Rüland-S 2021].

2. Anisotropic Calderón problem

In [Lassas-Liimatainen-S 2019], new proof of [Lassas-Uhlmann 2002]: if (M, g) is a compact Riemannian manifold with boundary, recover a real-analytic metric g up to isometry from DN map for Δ_g via Poisson embedding:

$$\text{points of } M^{\mathrm{int}} \leftrightsquigarrow \text{Poisson kernels in } C^\infty(\partial M)$$

Proof is heavily based on Runge approximation:

- harmonic functions separate points
- can prescribe Taylor expansions of harmonic functions

3. Monotonicity methods for Helmholtz

Shape detection for positive (e.g. conductivity) equations, based on monotonicity inequality $\sigma_0 \leq \sigma_1 \implies \Lambda_{\sigma_0} \geq \Lambda_{\sigma_1}$ [Tamburrino-Rubinacci 2002, Harrach 2008, ...]. Non-positive case:

Theorem (Harrach-Pohjola-S 2019)

If Λ_q is ND map for Helmholtz equation $(\Delta + k^2 q)u = 0$ in Ω, then

$$q_1 \leq q_2 \implies \Lambda_{q_1} \leq_{\text{fin}} \Lambda_{q_2}$$

meaning that $\Lambda_{q_2} - \Lambda_{q_1}$ has finitely many negative eigenvalues.

- works for imaging problems with positive frequency
- uses Runge approximation with constraints on f

Complex geometrical optics

Runge type results use that *γ is known near $\partial\Omega$*, or employ *real-analyticity*. They do not recover

- conductivities in $C^\infty(\overline{\Omega})$, which may oscillate near $\partial\Omega$
- inclusions inside inclusions (cf. [Greenleaf et al 2017]).

Complex geometrical optics solutions [Sylvester-Uhlmann 1987]

$$u = e^{\rho \cdot x}(1 + r), \qquad \rho \in \mathbb{C}^n, \qquad \rho \cdot \rho = 0$$

where $\|r\|_{L^2(\Omega)} \to 0$ as $|\rho| \to \infty$.

These are small in $\{\mathrm{Re}(\rho) \cdot x < 0\}$, large in $\{\mathrm{Re}(\rho) \cdot x > 0\}$, and oscillate in the direction of $\mathrm{Im}(\rho)$. Unlike in Runge approximation, solutions are *controlled in all of Ω*. They yield the Fourier transform of the unknown coefficient.

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1. Calderón problem
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3. Fractional equations

Fractional Laplacian

We will study an inverse problem for the *fractional Laplacian*

$$(-\Delta)^s, \quad 0 < s < 1,$$

defined via the Fourier transform by

$$(-\Delta)^s u = \mathscr{F}^{-1}\{|\xi|^{2s}\hat{u}(\xi)\}.$$

This operator is *nonlocal*: it does not preserve supports, and computing $(-\Delta)^s u(x)$ involves values of u far away from x.

Fractional Laplacian

Different models for diffusion:

$\partial_t u - \Delta u = 0$	normal diffusion/BM
$\partial_t u + (-\Delta)^s u = 0$	superdiffusion/Lévy flight
$\partial_t^\alpha u - \Delta u = 0$	subdiffusion/CTRW

The *fractional Laplacian* is related to

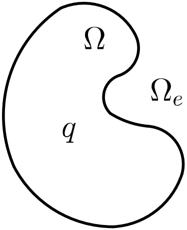
- anomalous diffusion involving long range interactions (turbulent media, population dynamics, elasticity)
- Lévy processes in probability theory and finance

Many results for time-fractional inverse problems [Jin-Rundell, survey 2015].

Fractional Laplacian

Let $\Omega \subset \mathbb{R}^n$ bounded, $q \in L^\infty(\Omega)$. Since $(-\Delta)^s$ is nonlocal, the Dirichlet problem becomes

$$\begin{cases} ((-\Delta)^s + q)u = 0 & \text{in } \Omega, \\ u = f & \text{in } \Omega_e \end{cases}$$



where $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ is the *exterior domain*.

Given $f \in H^s(\Omega_e)$, look for a solution $u \in H^s(\mathbb{R}^n)$. DN map

$$\Lambda_q : H^s(\Omega_e) \to H^{-s}(\Omega_e), \quad \Lambda_q f = (-\Delta)^s u|_{\Omega_e}.^1$$

Inverse problem: given Λ_q, determine q.

---

[1] the work required to maintain Dirichlet data f in Ω_e

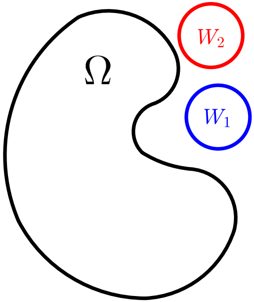
Main result

Theorem (Ghosh-S-Uhlmann 2020)

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $0 < s < 1$, and let $q_1, q_2 \in L^\infty(\Omega)$. If $W_j \subset \Omega_e$ are open sets, and if

$$\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}, \quad f \in C_c^\infty(W_1),$$

then $q_1 = q_2$ in Ω.



Main features:

- local data for *arbitrary* $W_j \subset \Omega_e$
- same method works for *all* $n \geq 1$
- new mechanism for (nonlocal) inverse problems
- works with single measurement [Ghosh-Rüland-S-Uhlmann 2020]

Extensions

Low regularity: potentials in $L^{\frac{n}{2s}}(\Omega)$, or roughly in $W^{-s,\frac{n}{s}}(\Omega)$, are uniquely determined [Rüland-S 2017].

Anisotropic case: the DN map for $((-\nabla \cdot A(x)\nabla)^s + q)u = 0$ determines q uniquely, if $A(x)$ is a known C^∞ positive matrix [Ghosh-Lin-Xiao 2017].

And many others...

Main tools 1: uniqueness

The fractional equation has strong uniqueness properties:

Theorem

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if both u and $(-\Delta)^s u$ vanish in some open set, then $u \equiv 0$.

Essentially due to [M. Riesz 1938], also have strong unique continuation results [Fall-Felli 2014, Rüland 2015].

Such a result could never hold for the Laplacian:
if $u \in C_c^\infty(\mathbb{R}^n)$, then both u and Δu vanish in a large set.

Main tools 1: uniqueness

Theorem

If $u \in H^{-r}(\mathbb{R}^n)$ for some $r \in \mathbb{R}$, and if $u|_W = (-\Delta)^s u|_W = 0$ for some open set $W \subset \mathbb{R}^n$, then $u \equiv 0$.

Proof (sketch).

If u is nice enough, then

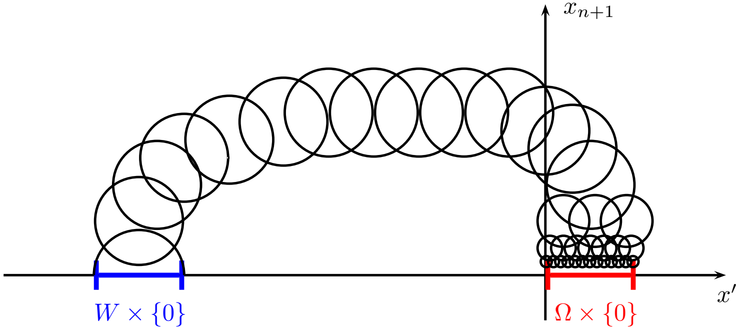
$$(-\Delta)^s u \sim \lim_{y \to 0} y^{1-2s} \partial_y w(\,\cdot\,, y)$$

where $w(x, y)$ is the *Caffarelli-Silvestre extension* of u:

$$\begin{cases} \mathrm{div}_{x,y}(y^{1-2s} \nabla_{x,y} w) = 0 & \text{in } \mathbb{R}^n \times \{y > 0\}, \\ w|_{y=0} = u. & \end{cases}$$

Thus $(-\Delta)^s u$ is obtained from a *local equation*, which is degenerate elliptic with A_2 weight y^{1-2s}. Carleman estimates [Rüland 2015] and $u|_W = (-\Delta)^s u|_W = 0$ imply uniqueness.

Main tools 1: propagation of smallness



1. $(w, \partial_y w)$ small on $W \times \{0\} \implies w$ small in $W \times (0, 1)$
   (boundary Carleman / interpolation inequality)
2. w small in $W \times (0, 1) \implies w$ small in $\Omega \times (h, 1)$
   (three balls inequality + chain of balls argument)
3. w small in $\Omega \times (h, 1) \implies w$ small in $\Omega \times \{0\}$
   (Sobolev + trace estimates, optimize h with $\sim |\log(h)|$ balls)

Main tools 2: approximation

Solutions of $\Delta u = 0$ (harmonic functions) in $\Omega \subset \mathbb{R}^n$ are *rigid*:

- if $n = 1$, then $u'' = 0 \implies u(x) = ax + b$
- u has no interior minima or maxima (*maximum principle*)
- if $u|_B = 0$ in $B \subset \Omega$, then $u \equiv 0$ (*unique continuation*)

Moreover, if $u_j \to f$ in $L^2(\Omega)$ where $\Delta u_j = 0$, then also $\Delta f = 0$ (*harmonic functions can only approximate harmonic functions*).

In contrast, solutions of $(-\Delta)^s u = 0$ turn out to be *flexible*.

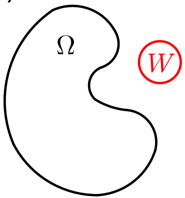
Main tools 2: approximation

Theorem (Ghosh-S-Uhlmann 2020)
Any $f \in L^2(\Omega)$ can be approximated in $L^2(\Omega)$ by *solutions $u|_\Omega$*, where

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \qquad \operatorname{supp}(u) \subset \overline{\Omega} \cup \overline{W}.$$

If everything is C^∞, can approximate in $C^\infty(\overline{\Omega})$.[1]

Earlier [Dipierro-Savin-Valdinoci 2017]: C^k approximation by solutions of $(-\Delta)^s u = 0$ in B_1, but with no control over $\operatorname{supp}(u)$.



[1] with special behaviour near $\partial\Omega$

Main tools 2: approximation

The approximation property is also valid e.g. for

- $\partial_t u + (-\Delta)^s u = 0$
- $\partial_t^2 u + (-\Delta)^s u = 0$
- $m(D_x)u + (-\Delta_y)^s u = 0$ where $m(D_x)$ is a Fourier multiplier

[Dipierro-Savin-Valdinoci 2017, Rüland-S 2017]

Control theory results [Warma et al...].

Main tools 2: approximation

The approximation property follows by duality from the uniqueness result.

This uses Fredholm properties of the solution operator for

$$\begin{cases} ((-\Delta)^s + q)u = F & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e, \end{cases}$$

mapping $F \in H^{\alpha-2s}(\Omega)$ to u in the *special space $H^{s(\alpha)}(\overline{\Omega})$*, adapted to the fractional Dirichlet problem, for $\alpha > 1/2$ [s-transmission property, Hörmander 1965, Grubb 2015]. One has

$$H^{\alpha}_{\mathrm{comp}}(\Omega) \subset H^{s(\alpha)}(\overline{\Omega}) \subset H^{\alpha}_{\mathrm{loc}}(\Omega)$$

but solutions in $H^{s(\alpha)}(\overline{\Omega})$ may have singularities near $\partial\Omega$.

Summary

1. The *Runge property* for second order PDE allows one to approximate solutions in $U \subset \Omega$ using solutions in Ω.
2. Runge approximation for the Calderón problem works in special cases. In general, need complex geometric optics.
3. The fractional operator $(-\Delta)^s$, $0 < s < 1$, is *nonlocal*. The DN map takes exterior Dirichlet values $u|_{\Omega_e}$ to exterior Neumann values $(-\Delta)^s u|_{\Omega_e}$.
4. Fractional equations may have *strong uniqueness and approximation properties*, replacing complex geometric optics and leading to strong results in inverse problems.