Inverse Problem for the Schrödinger Equation with Non–self–adjoint Matrix Potential

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Outline

Introduction

Statement of the Problem

For this work, we consider the following dynamical system

$$\begin{cases} iu_t - u_{xx} + Q(x)u = 0, & 0 \le x \le \ell,\ 0 < t < T, \\ u(x,0) = 0, & 0 \le x \le \ell, \\ u_x(0,t) = f(t),\ u(\ell,t) = 0, & 0 < t < T, \end{cases} \tag{1}$$

where $\ell > 0$, $T > 0$ are given, and $Q \in C\big([0,\ell];\mathbb{C}^{N\times N}\big)$, $Q \neq Q^*$, is a matrix potential. The vector function $f \in \mathcal{F}^T = L^2(0,T;\mathbb{C}^N)$ is referred to as the *boundary control*. The inner product in $\mathcal{F}^T$ is defined by

$$(f,g)_{\mathcal{F}^T} := \int_0^T \langle f(t), g(t)\rangle \, dt,$$

where $\langle \cdot,\cdot\rangle$ is a standard scalar product in $\mathbb{C}^N$.

Introduction

Statement of the Problem

We denote the solution to (1) corresponding to f by u^f. We introduce the response operator R^T by

$$\begin{aligned} &R^T : \mathcal{F}^T \to \mathcal{F}^T, \\ &(R^T f)(t) = u^f(0, t), \quad 0 < t < T. \end{aligned} \tag{2}$$

The inverse problem is to recover $Q(x)$ for $0 < x < \ell$ from R^T.

Introduction

Statement of the Problem

- For the scalar Schrödinger equation, this problem was solved by Avdonin et al (2002, 2005).
- However, our problem is different and more difficult because Q is a matrix and non–self–adjoint.
- It is a new problem, and is a good example to show the Boundary Control (BC) method and how to combine both dynamical and spectral approaches to solve inverse problems.

Introduction

Initial Setup

- Since Q is non–self–adjoint, we consider the operators

$$A = -\frac{\partial^2}{\partial x^2} + Q(x),$$
$$A^* = -\frac{\partial^2}{\partial x^2} + Q^*(x).$$

Introduction

Initial Setup

- Since Q is a matrix, we have no information about the eigenvalues of A and A^* with respect to their multiplicity and the dimension of their associated eigenspaces.
- For this talk, we will assume that for each eigenvalue λ_k with algebraic multiplicity M_k, that there are M_k corresponding eigenfunctions.
- However, the general case has been solved.

Introduction

Initial Setup

Along with System (1), we consider the following associated dynamical control system:

$$\begin{cases} iv_t - v_{xx} + Q^*(x)v = 0, & 0 \le x \le \ell,\ 0 < t < T, \\ v(x,0) = 0, & 0 \le x \le \ell, \\ v_x(0,t) = g(t),\ v(\ell,t) = 0, & 0 < t < T, \end{cases} \tag{3}$$

with solution v^g. The response operator for this system will be denoted $R_{\#}^T$ with

$$\begin{aligned} &R_{\#}^T : \mathcal{F}^T \to \mathcal{F}^T, \\ &(R_{\#}^T g)(t) = v^g(0,t), \quad 0 < t < T. \end{aligned} \tag{4}$$

Introduction

Initial Setup

Definition (Spaces for operators)

We first define the underlying space as $\mathcal{H} = L^2((0,\ell);\mathbb{C}^N)$. We consider the Sobolev spaces $H^1([0,\ell];\mathbb{C}^N)$, $H^2([0,\ell];\mathbb{C}^N)$, and we set

1. $\mathcal{H}_1 = \{f \in H^1([0,\ell];\mathbb{C}^N) \mid f(\ell) = 0\}$,
2. $\mathcal{H}_2 = \{f \in H^2([0,\ell];\mathbb{C}^N) \mid f'(0) = 0, f(\ell) = 0\}$.

The domain of A (A^*), denoted $\operatorname{dom} A$ ($\operatorname{dom} A^*$), is $\mathcal{H}_2$.

Introduction

Initial Setup

- For Systems (1) and (3), we introduce the *control operators*

$$W^T : \mathcal{F}^T \to \mathcal{H}, \quad (W^T f)(x) := u^f(x, T),$$
$$W_\#^T : \mathcal{F}^T \to \mathcal{H}, \quad (W_\#^T g)(x) := v^g(x, T). \tag{5}$$

Main Ideas of the Solution

The Boundary Control (BC) method

- The Boundary Control (BC) method is a technique that was first used to solve similar inverse problems for wave equations.
- It is known that controllability is equivalent to observability, which then is connected to inverse problems.
- However, the BC method revealed a direct connection between controllability and inverse problems.
- Because of two key properties: finite speed of propagation and causality, the BC method provides a purely dynamical approach to solving the inverse problem.

Main Ideas of the Solution

The Boundary Control (BC) method

- For the Schrödinger equation, we have infinite speed of propagation, and thus we cannot use a purely dynamical approach.
- Therefore, we use a spectral approach involving the Fourier method.
- However, the Fourier method requires Riesz bases of eigenfunctions of the underlying differential operators A and A^*.

Main Ideas of the Solution

Riesz Basis Property

- When the underlying operator is self–adjoint, we can use classical spectral theory for differential operators.
- Verifying the Riesz basis property for non–self–adjoint operators typically involves investigating the associated Sturm-Liouville (S-L) problem.
- Recently, Lunyov and Malamud (2015) developed an alternative approach using the first–order system representation of the S-L problem.
- Using this, we were able to prove that A and A^* possess Riesz bases of eigenfunctions, denoted $\{\varphi_{k,l}\}$ and $\{\psi_{k,l}\}$, respectively.

Main Ideas of the Solution

Spectral Controllability

- Now that we established the existence of a Riesz basis, we need to be able to drive the system to each eigenfunction. This is known as *spectral controllability*.

Definition (Spectral Controllability)

Systems (1) and (3) are spectrally controllable if for each $k \in \mathbb{N}$, $1 \leq l \leq M_k$, there exist controls $f_{k,l}$ and $g_{k.l}$ such that

$$\left(W^T f_{k,l}\right)(x) = \varphi_{k,l}(x),$$
$$\left(W_{\#}^T g_{k,l}\right)(x) = \psi_{k,l}(x).$$

Main Ideas of the Solution

Spectral Controllability

- The most common approach is by the method of moments and analyze the resulting exponential family.
- However, the study of vector exponentials is still underdeveloped and we cannot proceed in this way.
- Instead, we use a technique called the "control transmutation method" used by Miller (2005).
- This method uses controllability of the associated wave system to construct the necessary control functions, and we prove spectral controllability of our system.
- Combined with the work of Ervedoza and Zuazua (2010), we can obtain control functions from a smoother space, $H_0^1(0, T; \mathbb{C}^N)$.

Main Ideas of the Solution

Recovering Spectral Data

- Now, we'll begin the process of recovering spectral data from the response operators R^T and $R^T_{\#}$.
- To show the connection between response operators, we introduce the operator J^T in $\mathcal{F}^T$ by the rule

$$(J^T f)(t) := f(T - t), \quad 0 \leq t \leq T.$$

Lemma

The following identity holds.

$$(R^T_{\#})^* J^T = J^T R^T \tag{6}$$

Main Ideas of the Solution

Recovering Spectral Data

- From the control operators, we construct the *connecting operator* $C^T : \mathcal{F}^T \to \mathcal{F}^T$ by its quadratic form

$$(C^T f, g)_{\mathcal{F}^T} = \left(W^T f, W^T_{\#} g\right)_{\mathcal{H}} = \left(u^f(x,T), v^g(x,T)\right)_{\mathcal{H}}. \tag{7}$$

- It is an important fact in the BC method that C^T can also be expressed in terms of the dynamical data.

Lemma

$$C^T = i[(R^T_{\#})^* - R^T] = i[J^T R^T J^T - R^T] \tag{8}$$

Main Ideas of the Solution

Recovering Spectral Data

We formulate the main result.

Theorem

The spectrum of A and (non-normalized) controls $\{f_{k,l}\}$ are the spectrum and the eigenfunctions of the following generalized spectral problem:

$$C^T \dot{f}_{k,l} - i\lambda_k C^T f_{k,l} = 0. \tag{9}$$

Main Ideas of the Solution

Recovering Spectral Data

Similarly, we obtain a generalized spectral problem for A^*:

Remark

The spectrum of A^* and (non-normalized) controls $\{g_{k,l}\}$ are the spectrum and the eigenfunctions of the following generalized spectral problem:

$$\left(C^T\right)^* \dot{g}_{k,l} + i\overline{\lambda_k}\left(C^T\right)^* g_{k,l} = 0. \tag{10}$$

Main Ideas of the Solution

Recovering Spectral Data

- From Equations (9) and (10), we obtain the spectrum $\{\lambda_k\}_{k=1}^{\infty}$.
- However, the controls we obtain may not correspond to a biorthogonal family, or even be linearly independent.
- This is a consequence of our system being a vector equation.
- It is a linear algebra exercise to construct families of controls that correspond to biorthogonal families of eigenfunctions.

Main Ideas of the Solution

Recovering Spectral Data

- We then obtain new families of controls $\{f_{k,l}\}$ and $\{g_{k,l}\}$, but their corresponding eigenfunctions may be different than $\{\varphi_{k,l}\}$ and $\{\psi_{k,l}\}$, so we denote these new eigenfunctions by

$$(W^T f_{k,l})(x) = \hat{\varphi}_{k,l}(x),$$
$$(W_{\#}^T g_{k,l})(x) = \hat{\psi}_{k,l}(x).$$

- In addition, we observe

$$(R^T f_{k,l})(T) = u^{f_{k,l}}(0, T) = \hat{\varphi}_{k,l}(0) =: \Phi_{k,l},$$
$$(R_{\#}^T g_{k,l})(T) = v^{g_{k,l}}(0, T) = \hat{\psi}_{k,l}(0) =: \Psi_{k,l}.$$

Main Ideas of the Solution

Recovering Spectral Data

- In particular, the response operator recovers the traces of these eigenvectors, which is part of the spectral data we desire.

Definition

The set $D := \{\lambda_k, \Phi_{k,l}, \Psi_{k,l}\}$, $k \in \mathbb{N}$, $1 \leq l \leq M_k$ is *spectral data* of System (1).

Main Ideas of the Solution

Recovering the Potential

- Now we can use spectral data to recover the potential. We construct the auxiliary wave system

$$\begin{cases} w_{tt} - w_{xx} + Q(x)w = 0, & 0 \leq x \leq \ell,\ 0 < t < T, \\ w(x,0) = w_t(x,0) = 0, & 0 \leq x \leq \ell, \\ w_x(0,t) = f(t),\ w(\ell,t) = 0, & 0 < t < T, \end{cases} \tag{11}$$

  with response operator $(R_w^T f)(t) := w^f(0,t)$.
- We also consider the associated wave system with matrix potential Q^* with solution $w_\#^g(x,t)$.
- We define the connecting operator, C_w^T, as the scalar product of the corresponding solutions of these two wave systems.

Main Ideas of the Solution

Recovering the Potential

- The next step is to express C_w^T in terms of the spectral data we obtained from (1).
- Using the Fourier method, we represent the solution, $w^f(x,t)$, in the form

$$w^f(x,t) = \sum_{k,l} b_{k,l}(t)\hat{\varphi}_{k,l}(x),$$

where

$$b_{k,l}(t) = -\int_0^t \left[\Psi_{k,l} f(\tau)\right] \frac{\sin\sqrt{\lambda_k}(t-\tau)}{\sqrt{\lambda_k}}\, d\tau. \tag{12}$$

Main Ideas of the Solution

Recovering the Potential

- Similarly, we represent $w^g_{\#}(x,t)$ in the form

$$w^g_{\#}(x,t) = \sum_{k,l} c_{k,l}(t)\hat{\psi}_{k,l}(x),$$

where

$$c_{k,l}(t) = -\int_0^t \left[\Phi_{k,l}g(s)\right] \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}}\, ds. \qquad (13)$$

Main Ideas of the Solution

Recovering the Potential

- From here, we compute

$$\begin{aligned}(C_w^T f, g)_{\mathcal{F}^T} &= \Big(w^f(x,T), w_{\#}^g(x,T)\Big)_{\mathcal{H}} \\ &= \sum_{k,l}\Bigg\{\int_0^T \Big[\Psi_{k,l} f(t)\Big]\frac{\sin\sqrt{\lambda_k}(T-t)}{\sqrt{\lambda_k}}\,dt \\ &\qquad \times \int_0^T \Big[\Phi_{k,l} g(s)\Big]\frac{\sin\sqrt{\overline{\lambda_k}}(T-s)}{\sqrt{\overline{\lambda_k}}}\,ds\Bigg\},\end{aligned}$$

- Hence, C_w^T is completely determined by the spectral data.

Main Ideas of the Solution

Recovering the Potential

- Let e_j is the j-th standard basis vector in $\mathbb{C}^N$ and $y_j(x)$ be the solution to the boundary value problem

$$\begin{cases} y''(x) - Q(x)y(x) = 0, & 0 \leq x \leq \ell, \\ y(0) = 0, \ y'(0) = e_j, \end{cases} \tag{14}$$

- Let p_j^T be the control function such that

$$w^{p_j^T}(x, T) = \begin{cases} y(x), & x \leq T, \\ 0, & x > T. \end{cases} \tag{15}$$

Main Ideas of the Solution

Recovering the Potential

- The function p_j^T satisfies the equation

$$\big(C_w^T p_j^T\big)(t) = (T - t)e_j, \quad t \in (0, T).$$

- The BC method states that C_w^T is boundedly invertible. Hence, this equation has a unique solution for all $T \leq \ell$.
- We define $\mu_j(T) := p_j^T(+0)$ and note that $-p_j^T(+0) = y_j(T)$.
- Thus, $\mu_j(T)$ is twice differentiable with respect to T.

Main Ideas of the Solution

Recovering the Potential

- We then construct the $N \times N$ matrix $M(T)$ by

$$M(T) = \left[\; \mu_1(T) \; \middle| \; \mu_2(T) \; \middle| \; \cdots \; \middle| \; \mu_n(T) \; \right]$$

- Thus,

$$Q(T) = M''(T)M^{-1}(T),$$

  and M is invertible except for finitely many points.
- By varying T in $(0, \ell)$, we obtain $Q(\cdot)$ in that interval.
- When $M(T)$ is singular, we recover $Q(T)$ by continuity.
- This completes the process of recovering the potential.

Concluding Remarks

- We have developed a strategy to recover the non–self–adjoint matrix potential for the Schrödinger system.
- In the process, we proved the Riesz basis property of the operator A and the spectral controllability of the system.
- We also developed an algorithm to construct the control functions corresponding to the eigenfunctions of A.
- Our inverse data, R^T, is a Neumann–to–Dirichlet mapping.
- However, our approach can be extended to the Dirichlet–to–Neumann (D-N) mapping, which are also common in dynamical inverse problems.

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