Boundary controllability of a one-dimensional phase-field system with one control force

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General objective:

Controllability properties of a one-dimensional phase-field system of Caginalp type with one control force.

Phase-field system: it is a model describing the transition between the solid and liquid phases in solidification/melting processes of a material occupying a domain.

Contents

1. Introduction. Statement of the problem

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Fix $T > 0$. Notation: $Q_T := (0, \pi) \times (0, T)$

$$(1) \qquad \begin{cases} \tilde{\theta}_t - \xi \tilde{\theta}_{xx} + \dfrac{1}{2}\rho\xi\tilde{\phi}_{xx} + \dfrac{\rho}{\tau}\tilde{\theta} = f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\phi}_t - \xi\tilde{\phi}_{xx} - \dfrac{2}{\tau}\tilde{\theta} = -\dfrac{2}{\rho}f(\tilde{\phi}) & \text{in } Q_T, \\ \tilde{\theta}(0,\cdot) = v,\ \tilde{\phi}(0,\cdot) = c,\ \tilde{\theta}(\pi,\cdot) = 0,\ \tilde{\phi}(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde{\theta}(\cdot,0) = \tilde{\theta}_0,\ \tilde{\phi}(\cdot,0) = \tilde{\phi}_0 & \text{in } (0,\pi). \end{cases}$$

$\tilde{\theta} = \tilde{\theta}(x,t)$: the temperature of the material;

$\tilde{\phi} = \tilde{\phi}(x,t)$: phase-field function used to identify the solidification level of the material; $c \in \{-1, 0, 1\}$;

f: nonlinear term: $f(\tilde{\phi}) = -\dfrac{\rho}{4\tau}\left(\tilde{\phi} - \tilde{\phi}^3\right)$.

$\rho > 0, \tau > 0, \xi > 0$: latent heat, relaxation time; thermal diffusivity.

$v \in L^2(0,T)$: control. $\tilde{\theta}_0, \tilde{\phi}_0$: initial data.

1. Introduction. Statement of the problem

The phase function $\tilde{\phi}$ describes the phase transition of the material (solid or liquid): $\boxed{\tilde{\phi} = 1}$ solid state of the material; $\boxed{\tilde{\phi} = -1}$ liquid state.

G. CAGINALP, *An analysis of a phase field model of a free boundary*, Arch. Rational Mech. Anal. **92** (1986), no. 3, 205–245.

1. Introduction. Statement of the problem

Objective

Null controllability of system (1) at time $T > 0$: for any $\tilde{\theta}_0$ there exists a control $v \in L^2(0, T)$ such that system (1) has a solution $(\tilde{\theta}, \tilde{\phi})$ satisfying

$$\tilde{\theta}(\cdot, T) = 0 \quad \text{in} \quad \Omega.$$

Remark

- The temperature $\tilde{\theta}$ of the material could be zero with the material in solid ($\tilde{\phi} = 1$) or liquid phase ($\tilde{\phi} = -1$). Thus, null controllability of the temperature with $\tilde{\phi}(\cdot, T) = 1$ or $\tilde{\phi}(\cdot, T) = -1$ in $(0, \pi)$.
- The phase-field variable $\tilde{\phi}$ does not have a direct physical meaning. This is the reason why we control the temperature $\tilde{\theta}$ which, in fact, is the unique variable with physical meaning.

Remark

Three main difficulties:

1. Only **one control force** $v \in L^2(0,T)$ and two variables to be controlled $(\tilde{\theta}, \tilde{\phi})$ ($\tilde{\phi}$ is indirectly controlled).
2. **Boundary control**: the control v is exerted at point $x = 0$ by means of the boundary Dirichlet condition for the temperature $\tilde{\theta}$.
3. **Non-linear problem**: Only local controllability results (positive controllability result when the initial datum $(\tilde{\theta}_0, \tilde{\phi}_0)$ is near to the desired final state $(0, c)$).

1. Introduction. Statement of the problem

$$(1)\quad \begin{cases} \tilde\theta_t - \boxed{\xi}\,\tilde\theta_{xx} + \frac{1}{2}\rho\xi\tilde\phi_{xx} + \frac{\rho}{\tau}\tilde\theta = f(\tilde\phi) & \text{in } Q_T, \\ \tilde\phi_t - \boxed{\xi}\,\tilde\phi_{xx} - \frac{2}{\tau}\tilde\theta = -\frac{2}{\rho}f(\tilde\phi) & \text{in } Q_T, \\ \tilde\theta(0,\cdot) = v,\ \tilde\phi(0,\cdot) = c,\ \tilde\theta(\pi,\cdot) = 0,\ \tilde\phi(\pi,\cdot) = c & \text{on } (0,T), \\ \tilde\theta(\cdot,0) = \tilde\theta_0,\ \tilde\phi(\cdot,0) = \tilde\phi_0 & \text{in } (0,\pi). \end{cases}$$

Performing the change of variable $\boxed{(\theta,\phi) = (\tilde\theta, \tilde\phi - c)}$, system (1) becomes

$$(2)\quad \begin{cases} \theta_t - \xi\theta_{xx} + \frac{1}{2}\rho\xi\phi_{xx} - \frac{\rho}{2\tau}\phi + \frac{\rho}{\tau}\theta = g(\phi) & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \frac{1}{\tau}\phi - \frac{2}{\tau}\theta = -\frac{2}{\rho}g(\phi) & \text{in } Q_T, \\ \theta(0,\cdot) = v, \quad \phi(0,\cdot) = \theta(\pi,\cdot) = \phi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,0) = \theta_0, \quad \phi(\cdot,0) = \phi_0 & \text{in } (0,\pi), \end{cases}$$

where $(\theta_0,\phi_0) = (\tilde\theta_0, \tilde\phi_0 - c)$ and $\boxed{g(\phi) = \pm\frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3}$.

1. Introduction. Statement of the problem

System can be written:

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y_0 = (\theta_0, \phi_0)$, $y = (\theta, \phi)$, $F(y) = \begin{pmatrix} g(y_2) \\ -\frac{2}{\rho} g(y_2) \end{pmatrix}$, and

$$(3) \quad D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Objective

Local null controllability of system (2) at time $T > 0$: There exists $\varepsilon > 0$ s.t. for any y_0 with $\|y_0\| \leq \varepsilon$, there exists a control $v \in L^2(0,T)$ such that system (2) has a solution $y = (\theta, \phi)$ satisfying

$$\boxed{y(\cdot, T) = 0} \quad \text{in} \quad (0,\pi).$$

1. Introduction. Statement of the problem

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Strategy

1. Null controllability at time $T > 0$ of a homogenous linearized version of system (2) with an **explicit expression of the control cost** with respect to T.
2. Null controllability at time T of a non-homogenous linearized version of system (2).
3. A Fixed-Point argument will imply the local null controllability result at time T for system (2).

2. Controllability of the homogenous linear system

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Let us consider the system

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$,

$$(3)\qquad D = \begin{pmatrix} \xi & -\frac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \frac{\rho}{\tau} & -\frac{\rho}{2\tau} \\ -\frac{2}{\tau} & \frac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Proposition

Assume $y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi;\mathbb{R}^2)$ and $v \in L^2(0,T)$. Then, system (4) admits a unique solution $y = (\theta, \phi) \in L^2(Q_T;\mathbb{R}^2) \cap C^0([0,T];H^{-1})$ which depends continuously on the data.

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Theorem (Approximate controllability)

Fix $T > 0$. Then, system (4) is approximately controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time T if and only if

$$(5) \qquad \boxed{\xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0}, \quad \forall k, \ell \geq 1, \quad \ell > k.$$

$$(4)\qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Remark

- Condition

  $$(5)\quad \xi^2\tau^2(\ell^2-k^2)^2 - 2\xi\rho\tau(\ell^2+k^2) - 2\rho - 1 \neq 0, \quad \forall k,\ell \geq 1, \quad \ell > k,$$

  is equivalent to the property: *"The eigenvalues of the vectorial operators*

  $$(6)\qquad L = -D\partial_{xx} + A \quad \textit{and} \quad L^* = -D^*\partial_{xx} + A^*,$$

  have geometric multiplicity equal to one".
- Since NC $\Rightarrow$ AC, then (5) is a necessary condition for the null controllability of this system at time $T > 0$.
- Also, (5) only need to be checked a finite number of times.

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Null controllability with a bound of the control cost?? Yes, but we need a gap condition for the eigenvalues of the vectorial operator $\boxed{L = -D\partial_{xx} + A}$

Strategy

Apply the **moment method** to system (4). We will use:

- the eigenvalues of L and L^*: $\boxed{\lambda_k^{(1)}, \ \lambda_k^{(2)}}$;
- the eigenfunctions of L: $\boxed{\Psi_k^{(1)}, \ \Psi_k^{(2)}}$; (basis of $H^{-1}(0,\pi;\mathbb{R}^2)$).
- the eigenfunctions of L^*: $\boxed{\Phi_k^{(1)}, \ \Phi_k^{(2)}}$; (basis of $H_0^1(0,\pi;\mathbb{R}^2)$).

$$
(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

The condition $y(\cdot,T) = 0$ is equivalent to the existence of $v \in L^2(0,T)$

$$
\int_0^T B^* D^* \varphi_{k,x}^{(j)}(0,t) v(t)\, dt = -\left\langle y_0, \varphi_k^{(j)}(\cdot,0)\right\rangle, \forall k \geq 1,\ j = 1,2,
$$

where $\varphi_k^{(j)} = e^{-\lambda_k^{(j)}(T-t)}\Phi_k^{(j)}$ is the solution of the adjoint system to (4) associated to $\Phi_k^{(j)}$.

That is,

$$
(-1)^{j+1}\sqrt{\frac{2}{\pi}}\frac{k\xi}{\sqrt{\tau r_k}}\int_0^T e^{-\lambda_k^{(j)}t} v(T-t)\, dt = -e^{-\lambda_k^{(j)}T}\left\langle y_0, \Phi_k^{(j)}\right\rangle,
$$

$k \geq 1, j = 1,2.$

$$(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

The moment problem

Find $v \in L^2(0,T)$ such that

$$\int_0^T e^{-\lambda_k^{(j)} t} v(T-t)\,dt = c_k^{(j)}, \quad \forall k \geq 1,\ j = 1,2,$$

where $c_k^{(j)}$ is a bounded sequence.

Biorthogonal Families: ([FATTORINI, RUSSELL] Arch. Rat. Mech. Anal. (1971)).

Two ingredients:

1. **Biorthogonality:** $\sum_{k\geq 1}\left(\frac{1}{\lambda_k^{(1)}}+\frac{1}{\lambda_k^{(2)}}\right)<\infty \Longrightarrow \exists$ a family (not unique) $\left\{q_k^{(1)}, q_k^{(2)}\right\}_{k\geq 1} \subset L^2(0,T)$ s.t. $\int_0^T e^{-\lambda_k^{(i)}t} q_\ell^{(j)}\,dt = \delta_{k\ell}\delta_{ij}$, $\forall k,\ell \geq 1,\ i,j=1,2$

   Important: $\lambda_k^{(i)} \sim \xi k^2 + \frac{\rho+1}{2\tau} + O_i(k) > 0$. OK.

2. **Separability:** We can construct a biorthogonal family with a good bound for $\|q_k^{(i)}\|_{L^2(0,T)}$ for any $k\geq 1$, $i=1,2$.

The moment problem

Find $v \in L^2(0,T)$ such that

$$\int_0^T e^{-\lambda_k^{(j)} t} v(T-t)\,dt = c_k^{(j)}, \quad \forall k \geq 1,\ j = 1,2,$$

where c_{kj} is a bounded sequence.

Formal solution:

$$v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right)$$
$$= \sum_{k\geq 1} \left(e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle q_k^{(1)}(t) + e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle q_k^{(2)}(t) \right)$$

Is this series convergent in $L^2(0,T)$? It depends on $\boxed{\|q_k^{(i)}\|_{L^2(0,T)}}$

Bounds on Biorthogonal Families: ([Benabdallah, Boyer, G-B, Olive], SIAM J. Control Optim. **52** (2014), no. 5, 2970–3001).

Lemma (Separability and bounds)

Consider $\{\Lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ *s. t.* $\Lambda_k \neq \Lambda_n$*, for any* $k \neq n$*. Also assume that there exist an integer* $M \geq 1$ *and positive constants* p*,* δ *and* α *such that*

$$(7) \qquad \begin{cases} |\Lambda_k - \Lambda_n| \geq \delta \left|k^2 - n^2\right|, \quad \forall k,n \in \mathbb{N},\ |k-n| \geq M, \\ \inf\limits_{k\neq n,\ |k-n|<M} |\Lambda_k - \Lambda_n| > 0, \\ |p\sqrt{r} - \mathcal{N}(r)| \leq \alpha, \quad \forall r > 0, \end{cases}$$

($\mathcal{N}(r) := \#\{k : \Lambda_k \leq r\}$ *is the counting function associated to* $\{\Lambda_k\}_{k\geq 1}$*).*
Then, $\exists T_0 > 0$ *and* $C > 0$ *s. t., for any* $T \in (0, T_0)$*, we can find* $\{q_k\}_{k\geq 1} \subset L^2(0,T)$ *biorthogonal to* $\{e^{-\Lambda_k t}\}_{k\geq 1}$

$$\|q_k\|_{L^2(0,T)} \leq Ce^{C\sqrt{\Lambda_k} + \frac{C}{T}}, \quad \forall k \geq 1.$$

Without loss of generality, assume $\boxed{T \in (0, T_0)}$. We had (approximate controllability):

$$(5) \quad \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k.$$

Also assume:

$$(8) \qquad \boxed{\xi \neq \frac{1}{j^2}\frac{\rho}{\tau}, \quad \forall j \geq 1}.$$

Conclusion: we can apply the previous lemma to $\left\{\lambda_k^{(1)},\ \lambda_k^{(2)}\right\}_{k\geq 1}$: $\exists$ a **biorthogonal family** $\left\{q_k^{(1)}, q_k^{(2)}\right\}_{k\geq 1}$ in $L^2(0,T)$ to the exponentials s.t.

$$\|q_k^{(i)}\|_{L^2(0,T)} \leq Ce^{C\sqrt{\lambda_k^{(i)}}+\frac{C}{T}}, \quad \forall k \geq 1, i = 1, 2.$$

$C > 0$ is independent of k, i and T.

Recall

The moment problem

Find $v \in L^2(0,T)$ such that

$$\int_0^T e^{-\lambda_k^{(j)} t} v(T-t)\,dt = c_k^{(j)}, \quad \forall k \geq 1,\ j = 1,2,$$

where c_{kj} is a bounded sequence.

$$v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right)$$
$$= \sum_{k\geq 1} \left(e^{-\lambda_k^{(1)} T} c_k^{(1)} \left\langle y_0, \Phi_k^{(1)} \right\rangle q_k^{(1)}(t) + e^{-\lambda_k^{(2)} T} c_k^{(2)} \left\langle y_0, \Phi_k^{(2)} \right\rangle q_k^{(2)}(t) \right)$$

$$\boxed{\left| C_k^{(i)} \right| \leq C\, k\, e^{-\lambda_k^{(i)} T} \|y_0\|_{H^{-1}}}, \quad \forall k \geq 1, i = 1,2\ (C > 0 \text{ independent of } k, i, T).$$

The series of v converges absolutely in $L^2(0,T)$:

$$\begin{cases} v(T-t) = \sum_{k\geq 1} \left(C_k^{(1)} q_k^{(1)}(t) + C_k^{(2)} q_k^{(2)}(t) \right), \\ \left| C_k^{(i)} \right| \leq C\, k\, e^{-\lambda_k^{(i)} T} \|y_0\|_{H^{-1}}, \quad \forall k \geq 1, \quad i = 1, 2, \\ \|q_k^{(i)}\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_k^{(i)}} + \frac{C}{T}}, \quad \forall k \geq 1, \quad i = 1, 2. \end{cases}$$

and

$$\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}} \|y_0\|_{H^{-1}}$$

for positive constants C_0 and M independent of T.

Conclusion

We have solved the moment problem and we had proved the null controllability result for system (4).

$$
(4) \qquad \begin{cases} y_t - Dy_{xx} + Ay = 0 & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Theorem

Let us us fix $T > 0$ and consider ξ, ρ and τ, positive real numbers satisfying

$$
\begin{cases} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \dfrac{1}{j^2}\dfrac{\rho}{\tau}, \quad \forall j \geq 1. \end{cases}
$$

*Then, system (4) is null controllable in $H^{-1}(0,\pi;\mathbb{R}^2)$ at time $T > 0$. Moreover, there exist two constants $C_0, M > 0$, only depending on ξ, ρ and τ, s. t. the control v can be constructed satisfying (**control cost**)*

$$
\boxed{\|v\|_{L^2(0,T)} \leq C_0\, e^{\frac{M}{T}}\, \|y_0\|_{H^{-1}}}
$$

Remark

1. Condition

$$(5)\quad \xi^2\tau^2(\ell^2-k^2)^2-2\xi\rho\tau(\ell^2+k^2)-2\rho-1\neq 0,\quad \forall k,\ell\geq 1,\quad \ell>k.$$

is necessary for the approximate controllability of the system. Therefore, it is also necessary for the null controllability of it.

Remark

2. In the case in which $\boxed{\xi=\frac{1}{j^2}\frac{\rho}{\tau}}$, for some $j\geq 1$, the eigenvalues of L concentrate: $\boxed{\inf_{k,\ell\geq 1,k\neq\ell}|\Lambda_k-\Lambda_\ell|=0}$. In this case the controllability problem for system (4) has a minimal time $T_0\in[0,\infty]$ of null controllability.

3. Null controllability of the non-homogenous linear system

3. Null controllability of the non-homogenous linear system

Let us now consider the system

$$(9)\qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T := (0,\pi)\times(0,T), \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y = (\theta, \phi)$, f is a given function (heat source) and

Objective

The source term f must have an exponential decay when $t \to T$:

$$e^{\frac{C}{T-t}} f \in L^2(Q_T),$$

for an appropriate positive constant C.

We follow
[Liu, Takahashi, Tucsnak], ESAIM Control Optim. Calc. Var. **19** (2013), no. 1, 20–42.

3. Null controllability of the non-homogenous linear system

$$(9) \qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

From the previous expression of the **control cost** for the homogenous problem, we define the functions

$$\boxed{\rho_{\mathcal{F}}(t) := e^{\frac{b^2(a+1)M}{(b-1)(T-t)}}}, \quad \boxed{\rho_0(t) := e^{\frac{aM}{(b-1)(T-t)}}}, \quad \forall t \in \left[T\left(1-\frac{1}{b^2}\right), T\right],$$

extended to $\left[0, T(1-1/b^2)\right]$ in a constant way ($a, b > 1$ are constants that will be chosen later).

With the previous functions, we also introduce the weighted Banach spaces

$$\mathcal{F} := \left\{f \in L^2(Q_T;\mathbb{R}^2) : \rho_{\mathcal{F}} f \in L^2(Q_T;\mathbb{R}^2)\right\},$$
$$\mathcal{V} := \left\{v \in L^2(0,T) : \rho_0 v \in L^2(0,T)\right\},$$
$$\mathcal{Y}_0 := \left\{y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T;\mathbb{R}^2)\right\},$$
$$\mathcal{Y} := \left\{y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T) \times C^0(\overline{Q}_T)\right\}.$$

3. Null controllability of the non-homogenous linear system

Theorem

Consider ξ, ρ *and* τ, *three positive real numbers satisfying* (5) *and* (8), *and* $T > 0$. *Then, for any* $\boxed{(y_0, f) \in H^{-1}(0,\pi;\mathbb{R}^2) \times \mathcal{F}}$ *(resp.,* $\boxed{(y_0, f) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \times \mathcal{F}}$*), there exists* $v \in \mathcal{V}$ *(which depends linearly on the data) such that*

$$\|v\|_{\mathcal{V}} \leq C\, e^{C\left(T+\frac{1}{T}\right)}\, \|(y_0, f)\|_{(H^{-1})^2\times\mathcal{F}}$$

and the solution y *of* (9) *associated to* (y_0, f) *satisfies* $y \in \mathcal{Y}_0$ *(resp.,* $y \in \mathcal{Y}$*) and*

$$\|y\|_{\mathcal{Y}_0} \leq C\, e^{C\left(T+\frac{1}{T}\right)}\, \|(y_0, f)\|_{(H^{-1})^2\times\mathcal{F}},$$

(resp.,

$$\|y\|_{\mathcal{Y}} \leq C\, e^{C\left(T+\frac{1}{T}\right)}\, \|(y_0, f)\|_{H^{-1}\times H_0^1\times\mathcal{F}}),$$

for a positive constant C *independent of* T.

3. Null controllability of the non-homogenous linear system

$$
(9) \qquad \begin{cases} y_t - Dy_{xx} + Ay = f & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}
$$

Remark

Observe that the solution y associated to $v \in \mathcal{V}$ satisfies $y \in \mathcal{Y}_0$ (resp., $y \in \mathcal{Y}$), with $\boxed{\rho_0(t) := e^{\frac{aM}{(b-1)(T-t)}}}$ and

$$
\begin{aligned}
\mathcal{V} &:= \left\{ v \in L^2(0,T) : \rho_0 v \in L^2(0,T) \right\}, \\
\mathcal{Y}_0 &:= \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T;\mathbb{R}^2) \right\}, \\
\mathcal{Y} &:= \left\{ y \in L^2(Q_T;\mathbb{R}^2) : \rho_0 y \in L^2(Q_T) \times C^0(\overline{Q}_T) \right\}.
\end{aligned}
$$

So, $\boxed{y(\cdot,T) = 0}$ in $(0,\pi)$.

4. Local null controllability for the phase-field system

Objective

Null controllability of system (1) at time $T > 0$ (maintaining the material solid or liquid at this time): for any $(\tilde{\theta}_0, \tilde{\phi}_0)$ there exists a control $v \in L^2(0,T)$ such that system (1) has a solution $(\tilde{\theta}, \tilde{\phi})$ satisfying

$$\tilde{\theta}(\cdot, T) = 0, \quad \tilde{\phi}(\cdot, T) = c \quad \text{in} \quad \Omega.$$

We also had the change $\boxed{(\theta, \phi) = (\tilde{\theta}, \tilde{\phi} - c)}$ and system (1) became

$$(2) \quad \begin{cases} \theta_t - \xi\theta_{xx} + \frac{1}{2}\rho\xi\phi_{xx} - \frac{\rho}{2\tau}\phi + \frac{\rho}{\tau}\theta = g(\phi) & \text{in } Q_T, \\ \phi_t - \xi\phi_{xx} + \frac{1}{\tau}\phi - \frac{2}{\tau}\theta = -\frac{2}{\rho}g(\phi) & \text{in } Q_T, \\ \theta(0,\cdot) = v, \quad \phi(0,\cdot) = \theta(\pi,\cdot) = \phi(\pi,\cdot) = 0 & \text{on } (0,T), \\ \theta(\cdot,0) = \theta_0, \quad \phi(\cdot,0) = \phi_0 & \text{in } (0,\pi), \end{cases}$$

where $(\theta_0, \phi_0) = (\tilde{\theta}_0, \tilde{\phi}_0 - c)$ and $\boxed{g(\phi) = \pm\frac{3\rho}{4\tau}\phi^2 + \frac{\rho}{4\tau}\phi^3}$.

4. Local null controllability for the phase-field system

In vectorial form, the system can be written:

$$(2) \qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

where $y_0 = (\theta_0, \phi_0)$, $y = (\theta, \phi)$, $F(y) = \begin{pmatrix} g(y_2) \\ -\frac{2}{\rho} g(y_2) \end{pmatrix}$, and

$$(3) \qquad D = \begin{pmatrix} \xi & -\dfrac{1}{2}\rho\xi \\ 0 & \xi \end{pmatrix}, \quad A = \begin{pmatrix} \dfrac{\rho}{\tau} & -\dfrac{\rho}{2\tau} \\ -\dfrac{2}{\tau} & \dfrac{1}{\tau} \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Objective

Local null controllability of system (2) at time $T > 0$: There exists $\varepsilon > 0$ s.t. for any y_0 with $\|y_0\| \leq \varepsilon$, there exists a control $v \in L^2(0,T)$ such that system (2) has a solution $y = (\theta, \phi)$ satisfying

$$\boxed{y(\cdot,T) = 0} \quad \text{in} \quad (0,\pi).$$

4. Local null controllability for the phase-field system

$$(2)\qquad \begin{cases} y_t - Dy_{xx} + Ay = F(y)\ (= f) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

The idea is to apply a **fixed-point argument** to system (2). To this end, we take

$$y_0 = (\theta_0, \phi_0) \in H^{-1}(0,\pi) \times H_0^1(0,\pi) \text{ with } \boxed{\|y_0\|_{H^{-1}\times H_0^1} \le \varepsilon},$$

with $\varepsilon > 0$ to be determined.
Now, take $f \in \mathcal{F} = \left\{ f \in L^2(Q_T;\mathbb{R}^2) : \rho_{\mathcal{F}} f \in L^2(Q_T;\mathbb{R}^2) \right\}$ with

$$\boxed{f \in \overline{B}_{\mathcal{F},\varepsilon} = \{ f \in \mathcal{F} : \|f\|_{\mathcal{F}} \le \varepsilon \}}.$$

and we use the null controllability result for the non-homogenous linear system (Theorem 3.1) for (y_0, f): $\boxed{\exists v_f \in \mathcal{V}, y_f = (\theta_f, \phi_f) \in \mathcal{Y}}$ solutions of the non-homogenous linear system and

$$\|y_f\|_{\mathcal{Y}} + \|v_f\|_{\mathcal{V}} \le C\, e^{C\left(T+\frac{1}{T}\right)} \left(\|y_0\|_{H^{-1}\times H_0^1} + \|f\|_{\mathcal{F}} \right) \le C\, e^{C\left(T+\frac{1}{T}\right)}\, \varepsilon,$$

4. Local null controllability for the phase-field system

$$(2) \quad \begin{cases} y_t - Dy_{xx} + Ay = F(y) & \text{in } Q_T, \\ y(0,\cdot) = Bv, \quad y(\pi,\cdot) = 0 & \text{on } (0,T), \\ y(\cdot,0) = y_0 & \text{in } (0,\pi). \end{cases}$$

Fixed-point operator

$$\mathcal{N}(f) = F(y_f) = \begin{pmatrix} \pm\frac{3\rho}{4\tau}{\phi_f}^2 + \frac{\rho}{4\tau}{\phi_f}^3 \\ \mp\frac{3}{2\tau}{\phi_f}^2 - \frac{1}{2\tau}{\phi_f}^3 \end{pmatrix}.$$

It is clear that if f is a fixed-point of $\mathcal{N}$, i.e., if $f = \mathcal{N}(f)$, then $y_f \in \mathcal{Y}$ is a solution of system (2) and $y(\cdot,T) = 0$ in $(0,\pi)$.

It is possible to choose parameters $a, b \in (1,\infty)$ and $\varepsilon > 0$ (depending on T) such that:

1. $\mathcal{N}(\overline{B}_{\mathcal{F},\varepsilon}) \subseteq \overline{B}_{\mathcal{F},\varepsilon}$.
2. $\mathcal{N}$ is a contraction mapping

4. Local null controllability for the phase-field system

Conclusion

We can apply the Banach Fixed-Point Theorem. This proves that the operator $\mathcal{N}$ has a fixed-point.

We have proved:

Theorem

Fix $T > 0$ and assume

$$\begin{cases} \xi^2\tau^2(\ell^2 - k^2)^2 - 2\xi\rho\tau(\ell^2 + k^2) - 2\rho - 1 \neq 0, \quad \forall k, \ell \geq 1, \quad \ell > k, \\ \xi \neq \dfrac{1}{j^2}\dfrac{\rho}{\tau}, \quad \forall j \geq 1, \end{cases}$$

Then, there exists $\varepsilon > 0$ such that, for any $(\tilde{\theta}_0, \tilde{\phi}_0) \in H^{-1} \times (c + H_0^1)$, with $\boxed{\|\tilde{\theta}_0\|_{H^{-1}} + \|\tilde{\phi}_0 - c\|_{H_0^1} \leq \varepsilon}$, there exists $v \in L^2(0,T)$ for which system (1) has a unique solution which satisfies $y(\cdot, T) = 0$ in $(0,T)$.

Reference

GONZÁLEZ.-BURGOS, SOUSA-NETO, *Boundary controllability of a one-dimensional phase-field system with one control force*, J. Differential Equations (2020).

Thank you for your attention