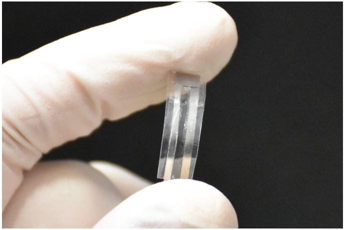
Introduction of Novel PDE Models of Certain Smart Material Systems and Diving into Related Controllability Issues

Ahmet Özkan Özer

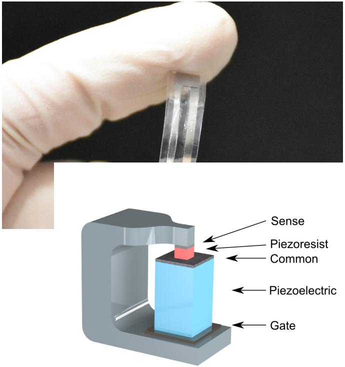
Department of Mathematics, Western Kentucky University
Bowling Green, Kentucky, USA

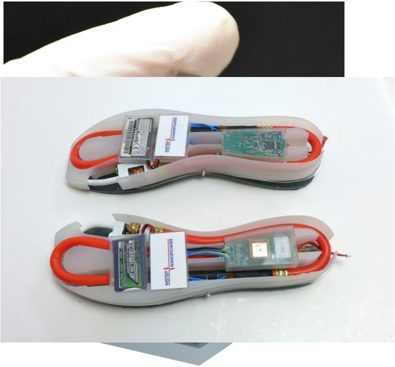
Seminario Control en Tiempos de Crisis 2020
December 2, 2020

Applications of smart-material systems.

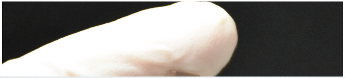


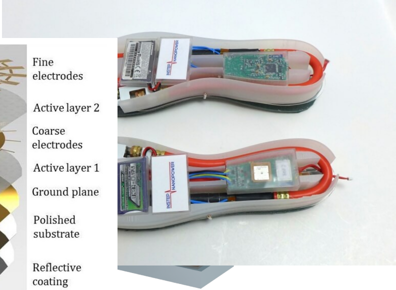
Applications of smart-material systems.

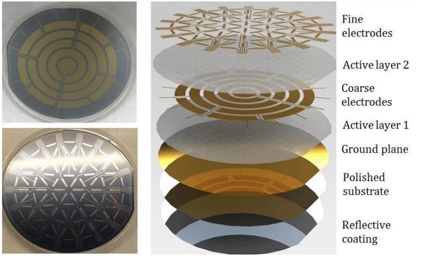



INSTEP NANOPOWER
INSTEP NANOPOWER

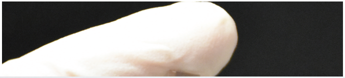
Applications of smart-material systems.

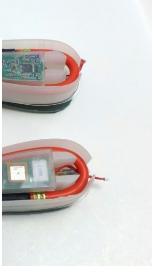


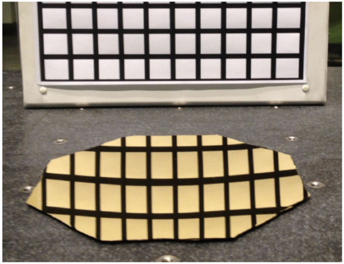


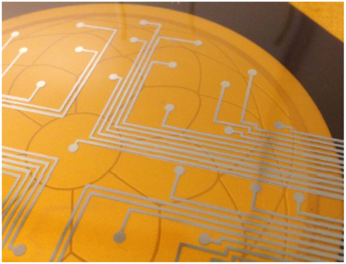


Applications of smart-material systems.

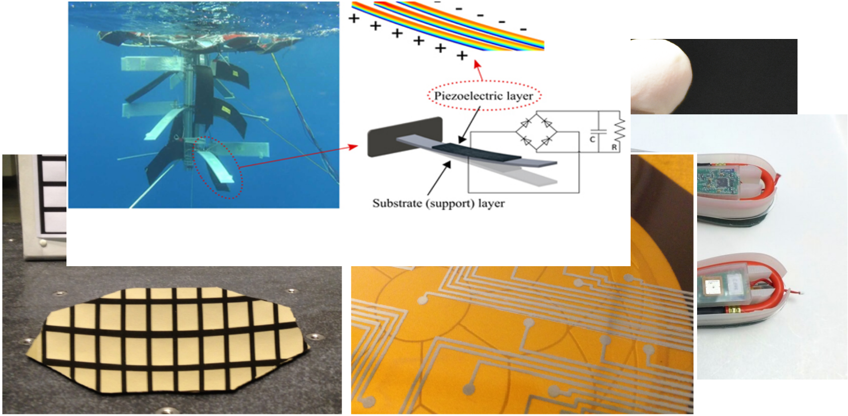




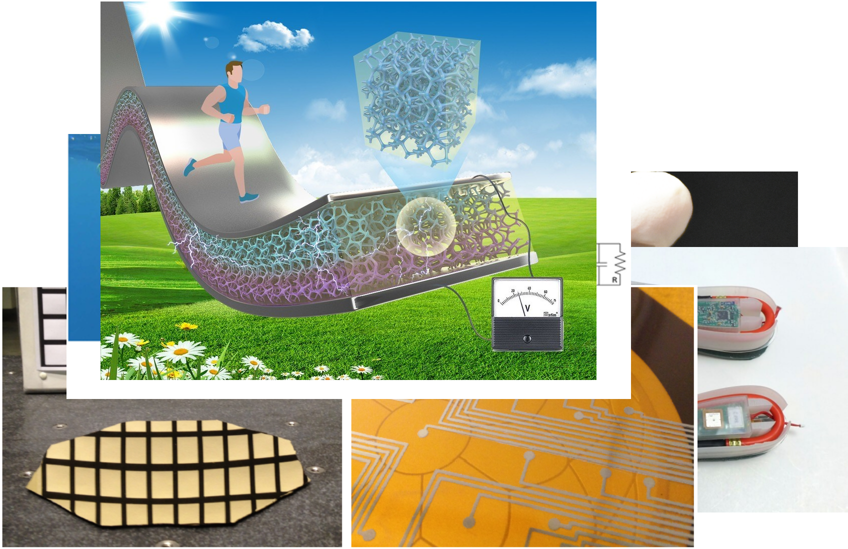




Applications of smart-material systems.



Applications of smart-material systems.



Outline

Vibrations on non-smart elastic beams

$v(x,t)$: Longitudinal displacement of the centerline of the beam
$w(x,t)$: Transverse displacement of the centerline of the beam
$\psi(x,t)$: Rotation of the beam
Notation: dot $= \frac{\partial}{\partial t}$, prime $= \frac{\partial}{\partial x}$.

Rayleigh (Kirchhoff) beam model

$$\begin{cases} \rho h\ddot{v} - \alpha h v_{xx} = 0, \\ \rho h\ddot{w} - \frac{\rho h^3}{12}\ddot{w}_{xx} + \frac{\alpha_1 h^3}{12} w_{xxxx} = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\ BC's : clamped, hinged, free, mixed \\ IC's \end{cases}$$

- $L, h > 0$: Length and thickness of the beam
- $\rho, \alpha, K > 0$: Material constants

$$(I - \frac{h^2}{12} D_x^2)\ddot{w} + Kw'''' = 0$$

$$\begin{cases} \rho h \ddot{v} - \alpha h v_{xx} = 0, \\ \rho h \ddot{w} + \frac{\alpha_1 h^3}{12} w_{xxxx} = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\ BC's : clamped, hinged, free, mixed \\ IC's \end{cases}$$

- $L, h > 0$: Length and thickness of the beam
- $\rho, \alpha, K > 0$: Material constants

$$\begin{cases} \rho h \ddot{v} - \alpha_1 h v_{xx} = 0, \\ \frac{\rho h^3}{12} \ddot{\psi} - \frac{\alpha_1 h^3}{12} \psi_{xx} + \alpha_3 h \left(w_x + \psi \right) = 0, \\ \rho h \ddot{w} - \alpha_1 h v_{xx} - \alpha_3 h \left(w_x + \psi \right)_x = 0, \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\ BC's : clamped, hinged, free, mixed \\ IC's \end{cases}$$

- $L, h > 0$: Length and thickness of the beam
- $\rho, \alpha_1, \alpha_3 > 0$: Material constants

Single piezoelectric beam model

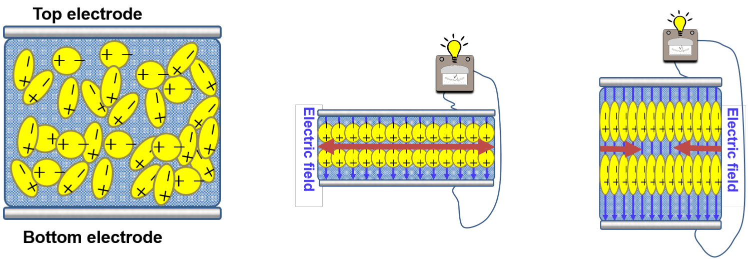


Figure: (a) A piezoelectric beam is an elastic beam with electrodes at their top and bottom surfaces, and connected to an external electric circuit. As voltage is applied to its electrodes, it actively (b) stretches or (c) shrinks in the longitudinal directions, therefore, causes charges to separate and line up in the vertical direction.

Actuation by what? voltage, charge, current, or mechanical?

Why voltage actuation?

- Traditionally piezoelectric beams are actuated by voltage: NASA, Tiersten'68, Jaffe et.al'71, Hagood'90, Banks&Smith'91, Rogacheva'94, many others...
- Easy to implement, simpler circuit design...it is a great advantage!
- Choice of models? finite or infinite dimensional? circuit model? Linear, nonlinear?
- Electrical hysteresis? accuracy of the model for low and high voltage profiles. Comstock'81, Newcomb'82, Hagood'90, Main & Garcia'97, Fleming'03...
- Nature of the control operator: bounded or **unbounded** in the Hilbert space? [Ozer & Morris- SICON'14, ESAIM-COCV'19].

- Electrostatic, quasi-static, or fully dynamic electro-magnetic assumptions? [Ozer & Morris'14- SICON, Ozer & Khenner'19-SPIE, Ozer'17 and 18-IEEE-TAC, Ozer'19-EECT, Ozer & Morris-ESAIM-COCV, Ozer'20-AMOP]

"Even though the $electro - static$ and $quasi - static$ approaches are sufficient for i.e. piezoelectric acoustic devices, electromagnetic waves generated by mechanical fields need to be accounted for in the calculation of radiated electromagnetic power from a vibrating piezoelectric device [Yang'06] "

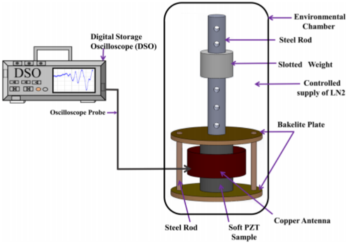
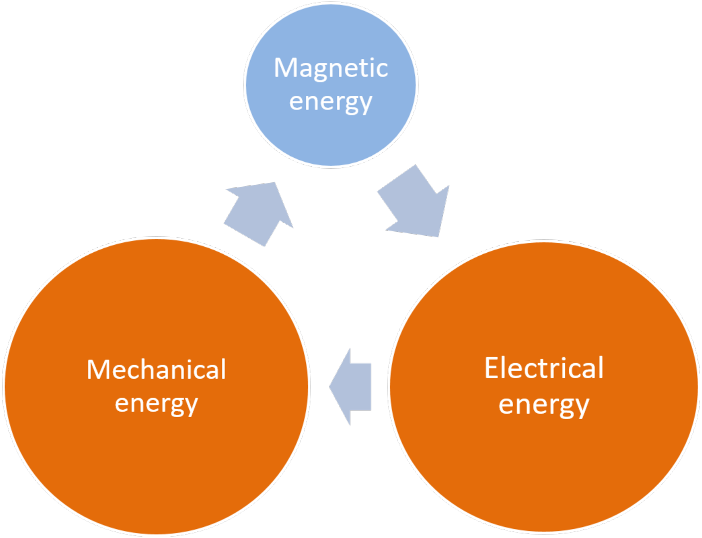


Fig. 1. Schematic diagram of experimental setup.

Figure: Electromagnetic Radiation from Soft PZT SP-5A Under Impact

Piezoelectricity? Magnetic energy is minor!



Going to basics...Why not?

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Frequency dependence of electromagnetic radiation from a finite vibrating piezoelectric body

Guangying Yang [a], Jianke Du [a], Ji Wang [a], Jiashi Yang [a,b,*]

[a] Piezoelectric Device Laboratory, School of Mechanical Engineering and Mechanics, Ningbo University, Ningbo, Zhejiang 315211, China
[b] Department of Mechanical and Materials Engineering, the University of Nebraska-Lincoln, Lincoln, NE 68588-0526, USA

ARTICLE INFO

Article history:
Received 12 February 2017
Received in revised form 30 March 2017
Accepted 3 April 2017
Available online 6 April 2017

Keywords:
Piezoelectric
Vibration
Electromagnetic
Radiation

ABSTRACT

We study electromagnetic radiation from a finite piezoelectric body in time-harmonic vibration through the analysis of piezoelectric crystal plate resonators. The polarization charge and free charge involved are obtained. To the lowest order, the charges are approximated by a vibrating electric dipole whose radiated power is then calculated. It is shown that the radiated power is formally proportional to the fourth power of the vibration frequency. As a consequence, the radiated power increases rapidly when the body becomes smaller, which is relevant in the miniaturization of resonant piezoelectric devices.

1. Introduction

Elastic waves and electromagnetic waves can interact in an elastic dielectric [1–4]. However, in the conventional theory of piezoelectricity [5,6], while the mechanical fields are governed by Newton's laws and are fully dynamic, the electric field is governed by electrostatics based on the quasistatic approximation in [4,7]. As a consequence, while the theory is capable of describing elastic wave phenomena, it cannot describe electromagnetic waves. The quasistatic approximation makes the conventional theory of piezoelectricity relatively simple. It is sufficient for the analysis of most conventional piezoelectric devices which are of millimeters in size or larger, and operate in the frequency range of MHz or lower. In these devices, e.g., conventional piezoelectric crystal plate res- high-frequency piezoelectric devices, the radiation damping due to electromagnetic waves generated by acoustic vibration through piezoelectric coupling becomes more pronounced [12]. At present there is only limited understanding of this effect. This is because the fully dynamic theory for describing coupled elastic and electromagnetic waves in piezoelectric crystals is much more complicated than the conventional theory of piezoelectricity. The theoretical solutions for radiation in [8–10] are for the relatively simple situation of unbounded plates in which the fields vary along the plate thickness only. Researchers also studied other fully dynamic [13,14] and quasistatic [15,16] problems with couplings among electric, magnetic and mechanical fields.

In this paper, instead of using directly the fully dynamic theory of coupled acoustic and electromagnetic fields, we propose a dif-

Piezoelectric beam assumptions

- Euler-Bernoulli small displacement assumptions.
- Edges are insulated (No fringing effects!).
- Assume transverse polarization in z direction, transverse isotropy.
- Activated by only external electric forces, i.e. **charge** or **current.** (**Voltage** is a different deal)!
- Linear constitutive equations.
    - No hysteresis (Electrical nonlinearity).

Full set of Maxwell's equations

"Dots to denote differentiation with respect to time"

$$\nabla \cdot D = \sigma_b, \qquad \text{(Electric Gauss's law)}$$
$$\nabla \cdot B = 0, \qquad \text{(Gauss's law of magnetism)}$$
$$\nabla \times E = -\dot{B}, \qquad \text{(Faraday's law)}$$
$$\frac{1}{\mu}(\nabla \times B) = i_b + \dot{D}. \qquad \text{(Ampére-Maxwell law)}$$

$$BC's : \quad \begin{cases} -D \cdot n = \sigma_s, \\ \phi = V, \\ \frac{1}{\mu}(B \times n) = i_s. \end{cases}$$

D	Electric displacement	E	Electric field
B	Magnetic field vector	μ	Permeability of beam
i_s	Surface current density	i_b	Body current density
σ_s	Surface charge density	σ_b	Body charge density
ϕ	Electric potential	V	Voltage.

Electrostatic → Quasi-static → Dynamic [1]

$$
\begin{aligned}
&\nabla \cdot D = \sigma_b, && \text{(Electric Gauss's law)}\\
&\nabla \cdot B = 0, && \text{(Gauss's law of magnetism)}\\
&\nabla \times E = -\dot{B}, && \text{(Faraday's law)}\\
&\mu(\nabla \times B) = i_b + \dot{D}. && \text{(Ampére-Maxwell law)}
\end{aligned}
$$

- Electrostatic: $B = \dot{D} = i_b = \sigma_b = 0 \quad \Rightarrow E = -\nabla\phi.$
- Quasi-static: $B \neq 0,\ \sigma_b = i_b = 0,\ \dot{D} \neq 0, \quad \Rightarrow E = -\nabla\phi - \dot{A}$ where A is the magnetic potential, and ϕ is the electric potential.
- *Dynamic* : Full set of Maxwell's equations.

[1] H.F. Tiersten, *Linear Piezoelectric Plate Vibrations* , Plenum Press, New York, 1969.

Piezoelectric beam (alternative) constitutive equations:

$$\begin{pmatrix} T \\ D \end{pmatrix} = \begin{bmatrix} \alpha & -\gamma^{\mathrm{T}}\beta \\ \gamma & \varepsilon \end{bmatrix} \begin{pmatrix} S \\ E \end{pmatrix}$$

$$\begin{cases} T_{11} = \alpha S_{11} - \gamma E_3 \\ T_{13} = -\gamma_1 E_1 \\ D_1 = \varepsilon_1 E_1 \\ D_3 = \gamma S_{11} + \varepsilon_3 E_3 \end{cases}$$

T	Stress tensor	S	Strain tensor
E	Electric field vector	D	Electric displacement vector
α	Elastic stiffness coefficient matrix	γ	Piezoelectric coefficient matrix
β	Impermittivity coefficient matrix		

Assumptions

- $E_1 \neq 0$, i.e. $E = (E_1, 0, E_3)$.
- $D = (D_1, 0, D_3)$ and $D_i(x, z, t) = D_i(x, t)$:

$$\Downarrow$$

$$B = \nabla \times A, \quad E = -\dot{A} - \nabla \phi.$$

φ : Electric potential
A : Electric vector potential

Quadratic-through thickness-Induced electric field effects

Let $\Omega = [0, L] \times [-h/2, h/2]$.
Assume quadratic-through-thickness along $z-$direction.

$$
\phi(x,z) = \phi^0(x) + z\phi^1(x) + \frac{z^2}{2}\phi^2(x)
$$

$$
\left(\begin{array}{c} A_1(x,z) \\ 0 \\ A_3(x,z) \end{array}\right) = \left(\begin{array}{c} A_1^0(x) + zA_1^1(x) + \frac{z^2}{2}A_1^2(x) \\ 0 \\ A_3^0(x) + zA_3^1(x) + \frac{z^2}{2}A_3^2(x) \end{array}\right).
$$

$$\mathbf{L} = \int_0^T \left[\mathbf{K} - (\mathbf{P} - \mathbf{E}) + \mathbf{B} + \mathbf{W}\right] \, dt$$

$$\mathbf{W} = \int_0^L \left(\cancelto{0}{-\sigma_b \left(\phi^0 + \frac{h^2}{24}\phi^2\right)} + \cancelto{0}{i_b^1 \left(A_1^0 + \frac{h^2}{24} A_1^2\right)} - \sigma_s \, \phi^1 + i_s^1 \, A_1^1 \right) \, dx$$

with surface continuity condition for each actuation :

$$\frac{di_s}{dx} = 0, \quad \dot{\sigma}_s = 0.$$

Admissible displacements: $\{v, w, \phi^0, \phi^1, \phi^2, A_1^1, A_1^2, A_1^3, A_3^1, A_3^2, A_3^3\}$.

Single beam: Bending is not coupled to stretching...

$$\phi^1 \to \phi, \quad \theta := A_1^1, \quad \eta := A_3^0 + \frac{h^2}{24} A_3^2, \quad \xi := \frac{\varepsilon_1 h^2}{12 \epsilon_{33}},$$

Single beam: Bending is not coupled to stretching...

$$\phi^1 \to \phi, \quad \theta := A_1^1, \quad \eta := A_3^0 + \frac{h^2}{24} A_3^2, \quad \xi := \frac{\varepsilon_1 h^2}{12\epsilon_{33}},$$

$$(Stretching) \begin{cases} \rho\ddot{v} - \alpha v_{xx} - \gamma\left(\phi + \dot{\eta}\right)_x = 0 \\ -\xi\left(\phi_{xx} + \dot{\theta}_x\right) + \dot{\eta} + \phi - \frac{\gamma}{\epsilon_{33}} v_x = -\frac{\gamma}{\epsilon_{33} h} \delta(x-L)\sigma_s(t) \\ \ddot{\theta} + \dot{\phi}_x - \frac{\mu}{\xi\epsilon_{33}}\left(\eta_x - \theta\right) = \frac{1}{\xi\epsilon_{33} h} \left(H(x) - H(x-L)\right) i_s(t) \\ \ddot{\eta} + \dot{\phi} - \frac{\gamma}{\epsilon_{33}} \dot{v}_x - \frac{\mu}{\epsilon_{33}}\left(\eta_{xx} - \theta_x\right) = 0 \end{cases}$$

$$\begin{cases} \alpha v_x + \gamma\left(\phi + \dot{\eta}\right) = 0 \quad \text{(Lateral force)} \\ \xi\epsilon_{33}\left(\dot{\theta} + \phi_x\right) = 0 \quad \text{(First charge moment)} \\ \mu\left(\theta - \eta_x\right) = 0 \quad \text{(Current)} \end{cases}$$

Notice that

- No way to include the current source $i_s(t)$ in the variational approach if there are no magnetic effects!
- The system is NOT WELL-POSED. The uniqueness fails:

Theorem (Morris & Ozer- IEEE-CDC'15)

For any scalar C^1 function $\chi = \chi(x, z, t)$, the Lagrangian $\mathbf{L}$ is invariant under the transformation

$$A \mapsto \tilde{A} := A + \nabla\chi$$
$$\phi \mapsto \tilde{\phi} := \phi - \dot{\chi}$$

A gauge needed to eliminate e-magnetic coupling!

- Remedy:$-$ $>$ Use a gauge:

$$\begin{cases} -\xi\theta_x + \eta = 0, & (\text{Coulomb} - \text{type}) \\ -\xi\theta_x + \eta = \frac{\xi\epsilon_{33}}{\mu}\dot{\phi}, & (\text{Lorenz} - \text{type}) \end{cases}$$

together with B.C's: $\theta(0) = \theta(L) = 0.$

  - Coulomb-type: E-M waves travel with infinite speed
  - Lorenz-type: E-M waves travel with finite speed

Letting $\phi^1 = \phi, \theta = A_1^1$, $\eta = A_3^0 + \frac{h^2}{24} A_3^2$, $\xi = \frac{\varepsilon_1 h^2}{12\epsilon_{33}}$,

$$(Stretching) \begin{cases} \rho\ddot{v} - \alpha v_{xx} - \gamma\left(\phi + \dot{\eta}\right)_x = 0 \\ -\xi\phi_{xx} + \xi\dot{\theta}_x + \dot{\eta} + \phi - \frac{\gamma}{\epsilon_{33}} v_x = -\frac{\gamma}{\epsilon_{33} h}\, \delta(x-L)\sigma_s(t) \\ \ddot{\theta} + \dot{\phi}_x - \frac{\mu}{\xi\epsilon_{33}}\left(\eta_x - \theta\right) = \frac{1}{\xi\epsilon_{33} h} \left(H(x) - H(x-L)\right) i_s(t) \\ \ddot{\eta} + \dot{\phi} - \frac{\gamma}{\epsilon_{33}} \dot{v}_x - \frac{\mu}{\epsilon_{33}}\left(\eta_{xx} - \theta_x\right) = 0 \end{cases}$$

$$\begin{cases} \alpha v_x + \gamma\left(\phi + \dot{\eta}\right) = 0 \quad \text{(Lateral force)} \\ \xi\epsilon_{33}\left(\dot{\theta} + \phi_x\right) = 0 \quad \text{(First charge moment)} \\ \mu\left(\theta - \eta_x\right) = 0 \quad \text{(Current)} \\ \theta(0) = \theta(L) = 0 \end{cases}$$

Define the operator $P_\xi := \left(-\xi D_x^2 + I\right)^{-1}$. It is well-known that P_ξ is a compact operator on $L^2(0, L)$. Also, P_ξ is a non-negative operator.

Define the operator $P_\xi := \left(-\xi D_x^2 + I\right)^{-1}$. It is well-known that P_ξ is a compact operator on $L^2(0,L)$. Also, P_ξ is a non-negative operator.

$$\begin{cases} \rho\ddot{v} - \alpha v_{xx} - \frac{\gamma^2}{\epsilon_{33}}(P_\xi v_x)_x - \gamma\dot{\eta}_x = -\frac{\gamma}{\epsilon_{33}h}\,\delta(x-L)\sigma_s(t) \\ \ddot{\theta} - \frac{\mu}{\epsilon_{33}}\theta_{xx} + \frac{\mu}{\xi\epsilon_{33}}\theta + \frac{\gamma}{\epsilon_{33}}(P_\xi\dot{v}_x)_x = \frac{1}{\xi\epsilon_{33}h}\,\left(H(x)-H(x-L)\right)i_s(t) \\ \ddot{\eta} - \frac{\mu}{\epsilon_{33}}\eta_{xx} + \frac{\mu}{\xi\epsilon_{33}}\eta - \frac{\gamma}{\epsilon_{33}}\left(\dot{v}_x - P_\xi(\dot{v}_x)\right) = 0, \\ -\xi\theta_x + \eta = 0, \quad (x,t)\in(0,L)\times\mathbb{R}^+ \\ \\ \begin{cases} v(0,t) = \alpha v_x(L,t) + \frac{\gamma^2}{\epsilon_{33}}P_\xi v_x(L,t) + \gamma\dot{\eta}(L,t) = 0, \\ \left\{\theta,\eta_x\right\}(0,t) = \left\{\theta,\eta_x\right\}(L,t) = 0, \\ (v,\theta,\eta,\dot{v},\dot{\theta},\dot{\eta})(x,0) = (v^0,\theta^0,\eta^0,v^1,\theta^1,\eta^1). \end{cases} \end{cases}$$

$$
\text{(Lorenz)}\begin{cases}
\rho\ddot{v} - \alpha v_{xx} - \gamma\left(\phi_x + \dot{\eta}_x\right) = 0 \\
\ddot{\phi} - \frac{\mu}{\epsilon_{33}}\phi_{xx} + \frac{\mu}{\xi\epsilon_{33}}\phi - \frac{\gamma\mu}{\xi{\epsilon_{33}}^2}v_x = \frac{\gamma\mu}{\xi{\epsilon_{33}}^2 h}\,\delta(x-L)\sigma_s(t) \\
\ddot{\theta} - \frac{\mu}{\epsilon_{33}}\theta_{xx} + \frac{\mu}{\xi\epsilon_{33}}\theta = \frac{1}{\xi\epsilon_{33}h}\,\left(H(x) - H(x-L)\right) i_s(t) \\
\ddot{\eta} - \frac{\mu}{\epsilon_{33}}\eta_{xx} + \frac{\mu}{\xi\epsilon_{33}}\eta - \frac{\gamma}{\epsilon_{33}}\dot{v}_x = 0, \\
-\xi\theta_x + \eta = \frac{\xi\epsilon_{33}}{\mu}\dot{\phi} \quad (x,t) \in (0,L)\times\mathbb{R}^+, \\
\begin{cases}
v(0,t) = \alpha v_x(L,t) + \gamma\phi(L,t) + \gamma\dot{\eta}(L,t) = 0, \\
\left\{\phi_x, \theta, \eta_x\right\}(0,t) = \left\{\phi_x, \theta, \eta_x\right\}(L,t) = 0, \\
(v,\phi,\theta,\eta,\dot{v},\dot{\phi},\dot{\theta},\dot{\eta})(x,0) = (v^0,\phi^0,\theta^0,\eta^0,v^1,\phi^1,\theta^1,\eta^1).
\end{cases}
\end{cases}
$$

Charge-control operator is UNBOUNDED in the energy space!
Current-control operator is in fact BOUNDED

$p(x,t)$: Total accumulated charge on the electrodes of the beam

$$\begin{cases} \rho h \ddot{v} - \alpha_1 h v_x x + \gamma\beta h p_{xx} = 0, & \\ \mu h \ddot{p} - \beta h p_{xx} + \gamma\beta h v_{xx} = 0, & \\ v(0,t) = p(0,t) = 0, & t > 0, \\ \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, & t > 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = \boxed{-\frac{V(t)}{h}}, & t > 0, \\ v(x,0) = v_0(x), \;\; v_t(x,0) = v_1(x), & x \in (0,L), \\ p(x,0) = p_0(x), \;\; p_t(x,0) = p_1(x), & x \in (0,L), \end{cases}$$

$$\rho h \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xx} + \frac{\alpha_1 h^3}{12} w_{xxxx} = 0, \quad + BC's + IC's$$

- $L, h > 0$: Length and thickness of the beam
- $\rho, \mu, \beta, \gamma, \alpha_1 > 0$: Material constants
- $V(t) =$ Voltage

Outline

$$\begin{aligned} \mathrm{E}(t) &= \frac{1}{2}\int_0^L \Big\{\mu\,|\theta-\eta_x|^2+\xi\epsilon_{33}|\dot\theta|^2+\epsilon_{33}\,|\dot\eta|^2+\rho|\dot v|^2 \\ &\quad +\alpha|v_x|^2+\frac{\gamma^2}{\epsilon_{33}}(P_\xi v_x)\bar v_x\Big\}\,dx. \end{aligned}$$

$$\mathrm{H} := \left[L^2(0,L))\times H_0^1(0,L)\times (L^2(0,L))^3\right]\bigcap\left\{\mathbf{y} : \xi(y_2)_x - y_3 = 0\right\}.$$

$$\begin{aligned} \langle \mathbf{y},\mathbf{z}\rangle_{\mathrm{H}} &= \int_0^L \{\mu y_1\bar z_1+\xi\epsilon_{33}y_2\bar z_2+\epsilon_{33}y_3\bar z_3+ \\ &\quad \alpha y_4\bar z_4+\frac{\gamma^2}{\epsilon_{33}}(P_\xi y_4)\bar z_4+\rho y_5\bar z_5\Big\}\,dx. \end{aligned}$$

Lemma

This form defines an inner product on the linear space H. *Moreover, E is the norm induced by this inner product and* H *is complete.*

Let

$$\mathbf{y} = [y_1, y_2, y_3, y_4, y_5]^{\mathrm{T}} = [\theta - \eta_x, \dot{\theta}, \dot{\eta}, v_x, \dot{v}]^{\mathrm{T}}.$$

$$\dot{\mathbf{y}} = A\mathbf{y} - \mathcal{B} i_s^1(t), \quad \mathbf{y}(x,0) = \mathbf{y_0} = (\theta^0 - \eta_x^0, \theta^1, \eta^1, v_x^0, v^1)$$

where $A =$

$$\begin{pmatrix} 0 & I & -D_x & 0 & 0 \\ \frac{-\mu}{\xi\epsilon_{33}} I & 0 & 0 & 0 & \frac{-\gamma}{\epsilon_{33}} D_x P_\xi D_x \\ \frac{-\mu}{\epsilon_{33}} D_x & 0 & 0 & 0 & \frac{\gamma}{\epsilon_{33}} (D_x - D_x P_\xi) \\ 0 & 0 & 0 & 0 & D_x \\ 0 & 0 & \frac{\gamma}{\rho} D_x & \frac{\alpha}{\rho} D_x + \frac{\gamma^2}{\rho\epsilon_{33}} P_\xi D_x & 0 \end{pmatrix} \quad \text{and}$$

$$\mathrm{Dom}(A) = \left[H_0^1(0,L) \times (H^2(0,L) \cap H_0^1(0,L)) \times (H^1(0,L))^2 \times H_L^1(0,L)\right]$$

$$\bigcap \left\{ \mathbf{y} \in \mathrm{H} : \left| \left(\alpha I + \frac{\gamma^2}{\epsilon_{33}} P_\xi \right) y_4 + \gamma y_3 \right|_{x=L} = 0 \right\},$$

and the B and B^* operators with the new state are

$$\langle \mathcal{B} u(t), \psi \rangle_{\mathrm{H}} = \frac{1}{\xi\epsilon_{33} h} \int_0^L u(t)\ \psi_2\ dx = u(t) \frac{1}{\xi\epsilon_{33} h} \int_0^L \psi_2\ dx = \langle u, \mathcal{B}^* \psi \rangle_{\mathcal{U}}$$

and $\mathcal{B}\mathcal{B}^* \in \mathcal{L}(\mathrm{H}, \mathrm{H})$.

Lemma

Let $\mathrm{Dom}(D_x^2) = \{w \in H^2(0,L) \ : \ w_x(0) = w_x(L) = 0\}$. *The operator* $\frac{1}{\xi}(P_\xi - I)$ *is continuous, self-adjoint and non-positive on* $\mathbb{L}^2$. *Moreover, for all* $w \in \mathrm{Dom}(P_\xi)$, $J = D_x^2 \ P_\xi = D_x^2(I - \xi D_x^2)^{-1} w$.

Lemma

The operator A *maps* $\mathrm{Dom}(A) \subset \mathrm{H}$ *to* H, *and is densely defined in* H.

Theorem

The operator $A : \mathrm{Dom}(A) \subset \mathrm{H} \to \mathrm{H}$ *satisfies* $A^* = -A$ *on* H, *and* A *is the generator of a unitary semigroup* $\{e^{At}\}_{t\geq 0}$ *on* H. *Letting* $T > 0$ *and* $i_s(t) \in L^2(0,T)$, *for any* $\mathbf{y_0} \in \mathrm{H}$, $\mathbf{y} \in C[[0,T];\mathrm{H}]$, *and there exists a positive constants* $c(T)$ *such that*

$$\|\mathbf{y}(T)\|_{\mathrm{H}}^2 \quad \leq \quad c(T) \left\{ \|\mathbf{y_0}\|_{\mathrm{H}}^2 + \|i_s\|_{L^2(0,T)}^2 \right\}.$$

Along the trajectories, the energy satisfies $\frac{d\mathrm{E}(t)}{dt} = i_s(t)\left(\int_0^L \psi_2(x)dx\right)$.
We investigate the asymptotic stability for the same B^*-feedback.
This leads to the feedback control $i_s^1(t) = -K_1 h^2 \xi^2 \epsilon_{33}^2 \int_0^L \dot{\theta}(z)dz$
where $K_1 > 0$ is arbitrary.

$$\begin{cases} \dot{\mathbf{y}} = \tilde{A}\mathbf{y} = A\mathbf{y} - K_1 h^2 \xi^2 {\epsilon_{33}}^2 BB^* \mathbf{y}, \\ \mathbf{y}(x,0) = \mathbf{y_0}. \end{cases}$$

Lemma

The infinitesimal generator $\tilde{A}$ satisfies $\tilde{A}^* = -\tilde{A}(-K_1)$ on $\tilde{\mathrm{H}}$. Moreover it is dissipative and it satisfies $\mathrm{Re}\left\langle \tilde{A}\mathbf{y}, \mathbf{y} \right\rangle_{\tilde{\mathrm{H}}} \leq 0$.

Theorem

$\tilde{\mathcal{A}} : \mathrm{Dom}(\tilde{A}) \to \tilde{\mathrm{H}}$ is the infinitesimal generator of a C_0-semigroup of contractions. Therefore for every $T \geq 0$, and $\mathbf{y_0} \in \mathrm{Dom}(\tilde{A})$ we have $\mathbf{y} \in C\left([0,T]; \mathrm{Dom}(\tilde{A})\right) \cap C^1\left([0,T]; \mathrm{H}\right)$. Moreover, the spectrum $\sigma(\tilde{A})$ of $\tilde{\mathcal{A}}$ has all isolated eigenvalues.

Lack of stabilizability - Coulomb gauge!

For real τ, define

$$A = \alpha\mu\xi\epsilon_{33}, \quad B = -(\alpha\epsilon_{33} + \gamma^2)(\mu - \epsilon_{33}\xi\tau^2) + \alpha\xi\mu\rho\epsilon_{33}\tau^2$$
$$C = -\rho\tau^2\epsilon_{33}\left(\mu - \epsilon_{33}\xi\tau^2\right),$$

$$a_1 = \sqrt{\frac{B + \sqrt{B^2 - 4AC}}{2A}}, \quad a_2 = \sqrt{\frac{B - \sqrt{B^2 - 4AC}}{2A}}.$$

Theorem (Ozer-AMOP'20, Ozer&Morris-ESAIM-COCV'20)

Let $a_1 = \frac{2n\pi}{L}, a_2 = \frac{2m\pi}{L}$, *and* $m^2 + n^2 > \frac{\mu\rho L^2}{16\alpha\xi\epsilon_{33}\pi^2}$ *for* $m, n \in \mathbb{N}$. *For* $\mathbf{y_0} \in \mathrm{H}$, *the semigroup* $\{e^{\tilde{A}t}\}_{t\geq 0}$ *is not asymptotically stable in* H, *i.e.* $\|e^{\tilde{A}t}\mathbf{y_0}\|_{\tilde{\mathrm{H}}} \nrightarrow 0, \quad t \to \infty$. *Furthermore, then the system* $\{\mathcal{A}, \mathcal{B}\}$ *is not asymptotically stabilizable by any bounded state feedback.*

Proof: Use Benchimol's Theorem.

How about charge actuation $\sigma_s(t)$instead?

In the case of charge-actuation: the control operator $\mathcal{B}$ is an unbounded operator with its adjoint $\mathbf{B}^*\psi = \frac{\gamma}{\epsilon_{33}h}\left(\psi_1(0) - \psi_1(L)\right)$.

B^* measurement is mechanical: The difference between tip velocities.

Eigenfunctions $\mathbf{y} \neq 0$ with $(y_1(L))^2 = 0$

and so $\mathcal{B}^*\mathbf{y} \equiv 0$ can be constructed by following the same argument as above for current control. It follows that the system is not stabilizable.

$$\mathrm{E}(t) = \frac{1}{2}\int_0^L \left\{\mu\left|\theta - \eta_x\right|^2 + \xi\epsilon_{33}|\dot{\theta} + \phi_x|^2 + \epsilon_{33}\left|\dot{\eta} + \phi\right|^2 + \alpha|v_x|^2 + \rho|\dot{v}|^2\right\} dx.$$

Define the states

$$\mathbf{y} = [y_1, y_2, y_3, y_4, y_5]^{\mathrm{T}} = [\theta - \eta_x, \dot{\theta} + \phi_x, \dot{\eta} + \phi, v_x, \dot{v}]^{\mathrm{T}}$$

By these choices of the states, note that the following compatibility condition (Coulomb-like) arises

$$\xi(y_2)_x - y_3 + \frac{\gamma}{\epsilon_{33}} y_4 = 0$$

Let $H_L^1(0,L) = \{f \in H^1(0,L) \ : \ f(0) = 0\}$. Define the linear space

$$\mathrm{H} = \Big\{\mathbf{y} \in (L^2(0,L))^5 : (y_2)_x \in L^2(0,L), \quad y_2(0) = y_2(L) = 0, \\ \xi(y_2)_x - y_3 + \tfrac{\gamma}{\epsilon_{33}} y_4 = 0\Big\}$$

and the bilinear form on $\mathrm{H} \times \mathrm{H}$:

$$a(\mathbf{y}, \mathbf{z}) \quad = \quad \int_0^L \{\mu y_1 \bar{z}_1 + \xi\epsilon_{33} y_2 \bar{z}_2 + \epsilon_{33} y_3 \bar{z}_3 + \alpha y_4 \bar{z}_4 + \rho y_5 \bar{z}_5\}\, dx.$$

Theorem

The energy $E(t)$ is the norm induced by this inner product, and H is a Hilbert space with this norm.

Define the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & -D_x & 0 & 0 \\ -\frac{\mu}{\xi\epsilon_{33}} I & 0 & 0 & 0 & 0 \\ -\frac{\mu}{\epsilon_{33}} D_x & 0 & 0 & 0 & \frac{\gamma}{\epsilon_{33}} D_x \\ 0 & 0 & 0 & 0 & D_x \\ 0 & 0 & \frac{\gamma}{\rho} D_x & \frac{\alpha}{\rho} D_x & 0 \end{pmatrix}$$

with

$$\mathrm{Dom}(\mathcal{A}) = \left[H^1_0(0,L) \times H^1_0(0,L) \times (H^1(0,L))^2 \times H^1_L(0,L)\right] \bigcap \left\{\mathbf{y} \in \mathrm{H} : \ (\alpha y_4 + \gamma y_3)\,(L) = 0\right\},$$

and the B and B^* operators with the new state are

$$\langle \mathcal{B} u(t), \psi \rangle_{\mathrm{H}} = \frac{1}{h} \int_0^L u(t)\ \psi_2\ dx = u(t) \frac{1}{h} \int_0^L \psi_2\ dx = \langle u, \mathcal{B}^* \psi \rangle_{\mathcal{U}}$$

Lemma

The operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \to \mathrm{H}$.

Theorem

For any $\mathbf{g} \in \mathrm{H}$ *there is* $\mathbf{y} \in \mathrm{Dom}(\mathcal{A})$ *so that* $\mathcal{A}\mathbf{y} = \mathbf{g}$. *That is,* $0 \in \rho(\mathcal{A})$.

Theorem

The operator $\mathcal{A}$ *satisfies* $\mathcal{A}^* = -\mathcal{A}$ *on* H, *and* $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathrm{H} \to \mathrm{H}$ *is the generator of a unitary semigroup* $\{e^{\mathcal{A}t}\}_{t\geq 0}$.

Theorem

Let $T > 0$, *and* $i_s(t) \in L^2(0,T)$. *For any* $\mathbf{y_0} \in \mathrm{H}$, $\mathbf{y} \in C[[0,T];\mathrm{H}]$, *and there exists a positive constants* $c(T)$ *such that*

$$\|\mathbf{y}(T)\|^2_{\mathrm{H}} \quad \leq \quad c(T)\left\{\|\mathbf{y_0}\|^2_{\mathrm{H}} + \|i_s\|^2_{L^2(0,T)}\right\}.$$

Similar to the Coulomb-gauge model:
For real τ, define

$$A = \alpha\mu\xi\epsilon_{33}, \quad B = -(\alpha\epsilon_{33} + \gamma^2)(\mu - \epsilon_{33}\xi\tau^2) + \alpha\xi\mu\rho\epsilon_{33}\tau^2$$
$$C = -\rho\tau^2\epsilon_{33}\left(\mu - \epsilon_{33}\xi\tau^2\right),$$

$$a_1 = \sqrt{\frac{B + \sqrt{B^2 - 4AC}}{2A}}, \quad a_2 = \sqrt{\frac{B - \sqrt{B^2 - 4AC}}{2A}}.$$

Theorem (Ozer'2020-AMOP)

Let $a_1 = \frac{2n\pi}{L}, a_2 = \frac{2m\pi}{L}$, *and* $m^2 + n^2 > \frac{\mu\rho L^2}{16\alpha\xi\epsilon_{33}\pi^2}$ *for* $m, n \in \mathbb{N}$. *For* $\mathbf{y_0} \in \mathrm{H}$, *the semigroup* $\{e^{\tilde{A}t}\}_{t\geq 0}$ *is not asymptotically stable in* H, *i.e.* $\|e^{\tilde{A}t}\mathbf{y_0}\|_{\tilde{\mathrm{H}}} \nrightarrow 0, \quad t \to \infty$. *Furthermore, then the system* $\{\mathcal{A}, \mathcal{B}\}$ *is not asymptotically stabilizable by any bounded state feedback.*

Proof: Use Benchimol's Theorem.

$\sigma_s(t) \neq 0, i_s(t) \equiv 0$: Lasiecka, Komornik, Zuazua, Guo...
$\sigma_s(t) \equiv 0, i_s(t) \neq 0$: Russell, Rao, Morgul, ...

$$\begin{cases} \rho \ddot{v} - \left(\alpha + \frac{\gamma^2}{\epsilon_{33}}\right) v_{xx} = 0, \\ v(0) = 0, \quad (\alpha + \frac{\gamma^2}{\epsilon_{33}}) v_x(L) = -\frac{\gamma}{\epsilon_{33} h} \sigma_s(t), \\ \dot{\sigma}_s(t) = i_s(t), \\ (v, \dot{v})(x, 0) = (v_0, v_1), \quad \dot{\sigma}_s(0) = \sigma_0. \end{cases}$$

Theorem (Wehbe-EJDE'03)

Let $T > 0$, *and* $i_s(t) = (\dot{v}(L,t) - K\sigma_s(t))$, $K \in \mathbb{R}^+$. *For any* $\mathbf{y_0} \in \mathrm{Dom}(\mathcal{A})$, *there exists a constant* $C(K) > 0$ *such that*

$$E(t) \leq E(0) \frac{2C}{t + C}, \forall t > 0.$$

Summary so far! Voltage control: Ozer-MCSS'15

	B^*-feedback	*Voltage control*
E-static	Velocity	E.S.
Q-static	Velocity	E.S.
Dynamic	*Induced charge*	*Not A.S.* (e-values on $i\mathbb{R}$)

	B^*-feedback	*Charge control*
E-static	Velocity	E.S.
Q-static	Velocity	E.S.
Dynamic	*Velocity*	*Not A.S.* (e-values on $i\mathbb{R}$)

	B^*-feedback	*Current control*
E-static	Charge and Tip v.	A.S
Q-static	Charge and Tip v.	A.S.
Dynamic	*Induced current*	*Not A.S.* (e-values on $i\mathbb{R}$)

Material parameters

ρ	Density	7600 kg/m^3
γ	Electromechanical coefficients	10^{-3} C/m^2
α_1	Stiffness constant	121×10^9 N/m^2
ε_{11}	permittivity constant	0.25×10^{-12} F/m
ϵ_{33}	permittivity constant	0.25×10^{-12} F/m
ξ	$\frac{\varepsilon_{11} h^2}{12\epsilon_{33}}$	8.3×10^{-10} $\frac{1}{\mathrm{m}^2}$
μ	Magnetic impermeability	1.2×10^{-6} H/m
h	Thickness of the beam	10^{-4} m
L	Length of the beam	1 m

n	m	$\lambda_3 = \imath\tau_+^{(3)}$	$\lambda_4 = \imath\tau_+^{(4)}$	$a_3 = \frac{n\pi}{L}$	$a_4 = \frac{m\pi}{L}$
2	4000	$7.589 \times 10^7 \imath$	$1.003 \times 10^8 \imath$	12.567	34,641.016
5	30	$7.589 \times 10^7 \imath$	$7.590 \times 10^7 \imath$	31.416	188.496
1	2	$7.589 \times 10^7 \imath$	$7.589 \times 10^7 \imath$	6.283	12.567

Table: Eigenvalues $\{\lambda_3, \lambda_4\}$ of $\mathcal{A}$ for the material parameters. The numbers are rounded to the nearest thousandth, and $\tau_+^{(j)} - \sqrt{\frac{\mu}{\xi\epsilon_{33}}} > 0$ for every $j = 3, 4$ where $\sqrt{\frac{\mu}{\xi\epsilon_{33}}} = 7.589 \times 10^7$.

Electro-magnetic and/or (remedial) mechanical feedback controllers

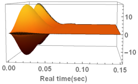
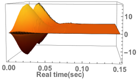
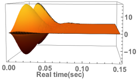
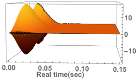
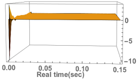
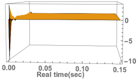
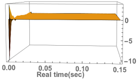
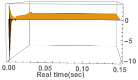
Case	Coulomb-gauge	Lorenz-gauge
I	$i_s(t) \equiv 0, g(t) \equiv 0$	$i_s(t) \equiv 0, g(t) \equiv 0.$
II	$i_s(t) = -K_1 \int_0^L \dot{\theta}\ dx$ $g(t) \equiv 0$	$i_s(t) = -K_1 \int_0^L \left(\phi_x + \dot{\theta}\right)\ dx$ $g(t) \equiv 0$
III	$i_s(t) \equiv 0$ $g(t) = -K_2 \dot{v}(L,t)$	$i_s(t) \equiv 0$ $g(t) = -K_2 \dot{v}(L,t)$
IV	$i_s(t) = -K_1 \int_0^L \dot{\theta}\ dx$ $g(t) = -K_2 \dot{v}(L,t)$	$i_s(t) = -K_1 \int_0^L \left(\phi_x + \dot{\theta}\right)\ dx$ $g(t) = -K_2 \dot{v}(L,t)$

Case	Electrostatic or quasi-static
I	$i_s(t), g(t) \equiv 0$
II	$i_s(t) = (\dot{v}(L,t) - K\sigma_s(t)), \quad g(t) \equiv 0$

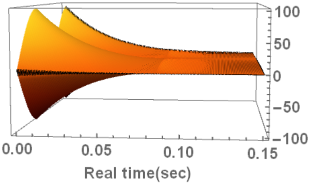
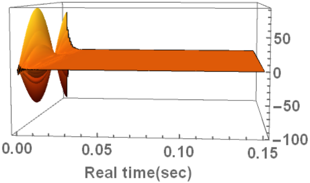
Utilizing the filtered semi-discrete finite differences (Zuazua, Tebou, ...)!

$$\Downarrow$$

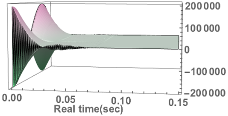
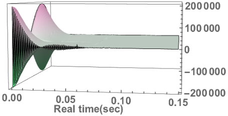
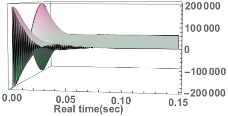
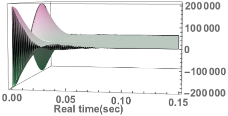
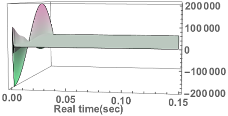
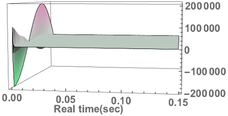
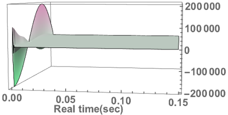
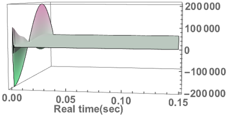
State: $v_x(x,t)$

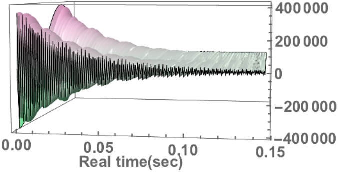
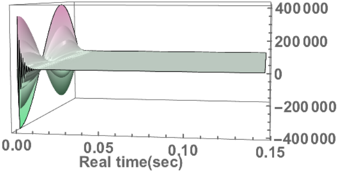
<table>
<tr><th>Case</th><th>Coulomb gauge</th><th>Lorenz gauge</th></tr>
<tr><td>I:No con.</td><td>
</td><td>
</td></tr>
<tr><td>II: Only elec. cont.</td><td>
</td><td>
</td></tr>
<tr><td>III: Only mech. cont.</td><td>
</td><td>
</td></tr>
<tr><td>IV: Two cont.</td><td>
</td><td>
</td></tr>
</table>

State: $v_x(x,t)$

Case	Electrostatic
I: No con.	
II: Only one cont.	

State: $\dot{v}(x,t)$

Case	Coulomb gauge	Lorenz gauge
I: No cont.		
II: Only elec. cont.		
III: Only mech. cont.		
IV: Two cont.s		

Case	Electrostatic
I: No cont.	
II: Only one cont.	

Voltage (O-Morris-SICON'14) vs. charge vs. current (O-Morris-ESAIM-COCV'20)

	B^*−feedback	*Voltage control*
E-static	Velocity	E.S.
Q-static	Velocity	E.S.
Dynamic	*Induced charge*	*Not A.S.* (e-values on $i\mathbb{R}$)

	B^*−feedback	*Charge control*
E-static	Velocity	E.S.
Q-static	Velocity	E.S.
Dynamic	*Velocity*	*Not A.S.* (e-values on $i\mathbb{R}$)

	B^*−feedback	*Current control*
E-static	Charge and Tip v.	A.S
Q-static	Charge and Tip v.	A.S.
Dynamic	*Induced current*	*Not A.S.* (e-values on $i\mathbb{R}$)

Outline

Various combos of Boundary & Distributed Delays -Multiple papers submitted

There is time delay between the controller/actuator and observer/sensor. The time delay has to be accounted for to design the feedback controller since as one considers a small perturbation delay in the output measurement, the stabilization of vibrations is at stake [Datko'93].

$$
\begin{cases}
\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + a_1 v_t + a_2 v_t(t-\tau) = 0, \\
\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \quad (x,t) \in \quad (0,L) \times (0,\infty) \\
\\
v(0,t) = p(0,t) = 0, \\
\alpha v_x(L,t) - \gamma\beta p_x(L,t) = -b_1 v_t(L,t) - b_2 v_t(L,t-\tau), \\
\beta p_x(L,t) - \gamma\beta v_x(L,t) = -c_1 p_t(L,t) - c_2 p_t(L,t-\tau), \quad t \geq 0, \\
\\
(v,p,v_t,p_t)(x,0) = (v_0(x), p_0(x), v_1(x), p_1(x)), \quad x \in [0,L],
\end{cases}
$$

- Various combos of $a_1, a_2, b_1, b_2, c_1, c_2$ are considered.
- The effect of the corresponding delay is investigated for the overall exponential stabilizability dynamics.
- Utilized the Lyapunov approach.

The following PDE model is derived through the variational approach and it is crucial for certain class of piezoelectric materials demonstrating time-dependent behavior in the form of colorredviscoelastic creep and dielectric relaxation.

$$\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_0^\infty \lambda_1(s) v_{xx}(t-s)ds = \ 0 \quad \text{in}$$

$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + \int_0^\infty \lambda_2(s) p_{xx}(t-s)ds = \ 0 \quad \text{in} \quad (0,L)\times(0,\infty)$$

with boundary conditions

$$v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, \quad t > 0,$$

where $\alpha := \alpha_1 + \gamma^2\beta$ with $\alpha_1, \beta, \gamma > 0$.

Consider the case where there is only a strain memory, $\lambda_1 = \lambda$ and $\lambda_2 \equiv 0$, together with nonlinear external forces, and electrical (current) damping

$$\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \int_0^\infty \lambda(s) v_{xx}(t-s)ds + f_1(v,p) = \ h_1(x) \quad \text{in}$$
$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + g(p_t) + f_2(v,p) = \ h_2(x) \quad \text{in} \quad (0,L)\times(0,\infty)$$

with boundary conditions

$$v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, \quad t > 0,$$

initial conditions

$$v(x,0) = v_0(x),\ v_t(x,0) = v_1(x),\ p(x,0) = p_0(x),\ p_t(0,x) = p_1(x),$$
$$v(x,-t) = v_2(x,t), \quad (x,t) \in (0,L)\times(0,\infty),$$

where v_0, v_1, v_2, p_0 and p_1 are functions that belong to appropriate spaces and α_1 satisfies $k_1 := \alpha_1 - \int_0^\infty \lambda(s)ds > 0$.

- Standard theory with classical constitutive equations for the relationship between the electric displacement, electric field, stress and strains do not account for these behaviors.
- This type of model has never been considered in the literature due to the existing complexity of PDE models for piezoelectric beams.
- The structure of the dynamical system associated with the solutions of this system allows using the "quasi-stability theory" in order to obtain the existence of global and exponential attractors.

Modeling : Full electromagnetic effects due to Maxwell's equations and with thermal effects by a thorough variational approach.
Let $A^\nu : D(A^\nu) \subset L^2(0,L) \to L^2(0,L)$ be the fractional power associated with operator A of order $\nu \in (0,1/2)$.

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + f_1(v,p) = h_1(x) & \text{in} \quad (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + A^\nu p_t + f_2(v,p) = h_2(x) & \text{in} \quad (0,L)\times(0,T), \\ c\theta_t - \kappa\theta_{xx} + \delta v_{xt} = 0 & \text{in} \quad (0,L)\times(0,T) \end{cases}$$

with clamped-free boundary and initial conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \\ \theta(0,t) = \theta(L,t) = 0, \ t > 0, \\ v(x,0) = v_0, \ v_t(x,0) = v_1, \\ p(x,0) = p_0, \ p_t(x,0) = p_1, \\ \theta(x,0) = \theta_0(x), \ \theta_t(0,x) = \theta_1(x), \ 0 < x < L. \end{cases}$$

Recent work with collaborators-submitted

- Studying the long-time dynamics of fractional piezoelectric beam with magnetic and thermal effects for the first time;
- Proving that the dynamical system generated by the system has a smooth global attractor with finite fractal dimension by the quasi-stability theory
- Obtaining the existence of a generalized exponential attractor in a scale of fractional spaces
- Establishing the stability of global attractors on the perturbation of the fractional exponent.

Outline

Mechanical effects:

- Mindlin-Timoshenko large displacement assumptions
  - Shear is taken into account
  - Longitudinal, bending, and, total rotation.
- Euler-Bernoulli large displacement assumptions
  - Only longitudinal and transverse vibrations are taken into account.

Electro-magnetic effects:

- Electrostatic, Quasi-static, Fully dynamic.
- Full set of Maxwell's equations to start with.
- Eliminate magnetic effects one-by-one to get to quasi-static or electro-static models!

Known DP models and results for nonlinear beams

$\underline{Why\ \ distributed\ \ parameter\ (DP)\ \ models?}$

Don't want the spill-over effect in designing a controller.

The spill-over effects is due to neglecting high-frequency modes in the controller design (Balas'78).

Known DP models and results for nonlinear beams

- First stab into this problem in the bilinear (affine) DP setting: A. Kugi, K. Schlacher'99
  - Hinged B.C's are considered. Longitudinal inertia is ignored.
  - Different control designs are proposed such as PD, H_∞, disturbance compensation.
- Port-Hamiltonian DP modelling, T. Voss, J. Sherpen'14
  - Clamped B.C's are considered.
  - Quasi-static model is not stabilizable.
  - Fully dynamic model is stabilizable.

Known DP models and results for nonlinear beams

- First stab into this problem in the bilinear (affine) DP setting: A. Kugi, K. Schlacher'99
  - Hinged B.C's are considered. Longitudinal inertia is ignored.
  - Different control designs are proposed such as PD, H_∞, disturbance compensation.
- Port-Hamiltonian DP modelling, T. Voss, J. Sherpen'14
  - Clamped B.C's are considered.
  - Quasi-static model is not stabilizable.
  - Fully dynamic model is stabilizable.
- No rigorous DP modeling and control results for
  - Cantilevered B.C's.
  - Electrostatic, quasi-static, fully dynamic models.
  - Numerical techniques!

Electrostatic - Nonlinear beam model- E-B assumptions

$$\begin{cases} \rho h \ddot{v} - \alpha_{11} h \left(v_x + \frac{1}{2} w_x^2\right)_x = 0 \\ \rho h \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xx} + \frac{\alpha_1 h^3}{12} w_{xxxx} - \\ \qquad \left[\left(\alpha_{11} h \left(v_x + \frac{1}{2} w_x^2\right) + \gamma_3 H_L \boxed{V(t)}\right) w_x\right]_x = 0, \\ \\ [v, w, w_x](0) = 0, \\ \left[\alpha_{11} h \left(v_x + \frac{1}{2} w_x^2\right)\right](L) = -\gamma_3 \boxed{V(t)}, \\ \left[\frac{\alpha_1 h^3}{12} w_{xx}\right](L) = -m(t), \\ \left[\frac{\rho h^3}{12} \ddot{w}_x - \frac{\alpha_1 h^3}{12} w_{xxx}\right](L) = g(t), \\ (v, w, \dot{v}, \dot{w})(x, 0) = (v_0, w_0, v_1, w_1). \end{cases}$$

- $v(x,t)$: stretching
- $w(x,t)$: bending
- $m(t), g(t)$: mechanical controllers; $V(t)$: Voltage
- Derivation: Full electro-magnetic effects (Maxwell's equations) $\mapsto$ Variational approach $\mapsto$ Discard magnetic effects.

How to stabilize? We try B^*-feedback law!

Euler-Bernoulli (E-B)	Mindlin-Timoshenko (M-T)
$V(t) = c_1 \left(\dot{v}(L,t) + \int_0^L w_x \dot{w}_x \; dx \right)$	$V(t) = c_4 \left(\cdots \text{ same } \cdots \right)$
$m(t) = -c_2 \dot{w}_x(L,t)$	$m(t) = -c_5 \dot{\psi}(L,t)$
$g(t) = c_3 \dot{w}_t(L,t)$	$g(t) = -c_6 \left(\cdots \text{ same } \cdots \right)$

Table: Stabilizing feedback controllers. Notice that the voltage controller $V(t)$ has the nonlinear term $\int_0^L w_x \dot{w}_x dx$. This is the contribution of nonlinearity to the $\mathcal{B}^*-$feedback law.

Energy is dissipative!

$$\frac{d\mathbf{E}(t)}{dt} = \begin{cases} -c_1 \left| \dot{v}(L,t) + \int_0^L w_x \dot{w}_x \; dx \right|^2 & \\ \qquad -c_2 |\dot{w}_x(L,t)|^2 - c_3 |\dot{w}(L,t)|^2 & \text{(E-B)} \\ -c_1 \left| \dot{v}(L,t) + \int_0^L w_x \dot{w}_x \; dx \right|^2 & \\ \qquad -c_2 |\dot{\psi}(L,t)|^2 - c_3 |\dot{w}(L,t)|^2 & \text{(M-T)} \end{cases} \leq 0.$$

- Analytic work for exponential stability is underway (preprint).

Nonlinear vs. Linear

Free the mechanical controllers $m(t) = g(t) \equiv 0$, and discard all nonlinear effects:

$$\begin{cases} \rho h \ddot{v} - \alpha_{11} h v_{xx} = 0, \\ \rho h \ddot{w} - \frac{\rho h^3}{12} \ddot{w}_{xx} + \frac{\alpha_1 h^3}{12} w_{xxxx} = 0, \\ \\ \left[v, w, w_x\right](0) = 0, \\ \alpha_{11} h v_x(L) = -\gamma_3 V(t), \\ \left[\frac{\alpha_1 h^3}{12} w_{xx}\right](L) = -\cancelto{0}{m(t)}, \\ \left[\frac{\rho h^3}{12} \ddot{w}_x - \frac{\alpha_1 h^3}{12} w_{xxx}\right](L) = \cancelto{0}{g(t)}, \\ (v, w, \dot{v}, \dot{w})(x, 0) = (v_0, w_0, v_1, w_1), \end{cases}$$

$V(t)$ can not control the bending motions anymore.

Exponential stabilization results for the linear model-B^*−feedback controllers

Let state-feedback be chosen as:

$$\mathbf{F}(t) = \begin{pmatrix} V(t) \\ m(t) \\ g(t) \end{pmatrix} = KB^*\varphi = \begin{pmatrix} -k_1\dot{v}(L) \\ k_2\dot{w}_x(L) \\ -k_3\dot{w}(L) \end{pmatrix} \quad (6)$$

- Exponential stability of the beam equation with $g(t), m(t) \neq 0$ (Rao'96, Guo'2002)
- $g(t)$ is not even necessary for exponential stability!
- Exponential stability of the wave equation (Triggiani'89, Zuazua'89)
- Exponential stability of the single piezo-beam model (Morris & Ozer-SICON'14)
- Lack of exp. stability of the fully dynamic model (A.O.Ozer-MCSS-'15); Exponential stability for certain number-theoretical conditions.
- Exp. stability of the three-layer laminate (Ozer-IEEE-TAC'17, EECT'18)

Goal: What kind of stability is obtained for the nonlinear models?

Consider the (E-B) model

Let $\mathbf{y} = [\mathbf{v}, \mathbf{w}, \dot{\mathbf{v}}, \dot{\mathbf{w}}]^{\mathrm{T}}$. Then,

$$\dot{\mathbf{y}} = (\mathcal{A} + \mathcal{N})\mathbf{y} + (\mathcal{B}_1 + \mathcal{B}_2\mathbf{y})\, u(t)$$

Define the natural energy space as

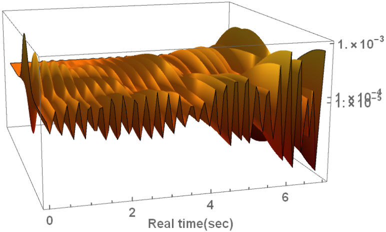
$$\mathrm{H} = H_L^1(0, L) \times H_L^2(0, L) \times L^2(0, L) \times H_L^1(0, L)$$

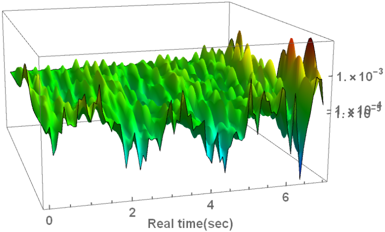
where $H_L^1(0, L) = \{z \in H^1(0, L) : z(0) = 0\}$,
$H_L^2(0, L) = \{z \in H^2(0, L) : z(0) = z_x(0) = 0\}$.

- $\mathcal{A}$ is an infinitesimal generator of a unitary semigroup on H,
- $\mathcal{N} : \mathrm{H} \to \mathrm{H}$ is locally Lipschitz.
- $\mathcal{B}_1 : \mathbb{C} \to \mathrm{H}$ is unbounded.
- $\mathcal{B}_2 : \mathbb{C} \to \mathrm{H}$ is bounded.

- Well-posedness is proved.
- Analytic work for exponential stability is underway (preprint).
- Different combos of boundary feedback and distributed damping are in consideration.
- Numerical work - Ozer& Khenner-SPIE-19. There are lots to be done!

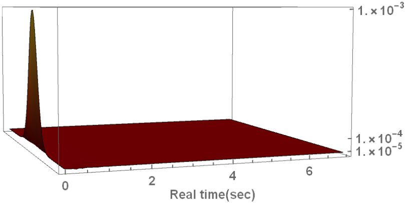
Euler Bernoulli: Uncontrolled



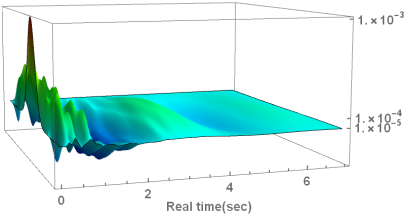


Euler Bernoulli: Fully controlled

Stretching - $v(x,t)$

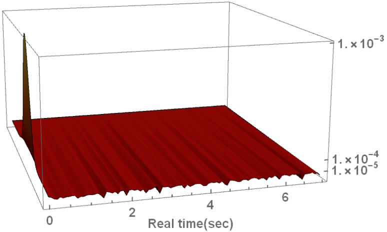


Bending - $w(x,t)$

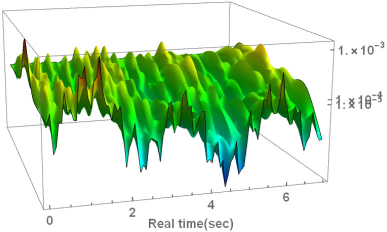


Euler Bernoulli: Partially controlled-I: $V(t) \neq 0, m(t), g(t) = 0$

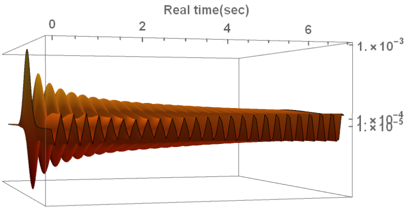
Stretching - $v(x,t)$



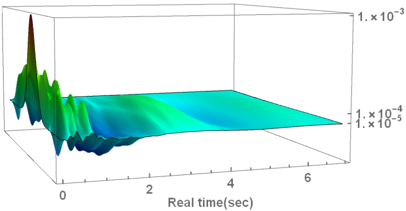
Bending - $w(x,t)$



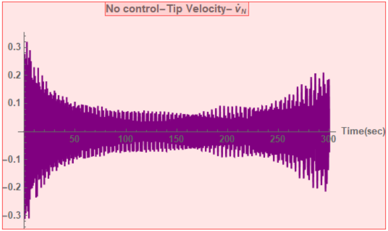
Stretching - $v(x,t)$

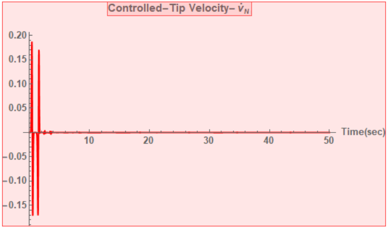


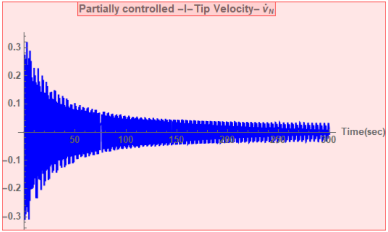
Bending - $w(x,t)$

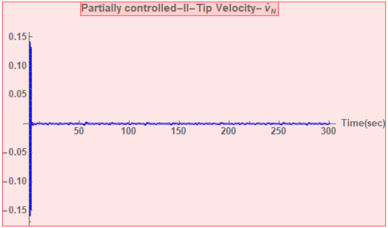


Euler Bernoulli: Tip velocities $\dot{v}(x,t)$-Scaled time

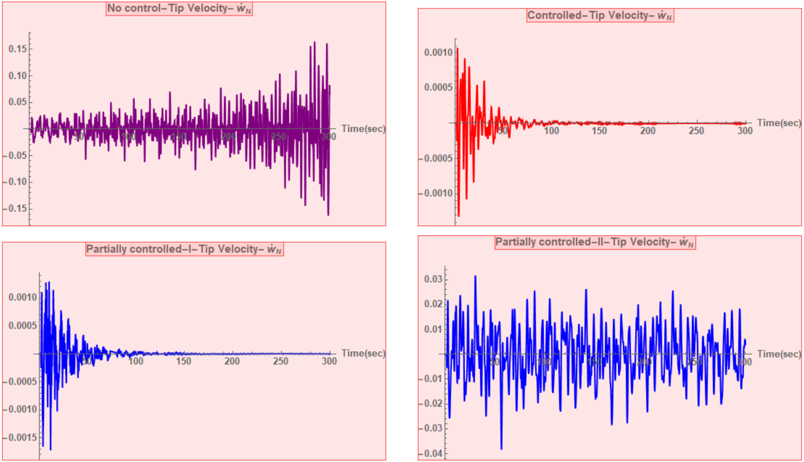




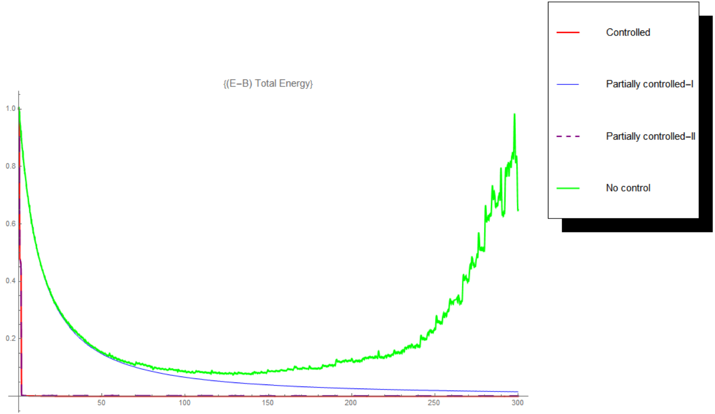




Euler Bernoulli: Tip velocities $\dot{w}(x,t)$-Scaled time



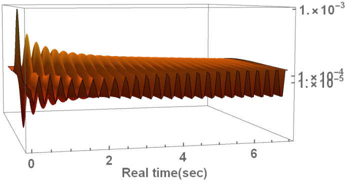
Normalized energies? (Scaled time)



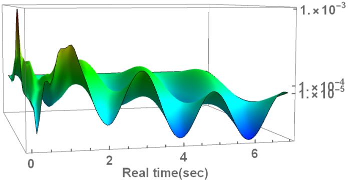
- Voltage controller $V(t)$ is strong.

Mindlin-Timoshenko: Uncontrolled

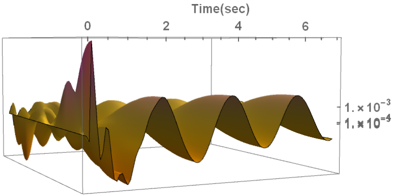
Stretching - $v(x,t)$



Bending - $w(x,t)$

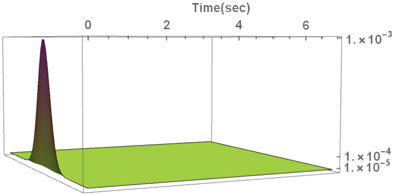


Rotation angle - $\psi(x,t)$

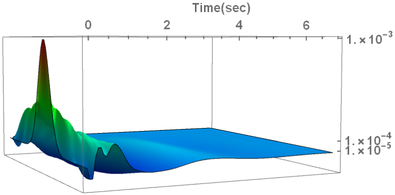


Mindlin-Timoshenko: Fully controlled

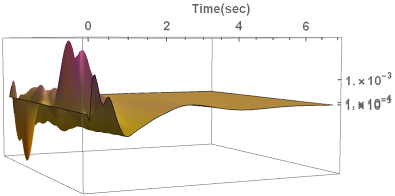
Stretching - $v(x,t)$



Bending - $w(x,t)$

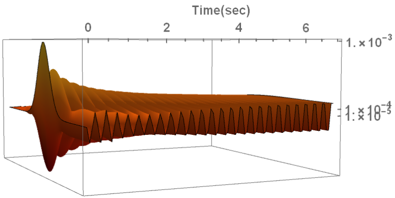


Rotation angle - $\psi(x,t)$

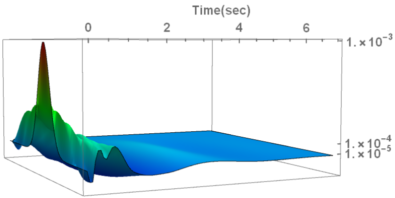


Mindlin-Timoshenko: Partially controlled: $V(t) = 0, \quad m(t), g(t) \neq 0$

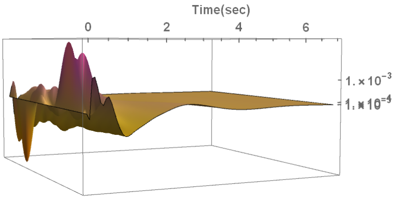
Stretching - $v(x,t)$



Bending - $w(x,t)$

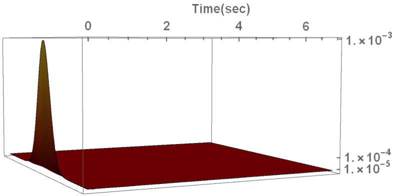


Rotation angle - $\psi(x,t)$

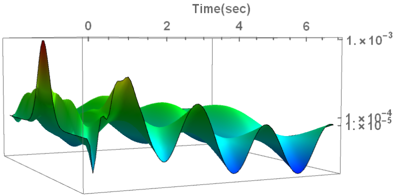


Mindlin-Timoshenko: Partially controlled: $V(t) \neq 0, m(t) = 0, g(t) = 0$

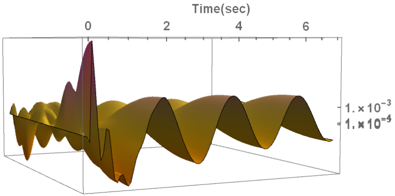
Stretching - $v(x,t)$



Bending - $w(x,t)$



Rotation angle - $\psi(x,t)$



Outline

Shocking — Known methods fail for boundary controlled systems!

- The continuous system is exp. stable but not the reduced model!
- First observed by H. T. Banks, K. Ito, C. Wang'91.
    - Many known techniques fail to mimic the stability behavior of the differential equations!
    - Finite Difference, Finite Element, Mixed-Finite Element, Galerkin, etc.
    - Stiff systems!

How about Filtered Finite Difference Method?

- Linear beam equation—L. Leon, E. Zuazua'02
- Linear wave equation— L.T. Tebou, E. Zuazua'07,
- Linear wave-equation— A. Marica, E. Zuazua'14,
- Nonlinear wave or beam equations (bounded feedback) —F. Alabau-Boussouira, Y. Privat, E. Trelat'17
- Filtering is necessary since the spurious (artifical or computer-generated) high frequency solutions destroy the approximated solution:
  For example, adding a damping term $dx^2\dot{u}_{xx}$ to the wave equation filters the artificial high-frequency solutions:

$$\ddot{u} - u_{xx} - (dx^2)\dot{u}_{xx} = 0, u(0,t) = 0, u_x(1,t) = \boxed{-\dot{u}(1,t).}$$

Various known techniques fail!

Example [Banks-Wang-90]: A one-dimensional wave equation (with boundary damping):

$$\begin{cases} \ddot{w} - w'' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ w(0,t) = 0, \quad w'(L,t) = -k\dot{w}(L,t), & t \in \mathbb{R}^+ \\ w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), & x \in (0,L) \end{cases}$$

Known that $\|w(x,t)\| \le C * e^{-\omega t}$. Equivalently, $\text{Max}(\text{Re}\{\lambda\}) < -\omega$.

Various known techniques fail!

Example [Banks-Wang-90]: A one-dimensional wave equation (with boundary damping):

$$\begin{cases} \ddot{w} - w'' = 0, & (x,t) \in (0,L) \times \mathbb{R}^+ \\ w(0,t) = 0, \quad w'(L,t) = -k\dot{w}(L,t), & t \in \mathbb{R}^+ \\ w(x,0) = w_0(x), \quad \dot{w}(x,0) = w_1(x), & x \in (0,L) \end{cases}$$

Known that $\|w(x,t)\| \le C * e^{-\omega t}$. Equivalently, $\text{Max}(\text{Re}\{\lambda\}) < -\omega$.
1) [Banks-Wang-90] Polynomial-based Galerkin approach:

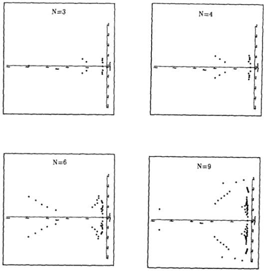
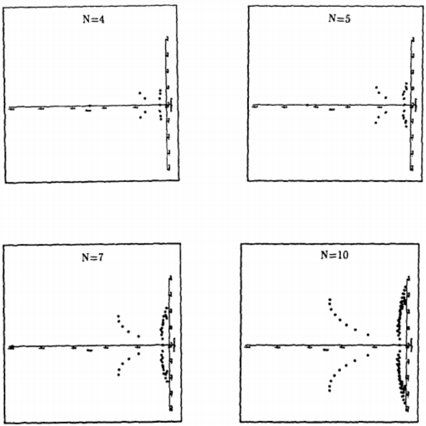


Figure 5.1: Locations of the eigenvalues of the matrix A^N for the polynomial based Galerkin method.

Various known techniques fail!

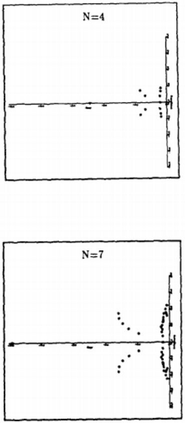
2) Linear spline based Galerkin approach:

Figure 5.2: Locations of the eigenvalues of the matrix A^N for the linear spline based Galerkin method.



3) Cubic spline based Galerkin approach:

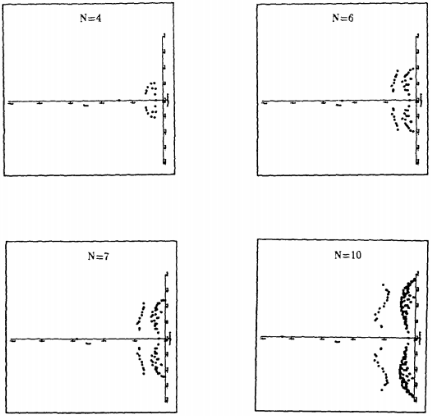
Figure 5.2: Locations of the eigenvalues of the matrix A^N for the linear spline based Galerkin method.



Various known techniques fail!

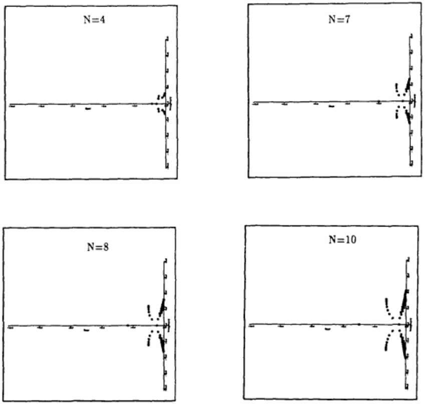
4) Finite Element approach:

Figure 5.4: Loaction of the eigenvalues of the matrix A^N for the finite element method.



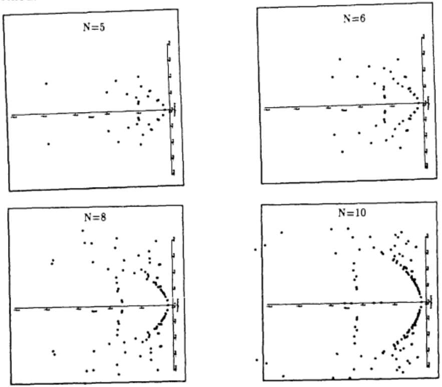
5) Finite Difference approach:

Figure 5.6: Locations of the eigenvalues of the matrix A^N for the finite-difference method.



6) Mixed Finite Element approach:

Figure 5.5: Location of the eigenvalues of the matrix A^N for the mixed finite element method.

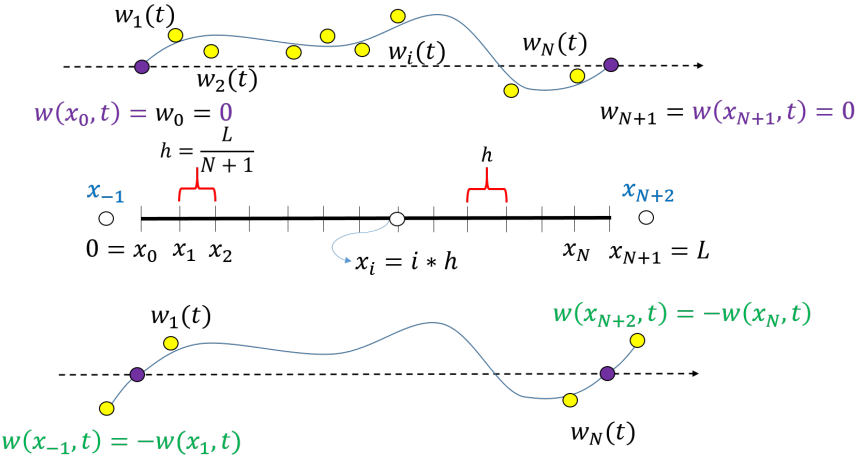


What has been done? Remedy?

- Filtering: (Infante & Zuazua'99, Leon & Zuazua'02, Tebou & Zuazua'06, Bugariu et al'15, Cindea et al'17)
  - *Direct Fourier Filtering*
  - Indirect Filtering by adding a viscosity term to the PDE
- Mixed Finite Element method or Polynomial based Galerkin methods: Glowinski et al'89, Castro & Micu'06
- Two-grid algorithms: Loreti & Mehrenberger, Negreanu & Zuazua'03
- Finite Difference Method without filtering: Liu & Guo'20

Semi-discrete Finite Difference Approximations - Rayleigh Beam

$$w(x_i, t) \approx w_i(t), \qquad i = 1,2,\dots,N$$



Discretized beam equation & Fictitious points

$$u_{xx} = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} + O(h^2),$$
$$u_{xxxx} = \frac{u(x_{i+2}) - 2u(x_{i+1}) + 6u(x_i) - 4u(x_{i-1}) + u(x_{i-2})}{h^4} + O(h^2).$$

$$\ddot{w}_j(t) - \alpha \frac{\ddot{w}_{j+1}(t) - 2\ddot{w}_j(t) + \ddot{w}_{j-1}(t)}{h^2}$$
$$+K \frac{w_{j+2}(t) - 4w_{j+1}(t) + 6w_j(t) - 4w_{j-1}(t) + w_{j-2}(t)}{h^4} = 0,$$

$$w_0 = w_{N+1} = 0,\ w_{-1} = -w_1,\ w_{N+2} = -w_N$$
$$w_j(0) = w_0^j,\ \dot{w}_j(0) = w_1^j, \quad j = 1, 2, \ldots, N.$$

Utilizing the eigenvalue analysis for the discrete $-\frac{d^2}{dx^2}$

Consider the central-finite difference approximation of the differential operator $-\frac{d^2}{dx^2}$ at x_j and the corresponding eigenvalue problem

$$-\frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{h^2} = \lambda\psi_j, \quad j = 1, 2, \ldots, N.$$

Letting $\vec{\psi} = [\psi_1, \psi_2, \ldots, \psi_N]$ and

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}$$

Denote the eigenvalues, i.e. $A_h\vec{\psi} = \lambda\vec{\psi}$ by

$$0 < \lambda_1(h) \le \lambda_2(h) \le \ldots \le \lambda_N(h).$$

Utilizing the eigenvalue analysis of A_h

Lemma

The eigenvalues λ_k *and eigenvectors* $\vec{\psi}^k = (\psi_{k,1}, \psi_{k,2}, \ldots, \psi_{k,N})$ *for* A_h *are*

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\left(\frac{\pi k h}{2L}\right), \quad k = 1, 2, \ldots, N,$$

$$\psi_{k,j} = \sin\left(\frac{j\pi k h}{L}\right), \quad k, j = 1, 2, \ldots, N.$$

Letting $\vec{w} = [w_1, w_2, \ldots, w_N]$, the model can be written as

$$\begin{cases} \ddot{\vec{w}} + \boxed{K(I + \alpha A_h)^{-1}(A_h)^2}\vec{w} = 0, \\ w_0 = w_N = 0, \ w_{-1} = -w_1, \ w_{N+2} = -w_N \\ \vec{w}(0) = \vec{w}_0, \ \dot{\vec{w}}(0) = \vec{w}_1. \end{cases}$$

Denote the eigenvalues of $K(I + \alpha A_h)^{-1}(-A_h)^2\vec{\varphi} = \tilde{\lambda}\vec{\varphi}$ by $0 < \tilde{\lambda}_1(h) \le \tilde{\lambda}_2(h) \le \ldots \le \tilde{\lambda}_N(h)$.

$$\tilde{\lambda}_k(h) = \frac{K}{\alpha}\lambda_k(h)\frac{1}{\frac{1}{\alpha\lambda_k(h)}+1}, \quad k = 1, 2, \ldots, N.,$$

and the corresponding eigenvectors $\vec{\varphi}^k = (\varphi_{k,1}, \varphi_{k,2}, \ldots, \varphi_{k,N})$ where

$$\varphi_{k,j} = \sigma_k \sin\left(\frac{j\pi kh}{L}\right), \quad k, j = 1, 2, \ldots, N.$$

and $\sigma_k = \sqrt{\frac{1}{K\lambda_k^3(h)\sum_{j=1}^{N}|\varphi_{k,j}|^2}}$ is the normalization constant. It is easy to check that $\lambda_k(h)h^2 < 4$ and therefore $\tilde{\lambda}_k(h)h^2 < 4\frac{K}{\alpha}$ for all $h > 0$. As well, $\tilde{\lambda}_N h^2 \to \frac{4K}{\alpha}$ as $h \to 0$.

Convergence fails!

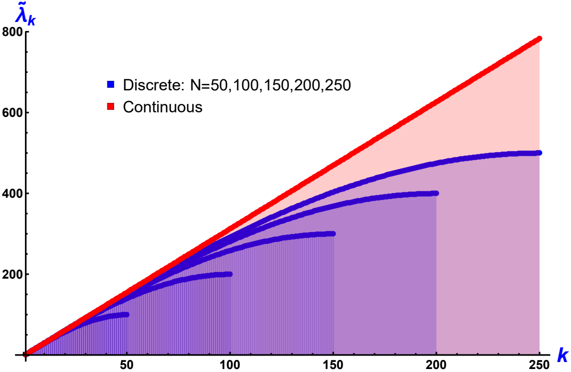


Figure: Discrete vs. continuous eigenvalues for $K = \alpha = 1$.

Uniform gap $\to$ 0... for larger k!!!

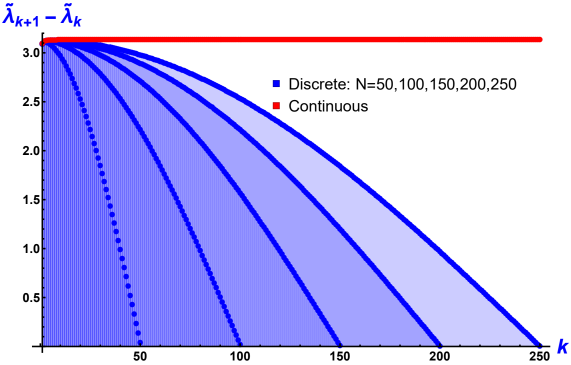


Figure: The uniform gap condition for the continuous eigenvalues does not hold anymore in the discrete case. For $K = \alpha = 1$, the gap $\tilde{\lambda}_{k+1}(h) - \tilde{\lambda}_k(h) \to 0$ as $h \to 0$.

First observed by Zuazua-Infante'99 for the wave equation.

Lack of Observability of the approximate model

Theorem (Ozer'19-IEEE-CDC)

Consider $\vec{w} = e^{i\sqrt{\tilde{\lambda}_N(h)}}\vec{\varphi}^N$. *For any* $T > 0$*, as* $h \to 0$

$$\sup_{\text{sol.s of approx.}} \frac{E_h(0)}{\int_0^T \left| \frac{w_{N+2} - 3w_{N+1} + 3w_N - w_{N-1}}{h^3} \right|^2 dt} \to \infty.$$

Theorem (Ozer-Hansen'11-MCSS)

One may also obtain that the operator $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ *is the generator of a unitary semigroup* $e^{\mathcal{A}t}$ *on* $\mathcal{H}$*. For given* $W_0 \in \mathcal{H}$*,* $W \in C[\mathbb{R}, \mathcal{H}]$*, and* $\dot{E}_0(t) = 0$*. Moreover, letting* $T > \frac{2L}{\sqrt{\frac{K}{\alpha}}}$*, there exists a constant* $C(T)$ *such that*

$$\underbrace{\boxed{\int_0^T |w'''(L,t)|^2 dt \geq C(T) \|W_0\|_E^2.}}_{\text{Observability inequality}}$$

Given $0 \leq \gamma < 4$, we introduce the class $\mathcal{C}_h(\gamma)$ of filtered solutions generated by the eigenvectors such that $\lambda h^2 \leq \gamma$. In particular,

$$C_h(\gamma) := \left\{ \vec{w}(t) = \sum_{\lambda(k)h^2 \leq \gamma} \left[a_k \sin\left(\sqrt{\tilde{\lambda}_k t}\right) + b_k \cos\left(\sqrt{\tilde{\lambda}_k t}\right)\right] \vec{\varphi}^k \right\}.$$

Theorem (Ozer-IEEE-CDC'19)

Assume that $0 < \gamma < 4$. *Then, there exists* $T(\gamma, h) = \frac{2\frac{\alpha}{K}(1+\frac{1}{\alpha\lambda_1})\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right)-\frac{\gamma h^2}{16}}}{1-\frac{\gamma}{4}} \geq 2L$ *such that for all* $T > T(\gamma, h)$ *there exists*

$$C(T,\gamma,h) = \frac{KL}{2\left[T\left(1-\frac{\gamma}{4}\right) - 2\frac{\alpha}{K}(1+\frac{1}{\alpha\lambda_1})\sqrt{L^2\left(1+\frac{3\gamma}{8\pi^2}\right)-\frac{\gamma h^2}{16}}\right]}$$

such that $E_h(0) \leq C(T,\gamma,h)\int_0^T \lambda_N^2 \left|\frac{w_N}{h}\right|^2 dt$ *holds for every solution in the class* $C_h(\gamma)$*, uniformly as* $h \to 0$.

$\boxed{\lambda_k(h)h^2 < \gamma}$ where γ is chosen:

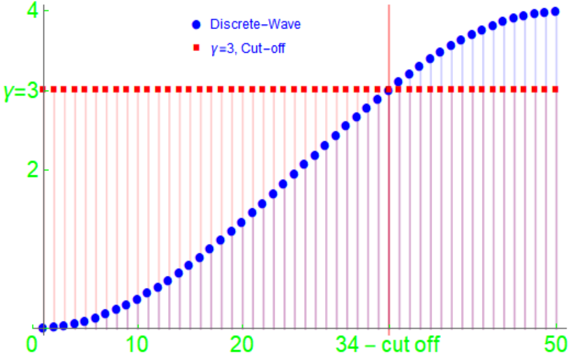


Figure: $N = 50, \gamma = 3$ corresponds that the eigen-frequencies $\tilde{\lambda}_{N\geq 34}$ are filtered

After direct filtering

$\boxed{\lambda_k(h)h^2 < \gamma}$ where γ is chosen:

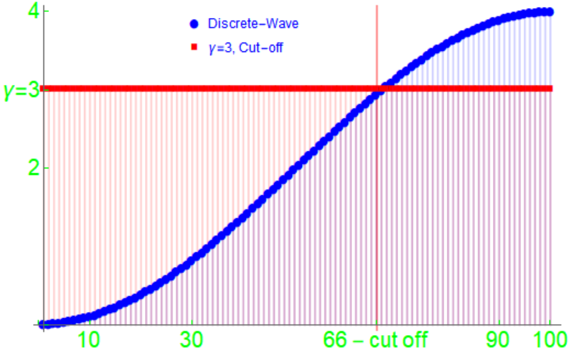


Figure: $N = 100, \gamma = 3$ corresponds that the eigen-frequencies $\tilde{\lambda}_{N\geq 66}$ are filtered

Ongoing research

- Mead-Marcus beam, Ozer'20-preprint

$$
\begin{aligned}
&\ddot{w} + w'''' - B^T v' = 0, \\
&-Cv'' + Pv = -Bw''', \quad (x,t) \in (0,L) \times \mathbb{R}^+ \\
&w(x,t), \;\; v'(x,t), \;\; w''(x,t)|_{x=0,L} = 0, \quad t \in \mathbb{R}^+ \\
&w(x,0) = w_0, \;\; \dot{w}(x,0) = w_1, \quad x \in (0,L)
\end{aligned}
$$

- Mead-Marcus beam, Ozer'20-preprint

$$\tilde{\lambda}_k(h) = (1 + B^T(CA_h + P)^{-1}B)\lambda_k^2, \quad k = 1, \ldots, N.,$$

and the corresponding eigenvectors $\vec{\varphi}^k = (\varphi_{k,1}, \varphi_{k,2}, \ldots, \varphi_{k,N})$ where

$$\varphi_{k,j} = \sigma_k \sin\left(\frac{j\pi kh}{L}\right), \quad k, j = 1, 2, \ldots, N.$$

Here notice that since A_h is a positive definite symmetric matrix, both $CA_h + P$ and $(CA_h + P)^{-1}$ are positive definite, and therefore the scalar $B^T(CA_h + P)^{-1}B$ is strictly positive.

- Mead-Marcus beam, Ozer'20-preprint

Let $\sigma_k = \sqrt{\frac{1}{(1+B^T(CA_h+P)^{-1}B)\lambda_k^4(h)\sum_{j=1}^{N}|\varphi_{k,j}|^2}}$ be the normalization constant. It is easy to check that $\lambda_k(h)h^2 < 4$ and therefore $\tilde{\lambda}_k(h)h^2 < 4(1 + B^T(CA_h + P)^{-1}B)$ for all $h > 0$. As well, $\tilde{\lambda}_N h^2 \to 4(1 + B^T(CA_h + P)^{-1}B)$ as $h \to 0$.

Outline

1. Fully Dynamic Single-layer piezoelectric beam models
   - Charge or Current-controlled
   - Voltage-controlled
2. Controllability results
   - Coulomb gauge fixing due to Maxwell's equations
   - Lorenz gauge fixing due to Maxwell's equations
   - How about Quasi-static or Electrostatic models?
   - Some Simulations
3. Results with Delay & Memory & Thermal effects & Fractional Damping
4. Nonlinear models vs. Linear models
5. Numerics - Lack of quality work in the literature
   - Toy Problem
6. Wolfram's Demonstration Projects

Other ongoing and future research

- Numerics for the piezoelectric beam, Ozer' & Wilson'20-preprint
- Numerics for the Laminate designs, Ozer'19-IFAC, Ozer'20-preprint
- *Observability* $\leftrightarrow$ *Controllability* $\leftrightarrow$ *Energy Harvesting*
- Wolfram Demonstration Projects (nontrivial laminate models)

Would like to be a Graduate Research Assistant in your second year?

- WKU-RCAP grant. Deadline is February 2020.
- KY NASA GF grant (Must be a US citizen). Deadline is April 2020.
- KY NSF EPSCoR RA Award. Deadline is ∼ April-May 2020.
- If you are interested, contact me at ozkan.ozer@wku.edu to have a Zoom meeting.
- If the grant is not funded, you still have GTA-ship. We still work on your MSc thesis.

Thanks for your attention.

- NSF EPSCoR Grant is greatly appreciated.

Any question?