Optimal control in finite strain elasticity and optimal design in electromechanics

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[4] Universidad Autónoma de Madrid

Online Seminar Control in Times of Crisis

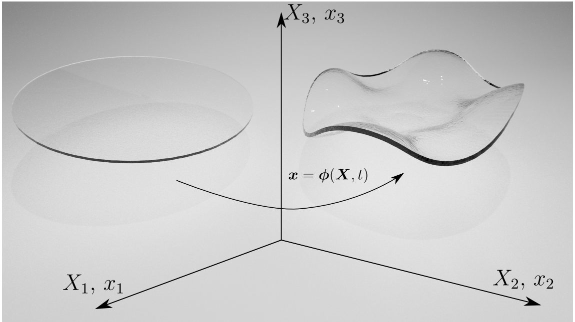
Outline

1. Control in finite elasticity
   - Finite Strain Elasticity
   - The optimal control problem
   - Numerical examples

2. Nonlinear electromechanics
   - Introduction
   - Background
   - The Design problem
   - Numerical examples

Outline

Large strain kinematics: $\boldsymbol{F}, \boldsymbol{H}, J$



$$\boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{\phi}; \qquad J = \det \boldsymbol{F}; \qquad \boldsymbol{H} = J\boldsymbol{F}^{-T}$$

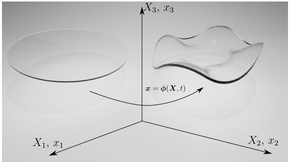
Finite Strain Elasticity: the BVP

- **Tensor cross product** [de Boer, 1982; Bonet, Gil and Ortigosa, 2015]:

$$(\boldsymbol{A} \times \boldsymbol{B})_{ij} = \mathcal{E}_{ikl}\mathcal{E}_{jmn}A_{km}B_{ln}$$

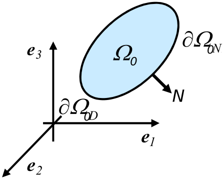
- The **area** and **volume** maps are redefined as:

$$\boxed{\boldsymbol{H} = \frac{1}{2}\boldsymbol{F} \times \boldsymbol{F}; \qquad J = \frac{1}{6}\left(\boldsymbol{F} \times \boldsymbol{F}\right) : \boldsymbol{F}}$$



- **Strong form (PDE)** in the Lagrangian formalism

$$\begin{aligned}
\nabla_0 \times \boldsymbol{F} &= 0; && \text{in } \Omega_0 && \Rightarrow \boldsymbol{F} = \nabla_0\phi\\
\nabla_0 \cdot \boldsymbol{P}(\boldsymbol{F}) + \boldsymbol{b}_0 &= \boldsymbol{0}; && \text{in } \Omega_0 && \boldsymbol{P} = \boldsymbol{\sigma H}\\
\phi &= \phi^*; && \text{on } \partial\Omega_{0,D} &&\\
\boldsymbol{PN} &= \boldsymbol{t}_0; && \text{on } \partial\Omega_{0,N} &&
\end{aligned}$$



- **Constitutive relation** through nonlinear stored energy functional $e(\boldsymbol{F})$:

$$\boxed{\boldsymbol{P}(\boldsymbol{F}) = \frac{\partial e(\boldsymbol{F})}{\partial \boldsymbol{F}}}$$

- How to define $e(\boldsymbol{F})$ being **mathematically and physically admissible?**

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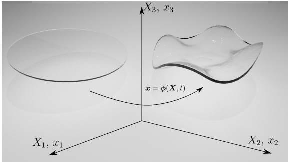

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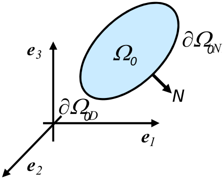

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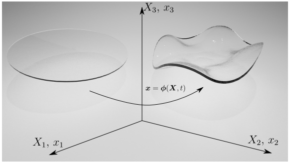

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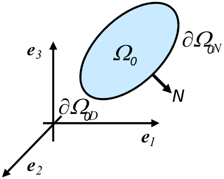

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Finite Strain Elasticity: polyconvexity

- **Polyconvexity conditon** [Ball, 1976; Schröder and Neff, 2004...]
- The internal strain energy functional e is expressed as a **CONVEX** multi-variable functional W in terms of **fibre**, **area** and **volume** maps:

$$e\left(\boldsymbol{F}\right) = \mathbb{W}\left(\boldsymbol{\mathcal{V}}\right); \qquad \boldsymbol{\mathcal{V}} = \left\{\boldsymbol{F}, \boldsymbol{H}(\boldsymbol{F}), J(\boldsymbol{F})\right\}$$

$$\mathbb{W}\left(\lambda\boldsymbol{\mathcal{V}}_1 + (1-\lambda)\,\boldsymbol{\mathcal{V}}_2\right) \leq \lambda\mathbb{W}\left(\boldsymbol{\mathcal{V}}_1\right) + (1-\lambda)\,\mathbb{W}\left(\boldsymbol{\mathcal{V}}_2\right) \quad \lambda \in [0,1]\,, \forall\boldsymbol{\mathcal{V}}_1, \boldsymbol{\mathcal{V}}_2 \in \boldsymbol{\mathcal{V}}$$

- **Polyconvex** constitutive models:

NH $\mathbb{W}\left(\boldsymbol{F}, J\right) = \dfrac{\mu}{2}||\boldsymbol{F}||^2 + f\left(J\right);$ $\qquad f\left(J\right) = -\mu \ln J + \dfrac{\lambda}{2}\left(J-1\right)^2$

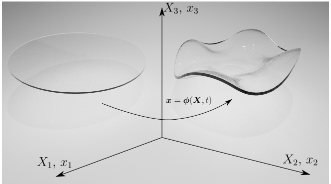
MR $\mathbb{W}\left(\boldsymbol{F}, \boldsymbol{H}, J\right) = \dfrac{\alpha}{2}||\boldsymbol{F}||^2 + \dfrac{\beta}{2}||\boldsymbol{H}||^2 + g\left(J\right);$ $\qquad g\left(J\right) = -\left(\alpha + 2\beta\right)\ln J + \dfrac{\lambda}{2}\left(J-1\right)^2$

Gent $\mathbb{W}\left(\boldsymbol{F}, J\right) = -\dfrac{\mu J_m}{2}\ln\left(1 - \dfrac{||\boldsymbol{F}||^2 - 3}{J_m}\right) + f\left(J\right);$

Finite Strain Elasticity: the weak form

- The **PDE** and boundary conditions

$$\boldsymbol{\nabla}_0 \times \boldsymbol{F} = \boldsymbol{0} \ in \ \Omega_0 \qquad \Rightarrow \qquad \boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{\phi}$$
$$\boldsymbol{\nabla}_0 \cdot \boldsymbol{P} + \boldsymbol{b}_0 = \boldsymbol{0} \ in \ \Omega_0; \qquad \boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{H}$$
$$\boldsymbol{\phi} = \overline{\boldsymbol{\phi}} \ on \ \partial\Omega_{0,D}$$
$$\boldsymbol{P}\boldsymbol{N} = \boldsymbol{T} \ on \ \partial\Omega_{0,N}$$



- For $\phi \in H^1(\Omega_0)$, multiplication by **test function** $\delta\phi \in H_0^1(\Omega_0)$ leads to the **weak form**

$$\mathcal{W}(\phi) = \underbrace{\int_{\Omega_0} \boldsymbol{P}(\boldsymbol{F}) : \boldsymbol{\nabla}_0 \delta\boldsymbol{\phi} \, dV}_{\text{Elastic work}} - \underbrace{\left(\int_{\Omega_0} \boldsymbol{f}_0 \cdot \delta\boldsymbol{\phi} \, dV + \int_{\partial\Omega_{0N}} \boldsymbol{t}_0 \cdot \delta\boldsymbol{\phi} \, dA \right)}_{\text{External work}} = 0$$

- **Stationary condition** of energy functional:

$$\Pi(\boldsymbol{\phi}) = \inf_{\boldsymbol{\phi}} \left\{ \underbrace{\int_{\Omega_0} \mathbb{W}(\boldsymbol{F}, \boldsymbol{H}, J) \, dV}_{\text{Elastic energy}} - \underbrace{\left(\int_{\Omega_0} \boldsymbol{f}_0 \cdot \boldsymbol{\phi} \, dV + \int_{\partial\Omega_{0N}} \boldsymbol{t}_0 \cdot \boldsymbol{\phi} \, dA \right)}_{\text{external energy}} \right\}$$

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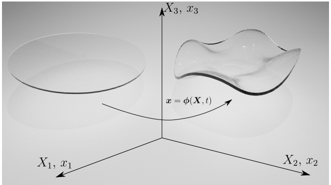

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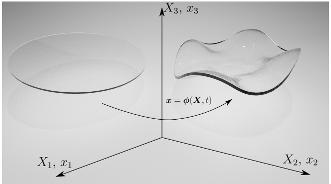

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Finite Strain Elasticity: existence of minimisers

- **Energy functional** to be minimised:

$$\Pi(\boldsymbol{\phi}) = \inf_{\boldsymbol{\phi}} \left\{ \underbrace{\int_{\Omega_0} \mathbb{W}(\boldsymbol{F}, \boldsymbol{H}, J)\, dV}_{\text{Elastic energy}} - \underbrace{\left(\int_{\Omega_0} \boldsymbol{f}_0 \cdot \boldsymbol{\phi}\, dV + \int_{\partial\Omega_{0N}} \boldsymbol{t}_0 \cdot \boldsymbol{\phi}\, dA \right)}_{\text{external energy}} \right\}$$

[1] Polyconvexity: for $\boldsymbol{\mathcal{V}} = \{\boldsymbol{A}, \boldsymbol{B}, c\}$, $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{3\times 3}$, $c \in \mathbb{R}^+$

$$\mathbb{W}\left(\lambda \boldsymbol{\mathcal{V}}_1 + (1-\lambda)\boldsymbol{\mathcal{V}}_2\right) \leq \lambda \mathbb{W}\left(\boldsymbol{\mathcal{V}}_1\right) + (1-\lambda)\,\mathbb{W}\left(\boldsymbol{\mathcal{V}}_2\right) \quad \lambda \in [0,1]\,, \forall \boldsymbol{\mathcal{V}}_1, \boldsymbol{\mathcal{V}}_2 \in \boldsymbol{\mathcal{V}}$$

[2] Coerciveness: for $p \geq 2$, $\frac{1}{p} + \frac{1}{q} \leq 1$, $r \geq 1$

$$\mathbb{W}\left(\boldsymbol{\mathcal{V}}\right) \geq \alpha \left(||\boldsymbol{F}||^p + ||\boldsymbol{H}||^q + J^r \right) + \beta \qquad \alpha > 0$$

[3] Limit behaviour:

$$\lim_{J \to 0^+} \mathbb{W}\left(\boldsymbol{F}, \boldsymbol{H}, J\right) = +\infty$$

Biomimetic actuation

- **Biomimetic**-type actuation

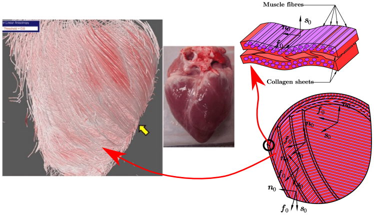
$$\mathbb{W}_{\text{fiber}}(\boldsymbol{F}, m) = \mathbb{W}(\boldsymbol{F}) + \mathbb{W}_{\text{m}}(\boldsymbol{F}, m)$$

$$\mathbb{W}_{\text{m}}(\boldsymbol{F}, m) = -\frac{m}{2}||\boldsymbol{F}\boldsymbol{a}||^2$$

$$\boldsymbol{a} : \boldsymbol{a}(\boldsymbol{X}) : \Omega_0 \to \mathbb{R}^3$$

$$m : m(\boldsymbol{X}) : \Omega_0 \to \mathbb{R}$$

$$\boldsymbol{P}(\boldsymbol{F}, m) = \partial_{\boldsymbol{F}}\mathbb{W}(\boldsymbol{F}) - m\boldsymbol{F}\boldsymbol{a} \otimes \boldsymbol{a}$$



- For Mooney-Rivlin model, **polyconvexity** and **coerciveness** guaranteed if:

$$\mathbb{W}(\boldsymbol{F}, \boldsymbol{H}, J) = \frac{\alpha}{2}||\boldsymbol{F}||^2 + \frac{\beta}{2}||\boldsymbol{H}||^2 + g(J)$$

$$\sup_{\boldsymbol{X} \in \Omega_0} m(\boldsymbol{X}) < \alpha$$

- **Stationary condition** of energy functional:

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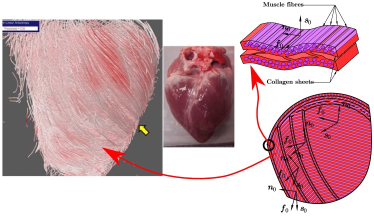

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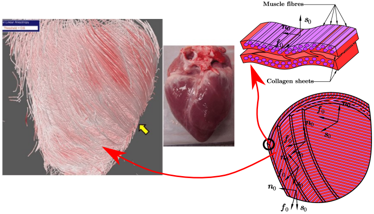

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The optimal control problem I

- **The Optimal Control** problem

$$
\begin{cases}
\text{Minimize in } m: & \mathcal{J}(m) \\
\text{subject to} & m \in L^2(\Omega_0) \\
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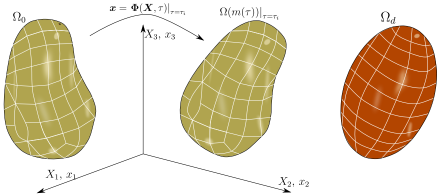

A **Hausdorff** distance-based objective function

$$\mathcal{J}(m) = \rho_H\left(\Omega(m), \Omega_d\right)$$

$$\rho_H\left(\Omega(m), \Omega_d\right) := \max\left\{ \sup_{\boldsymbol{x}\in\Omega_d} d_{\Omega(m)}\left(\boldsymbol{x}\right),\ \sup_{\boldsymbol{x}\in\Omega(m)} d_{\Omega_d}\left(\boldsymbol{x}\right) \right\}$$

$$d_{\Omega}\left(\boldsymbol{x}\right) := \inf_{\boldsymbol{y}\in\Omega} \|\boldsymbol{y} - \boldsymbol{x}\|$$

Regularised Hausdorff functional

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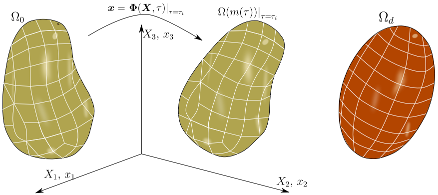

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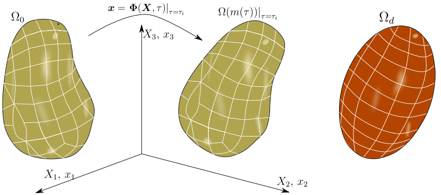

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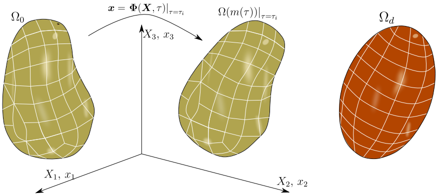
- **The Optimal Control** problem

$$\begin{cases} \text{Minimize in } m: & \mathcal{J}(m) \\ \text{subject to} & m \in L^2(\Omega_0) \\ & m(\boldsymbol{X}) \leq C \text{ a.e. } \boldsymbol{X} \in \Omega_0 \\ & \boldsymbol{\phi} \text{ is a minimizer of } \Pi(\boldsymbol{\phi}, m) \end{cases}$$



A **Hausdorff** distance-based objective function

$$\mathcal{J}(m) = \rho_H\left(\Omega(m), \Omega_d\right)$$

$$\rho_H\left(\Omega(m), \Omega_d\right) := \max\left\{\sup_{\boldsymbol{x}\in\Omega_d} d_{\Omega(m)}\left(\boldsymbol{x}\right),\ \sup_{\boldsymbol{x}\in\Omega(m)} d_{\Omega_d}\left(\boldsymbol{x}\right)\right\}$$

$$d_{\Omega}\left(\boldsymbol{x}\right) := \inf_{\boldsymbol{y}\in\Omega} \|\boldsymbol{y}-\boldsymbol{x}\|$$

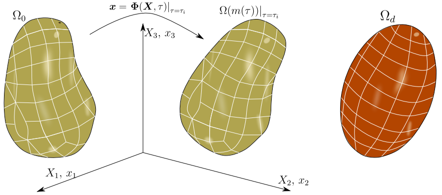
Regularised Hausdorff functional

$$\boxed{\mathcal{J}(m) = \tilde{\rho}_H\left(\Omega(m), \Omega_d\right)}$$

The optimal control problem II

- **The Optimal Control** problem

$$\begin{cases} \text{Minimize in } m: & \mathcal{J}(m) \\ \text{subject to} & m \in L^2(\Omega_0) \\ & m(\boldsymbol{X}) \leq C \text{ a.e. } \boldsymbol{X} \in \Omega_0 \\ & \boldsymbol{\phi} \text{ is a minimizer of } \Pi(\boldsymbol{\phi}, m) \end{cases}$$



$$\mathcal{L}\left(\overline{\boldsymbol{\phi}}, \overline{\boldsymbol{p}}, \overline{m}\right) = \mathcal{J}\left(\overline{\boldsymbol{\phi}}, \overline{m}\right) - \int_{\Omega_0} \boldsymbol{P}(\boldsymbol{\nabla}_0 \overline{\boldsymbol{\phi}}, \overline{m}) : \boldsymbol{\nabla}_0 \overline{\boldsymbol{p}}\, dX$$

$$\mathcal{J}\left(\overline{\boldsymbol{\phi}}, \overline{m}\right) = \tilde{\rho}_H\left(\overline{\boldsymbol{\phi}}(\Omega_0), \Omega_d\right) + \frac{M}{2}\int_{\Omega_0} \overline{m}^2(\boldsymbol{X})\, dX + \frac{\varepsilon}{2}\int_{\Omega_0} |\boldsymbol{\nabla}_0 \overline{m}(\boldsymbol{X})|^2\, dX$$

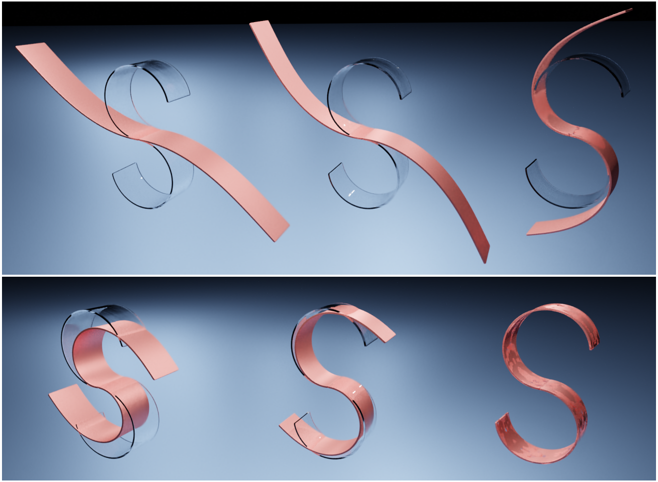
$$\frac{\partial \mathcal{L}}{\partial \overline{\boldsymbol{p}}}(\overline{\boldsymbol{\phi}}, \overline{\boldsymbol{p}}, \overline{m})(\boldsymbol{v}) = \frac{\partial \Pi}{\partial \overline{\boldsymbol{\phi}}}(\overline{\boldsymbol{\phi}}, \overline{m})(\boldsymbol{v}) = \int_{\Omega_0} \boldsymbol{P}(\boldsymbol{\nabla}_0 \overline{\boldsymbol{\Phi}}, \overline{m}) : \boldsymbol{\nabla}_0 \boldsymbol{v}\, dX = 0 \quad \forall \boldsymbol{v} \in H_0^1(\Omega_0)$$

$$\frac{\partial \mathcal{L}}{\partial \overline{\boldsymbol{\Phi}}}(\overline{\boldsymbol{\Phi}}, \overline{\boldsymbol{p}}, \overline{m})(\boldsymbol{v}) = \frac{\partial \tilde{\rho}_H}{\partial \overline{\boldsymbol{\Phi}}}(\overline{\boldsymbol{\Phi}})(\boldsymbol{v}) - \int_{\Omega_0} \boldsymbol{\nabla}_0 \overline{\boldsymbol{p}} : \boldsymbol{\mathcal{C}}(\boldsymbol{\nabla}_0 \overline{\boldsymbol{\Phi}}, \overline{m}) : \boldsymbol{\nabla}_0 \boldsymbol{v}\, dX = 0 \quad \forall \boldsymbol{v} \in H_0^1(\Omega_0)$$

$$\boldsymbol{\mathcal{C}}_{iIjJ} = \left(\boldsymbol{\nabla}^2_{\boldsymbol{FF}} W(\boldsymbol{F}, \boldsymbol{H}(\boldsymbol{F}), J(\boldsymbol{F}))\right)_{iIjJ} - \delta_{ij} m a_I a_J$$

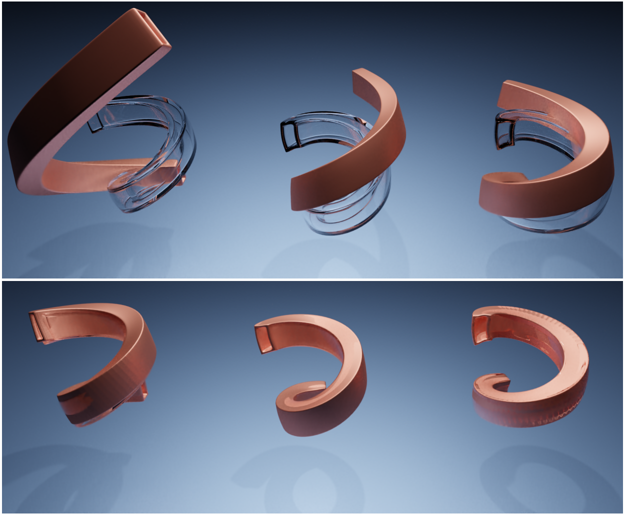
$$\frac{\partial \mathcal{L}}{\partial \overline{m}}(\overline{\boldsymbol{\Phi}}, \overline{\boldsymbol{p}}, \overline{m})(\hat{m}) = \int_{\Omega_0} \left[\hat{m}\left((\boldsymbol{F}\boldsymbol{a} \otimes \boldsymbol{a}) : \boldsymbol{\nabla}_0 \boldsymbol{p}\right) + M\, \overline{m}\, \hat{m} + \varepsilon\, \boldsymbol{\nabla}_0 \overline{m} \cdot \boldsymbol{\nabla}_0 \hat{m}\right] dX$$

Numerical example I



R. Ortigosa, J. Martínez-Frutos, C. Mora-Corral, P. Pedregal and F. Periago. *Optimal control of soft materials using a Hausdorff distance functional*, **SICON**, *accepted.*

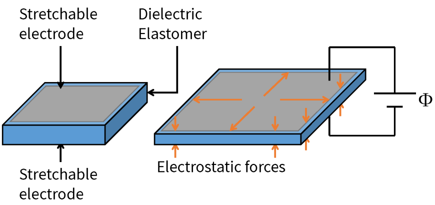
Numerical example II



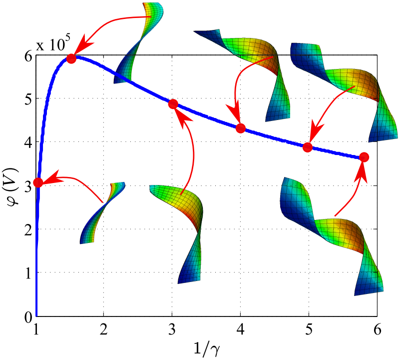
R. Ortigosa, J. Martínez-Frutos, C. Mora-Corral, P. Pedregal and F. Periago. *Optimal control of soft materials using a Hausdorff distance functional*, **SICON**, *accepted*.

Outline

Finite Strain Electromechanics: motivation



Physical principle: Coulomb forces



In-house simulation

Giant electrically induced deformations in EAP membrane
[**Suo 2013 (Harvard Lab), Max Plank Press 2016**]

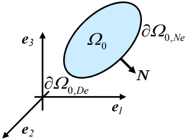
[2016 - ongoing]

Tunable EAP materials
Low voltage laminates
Space/subaquatic explorations

Finite Strain Electromechanics: the BVP

- Material strong form of **electrostatics** (e) in terms of $\boldsymbol{E}_0, \boldsymbol{D}_0$:

$$
\begin{aligned}
\boldsymbol{\nabla}_0 \times \boldsymbol{E}_0 &= \boldsymbol{0} \ in\ \Omega_0 && \Rightarrow && \boldsymbol{E}_0 = -\boldsymbol{\nabla}_0 \varphi = \boldsymbol{F}^T \boldsymbol{E} \\
\boldsymbol{\nabla}_0 \cdot \boldsymbol{D}_0 - \rho_0 &= 0 \ in\ \Omega_0; && && \boldsymbol{D}_0 = \boldsymbol{H}^T \boldsymbol{D} \\
\varphi &= \overline{\varphi} \ on\ \partial\Omega_{0,De} \\
\boldsymbol{D}_0 \cdot \boldsymbol{N} &= -\omega_0 \ on\ \partial\Omega_{0,Ne}
\end{aligned}
$$



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\boldsymbol{\nabla}_0 \cdot \boldsymbol{P} + \boldsymbol{b}_0 &= \boldsymbol{0} \ in\ \Omega_0; && && \boldsymbol{P} = \boldsymbol{\sigma} \boldsymbol{H} \\
\boldsymbol{\phi} &= \overline{\boldsymbol{\phi}} \ on\ \partial\Omega_{0,Dm} \\
\boldsymbol{P}\boldsymbol{N} &= \boldsymbol{T} \ on\ \partial\Omega_{0,Nm}
\end{aligned}
$$

- With a **coupling constitutive law** (Ogden & Dorfmann, Bustamante, McMeeking & Landis, Suo & Zhao)

$$
e(\boldsymbol{F}, \boldsymbol{D}_0) = e_m(\boldsymbol{F}) + e_c(\boldsymbol{F}, \boldsymbol{D}_0)
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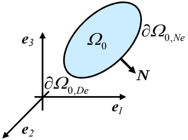

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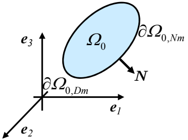

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$$
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- **Constitutive relations**:

$$
\boldsymbol{P} = \frac{\partial e(\boldsymbol{F}, \boldsymbol{D}_0)}{\partial \boldsymbol{F}} \qquad \boldsymbol{E}_0 = \frac{\partial e(\boldsymbol{F}, \boldsymbol{D}_0)}{\partial \boldsymbol{D}_0}
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Background

Finite Strain Electromechanics: the BVP

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$$

Finite Strain Electromechanics: polyconvexity

- Classical approach [Ogden & Dorfmann, Bustamante, McMeeking & Landis, Suo & Zhao, Vu & Steinman]

$$e\left(\boldsymbol{F},\boldsymbol{D}_0\right)=e_m\left(\boldsymbol{F}\right)+e_e\left(\boldsymbol{F},\boldsymbol{D}_0\right)$$

- For **polyconvex** reversible electromechanics:

$$\boxed{e\left(\boldsymbol{F},\boldsymbol{D}_0\right)=\mathbb{W}\left(\boldsymbol{\mathcal{V}}\right);\qquad \boldsymbol{\mathcal{V}}=\left\{\boldsymbol{F},\boldsymbol{H},J,\boldsymbol{D}_0,\boldsymbol{d}\right\};\qquad \boldsymbol{d}=\boldsymbol{F}\boldsymbol{D}_0}$$

$$\boxed{\mathbb{W}\left(\lambda\boldsymbol{\mathcal{V}}_1+(1-\lambda)\,\boldsymbol{\mathcal{V}}_2\right)\leq\lambda\mathbb{W}\left(\boldsymbol{\mathcal{V}}_1\right)+(1-\lambda)\,\mathbb{W}\left(\boldsymbol{\mathcal{V}}_2\right)\quad \lambda\in[0,1]\,,\forall\boldsymbol{\mathcal{V}}_1,\boldsymbol{\mathcal{V}}_2\in\boldsymbol{\mathcal{V}}}$$

- A possible **polyconvex** electrically enhanced model is:

$$\mathbb{W}\left(\boldsymbol{F},\boldsymbol{H},J,\boldsymbol{D}_0,\boldsymbol{d}\right)=\underbrace{\mu_1||\boldsymbol{F}||^2+\mu_2||\boldsymbol{H}||^2+f(J)}_{\text{Mooney-Rivlin model}}+\underbrace{\frac{1}{2\varepsilon_2}||\boldsymbol{D}_0||^2+\frac{1}{2\varepsilon_1}||\boldsymbol{d}||^2}_{\text{Electro-mechanical term}}$$

- Constitutive relations:

$$\boldsymbol{P}=\frac{\partial e\left(\boldsymbol{F},\boldsymbol{D}_0\right)}{\partial\boldsymbol{F}}\qquad \boldsymbol{P}=\frac{\partial\mathbb{W}}{\partial\boldsymbol{F}}+\frac{\partial\mathbb{W}}{\partial\boldsymbol{H}}\boldsymbol{\times}\boldsymbol{F}+\frac{\partial\mathbb{W}}{\partial J}\boldsymbol{H}+\frac{\partial\mathbb{W}}{\partial\boldsymbol{d}}\otimes\boldsymbol{D}_0$$

$$\boldsymbol{E}_0=\frac{\partial e\left(\boldsymbol{F},\boldsymbol{D}_0\right)}{\partial\boldsymbol{D}_0}\qquad \boldsymbol{E}_0=\frac{\partial\mathbb{W}}{\partial\boldsymbol{D}_0}+\boldsymbol{F}^T\frac{\partial\mathbb{W}}{\partial\boldsymbol{d}}$$

A. J. Gil and R. Ortigosa. *A new framework for large strain electromechanics based on convex multi-variable strain energies: Variational formulation and material characterisation*, **CMAME** 302 (2016) 293-328.

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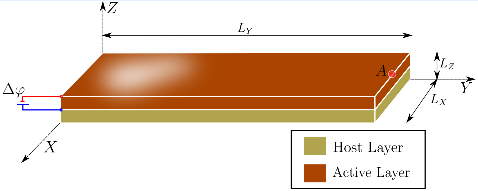
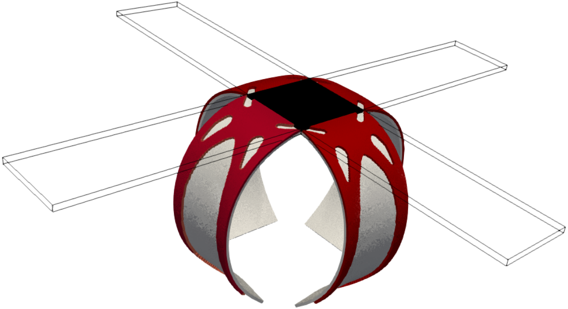
A. J. Gil and R. Ortigosa. *A new framework for large strain electromechanics based on convex multi-variable strain energies: Variational formulation and material characterisation*, **CMAME** 302 (2016) 293-328.

Background

Topology optimisation of Electro Active material

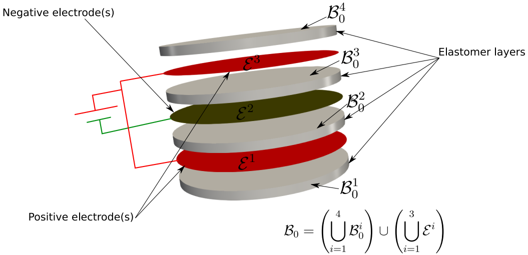




R. Ortigosa, J. Martínez-Frutos, D. Ruíz, A. Donoso and J. C. Bellido. *Density-based Topology Optimisation considering nonlinear electromechanics*, **SMO** *in print*.

Fabrication principle and the engineering objective I

- **Layer-by-layer** design layout: **engineering principles**



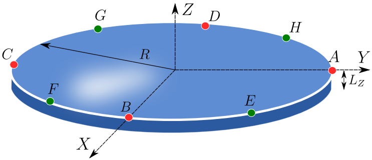
I. $\mathcal{B}_0$ is comprised of the union of N_L elastomeric layers, $\{\mathcal{B}_0^1, \dots, \mathcal{B}_0^{N_L}\}$.
II. Electrodes intercalated between adjacent elastomer layers.
III. Negligible thickness of electrodes, placed at *surface* regions $\{\mathcal{E}^1, \dots, \mathcal{E}^{N_L-1}\}$,

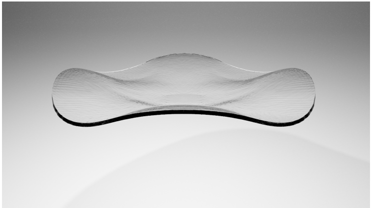
$$\mathcal{E}^i = \bar{\mathcal{B}}_0^i \cap \bar{\mathcal{B}}_0^{i+1}; \quad i = \{1, \dots, N_L - 1\}.$$

IV. Electrode regions subjected to same voltage value, either positive or negative.

Fabrication principle and the engineering objective II

- **Layer-by-layer** design layout: **the objective**

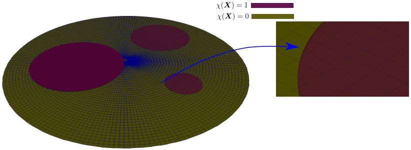


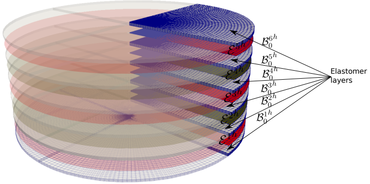


Electrode design Objective:
$$\begin{cases} \text{maximise } u_Z \text{ in points } \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\} \\ \text{minimise } u_Z \text{ in points } \{\mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}\} \end{cases}$$

The design problem: computational considerations I

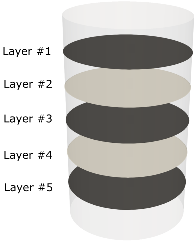
- **Layer-by-layer** design layout: **computational considerations**





- **Characteristic function** approach:

$$\forall \boldsymbol{\xi} \in \mathcal{E}^i, \begin{cases} \chi_{\mathcal{E}^i}(\boldsymbol{\xi}) = 1; & \text{if } \boldsymbol{\xi} \in \mathcal{E}^i_{\text{Elec}}; \\ \chi_{\mathcal{E}^i}(\boldsymbol{\xi}) = 0; & \text{if } \boldsymbol{\xi} \in \mathcal{E}^i \setminus \mathcal{E}^i_{\text{Elec}}. \end{cases}$$



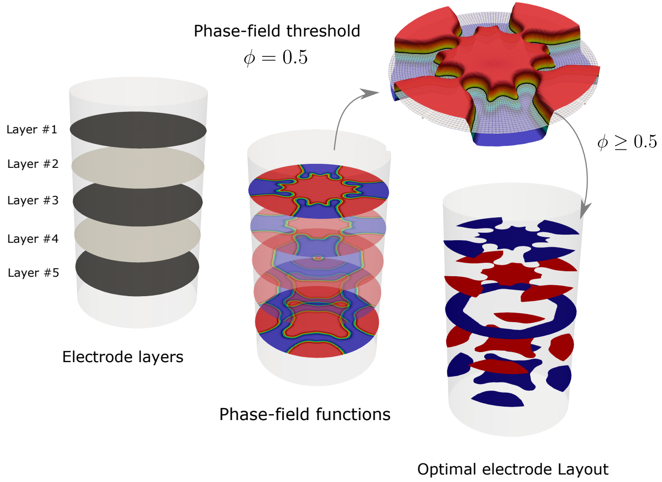
Electrode layers

Optimal electrode Layout

Regularisation of the problem II

- **Smooth representation** of electrodes within $\mathcal{E}^i$ via an in-surface **phase-field function** $\psi_{\mathcal{E}^i}$.

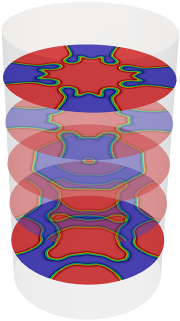
$$\mathbb{V}^{\psi_{\mathcal{E}^i}} = \left\{\psi_{\mathcal{E}^i} : \mathcal{E}^i \to [0,1]\,, \qquad \psi_{\mathcal{E}^i} \in H^1(\mathcal{E}^i)\right\}; \quad i = \{1,\dots,N_{L-1}\},$$



Regularisation of the problem III

I **Smooth representation** of electrodes within $\mathcal{E}^i$ via an in-surface **phase-field function** $\psi_{\mathcal{E}^i}$.

$$\mathbb{V}^{\psi_{\mathcal{E}^i}} = \left\{\psi_{\mathcal{E}^i} : \mathcal{E}^i \to [0,1], \quad \psi_{\mathcal{E}^i} \in H^1(\mathcal{E}^i)\right\}; \quad i = \{1, \dots, N_L -$$



II **Smooth extension of $\psi_{\mathcal{E}^i}$ into the volume** yielding the phase field function $\psi_{\mathcal{B}_0}$.

$$\Lambda(\psi_{\mathcal{E}}, \psi^*_{\mathcal{B}_0}) = \inf_{\psi_{\mathcal{B}_0} \in \mathbb{V}^{\psi_{\mathcal{B}_0}}} \left\{ \int_{\mathcal{B}_0} \frac{1}{2} |\nabla_0 \psi_{\mathcal{B}_0}|^2 \, dV \right\}$$

$$\mathbb{V}^{\psi_{\mathcal{B}_0}} = \left\{ \begin{array}{l} \psi_{\mathcal{B}_0} : \mathcal{B}_0 \to [0,1], \quad \psi_{\mathcal{B}_0} \in H^1(\mathcal{B}_0) \\ \text{s.t.} \quad \psi_{\mathcal{B}_0} = \psi_{\mathcal{E}^i} \text{on } \mathcal{E}^i, \; i = \{1, \dots, N_L - 1\} \end{array} \right\}$$

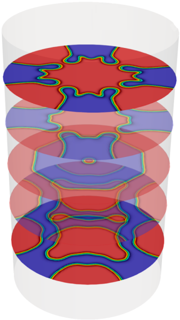
Regularisation of the problem III

I **Smooth representation** of electrodes within $\mathcal{E}^i$ via an in-surface **phase-field function** $\psi_{\mathcal{E}^i}$.

$$\mathbb{V}^{\psi_{\mathcal{E}^i}} = \left\{ \psi_{\mathcal{E}^i} : \mathcal{E}^i \to [0,1] , \quad \psi_{\mathcal{E}^i} \in H^1(\mathcal{E}^i) \right\} ; \quad i = \{1, \ldots, N_{L-}$$

II **Smooth extension of $\psi_{\mathcal{E}^i}$ into the volume** yielding the phase field function $\psi_{\mathcal{B}_0}$.

$$\boxed{\begin{aligned} \Lambda(\boldsymbol{\psi}_{\mathcal{E}}, \psi^*_{\mathcal{B}_0}) &= \inf_{\psi_{\mathcal{B}_0} \in \mathbb{V}^{\psi_{\mathcal{B}_0}}} \left\{ \int_{\mathcal{B}_0} \frac{1}{2} |\boldsymbol{\nabla}_0 \psi_{\mathcal{B}_0}|^2 \, dV \right\} \\ \mathbb{V}^{\psi_{\mathcal{B}_0}} &= \left\{ \begin{array}{l} \psi_{\mathcal{B}_0} : \mathcal{B}_0 \to [0,1] , \quad \psi_{\mathcal{B}_0} \in H^1(\mathcal{B}_0) \\ \text{s.t.} \quad \psi_{\mathcal{B}_0} = \psi_{\mathcal{E}^i} \text{on } \mathcal{E}^i, \ i = \{1, \ldots, N_L - 1\} \end{array} \right\} \end{aligned}}$$



Regularisation of the problem IV

III Energy Interpolation Scheme

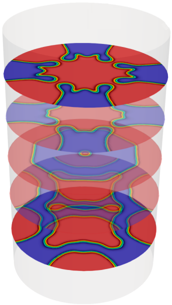
- Additive decomposition:

$$e(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}),\boldsymbol{F},\boldsymbol{D}_0) = e(\boldsymbol{F}) + \bar{e}_{em}(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}),\boldsymbol{F},\boldsymbol{D}_0)$$

- Conditions to be satisfied:

$$\bar{e}_{em}(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}),\boldsymbol{F},\boldsymbol{D}_0)\big|_{\psi_{\mathcal{B}_0}=1} = e_{em}(\boldsymbol{F},\boldsymbol{D}_0)$$

$$\bar{e}_{em}(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}),\boldsymbol{F},\boldsymbol{D}_0)\big|_{\psi_{\mathcal{B}_0}=0} = \frac{1}{\alpha}e_{em}(\boldsymbol{F},\boldsymbol{D}_0)$$



- Proposed scheme

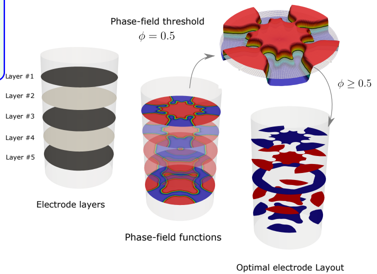
$$\boxed{\bar{e}_{em}(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}),\boldsymbol{F},\boldsymbol{D}_0) = \frac{1}{(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}))^p + \alpha(1-(\psi_{\mathcal{B}_0}(\boldsymbol{X},\boldsymbol{\psi}_{\mathcal{E}}))^p)}e_{em}(\boldsymbol{F},\boldsymbol{D}_0)}$$

The regularised problem

- The **PDE** and boundary conditions

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{\nabla}_0 \boldsymbol{\phi} && \text{in } \mathcal{B}_0\\
\text{DIV}\boldsymbol{P}(\psi_{\mathcal{B}_0}(\boldsymbol{X}, \boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{F}, \boldsymbol{E}_0) + \boldsymbol{f}_0 &= \boldsymbol{0} && \text{in } \mathcal{B}_0\\
\boldsymbol{P}\boldsymbol{N} &= \boldsymbol{t}_0 && \text{on } \partial_t \mathcal{B}_0\\
\boldsymbol{\phi} &= \bar{\boldsymbol{\phi}}^b && \text{on } \partial_{\boldsymbol{\phi}} \mathcal{B}_0
\end{aligned}
$$

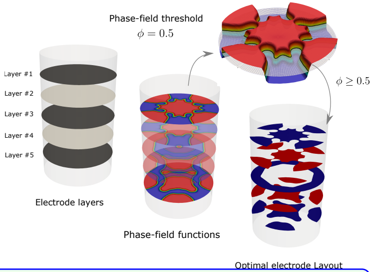
$$
\begin{aligned}
\boldsymbol{E}_0 &= -\boldsymbol{\nabla}_0 \varphi && \text{in } \mathcal{B}_0\\
\text{DIV}\boldsymbol{D}_0(\psi_{\mathcal{B}_0}(\boldsymbol{X}, \boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{F}, \boldsymbol{E}_0) &= \rho_0 && \text{in } \mathcal{B}_0\\
\boldsymbol{D}_0 \cdot \boldsymbol{N} &= -\omega_0 && \text{on } \partial_\omega \mathcal{B}_0\\
\varphi &= \bar{\varphi}^b && \text{on } \partial_\varphi \mathcal{B}_0\\
\varphi &= \pm\bar{\varphi} && \text{on } \mathcal{E}^{\pm}
\end{aligned}
$$



The optimal design problem I

- **The Optimal Design** problem

$$\begin{cases} \min\limits_{\psi_{\mathcal{E}^i}(\boldsymbol{X})} \mathcal{J}(\boldsymbol{\phi}) \\ \text{s.t} \begin{cases} \text{Statements } \mathcal{W}^{\text{mec}} = 0 \ \mathcal{W}^{\text{elec}} = 0 \\ \text{Constitutive model and energy} \\ \ \text{interpolation scheme} \\ \text{Laplacian extension into the volume} \\ 0 \leq \psi_{\mathcal{E}^i}(\boldsymbol{X}) \leq 1; \quad \psi_{\mathcal{E}^i}(\boldsymbol{X}) \in H^1(\mathcal{E}^i) \end{cases} \end{cases}$$



$$\boxed{\mathcal{L}(\psi_{\mathcal{B}_0}(\boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{\phi}, \varphi, \boldsymbol{p}_{\boldsymbol{\phi}}, p_{\varphi}) = \mathcal{J}(\boldsymbol{\phi}) - \mathcal{W}^{\text{mec}}(\psi_{\mathcal{B}_0}(\boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{\phi}, \varphi, \boldsymbol{p}_{\boldsymbol{\phi}}) - \mathcal{W}^{\text{ele}}(\psi_{\mathcal{B}_0}(\boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{\phi}, \varphi, p_{\boldsymbol{\varphi}})}$$

$$\mathcal{W}^{\text{mec}}(\psi_{\mathcal{B}_0}(\boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{\phi}, \varphi, \boldsymbol{p}_{\boldsymbol{\phi}}) = \int_{\Omega_0} \boldsymbol{P}(\psi_{\mathcal{B}_0}, \boldsymbol{F}, \boldsymbol{D}_0) : \boldsymbol{\nabla}_0 \boldsymbol{p}_{\boldsymbol{\phi}} \, dV - \left(\int_{\Omega_0} \boldsymbol{f}_0 \cdot \boldsymbol{p}_{\boldsymbol{\phi}} \, dV + \int_{\partial\Omega_{0Nm}} \boldsymbol{t}_0 \cdot \boldsymbol{p}_{\boldsymbol{\phi}} \, dA \right)$$

$$\mathcal{W}^{\text{elec}}(\psi_{\mathcal{B}_0}(\boldsymbol{\psi}_{\mathcal{E}}), \boldsymbol{\phi}, \varphi, p_{\boldsymbol{\varphi}}) = \int_{\Omega_0} \boldsymbol{D}_0(\psi_{\mathcal{B}_0}, \boldsymbol{F}, \boldsymbol{E}_0) \cdot \boldsymbol{\nabla}_0 p_{\varphi} \, dV + \left(\int_{\Omega_0} \rho_0 p_{\varphi} \, dV + \int_{\partial\Omega_{0Ne}} \omega_0 p_{\varphi} \, dA \right)$$

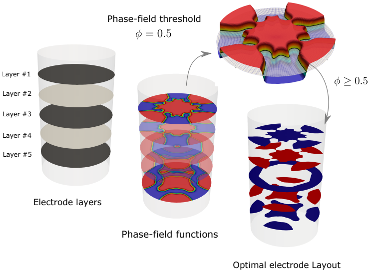
- **Sensitivity** of the Lagrangian

$$\boxed{D\mathcal{L}[\Delta\psi_{\mathcal{E}^i}] = \int_{\mathcal{E}^i} g_{\mathcal{E}^i} \, dA}$$

The optimal design problem II

- **The Optimal Design** problem

$$
\begin{cases}
\min\limits_{\psi_{\mathcal{E}^i}(\boldsymbol{X})} \; \mathcal{J}(\boldsymbol{\phi}) \\
\text{s.t} \begin{cases}
\text{Statements } \mathcal{W}^{\text{mec}} = 0 \;\; \mathcal{W}^{\text{elec}} = 0 \\
\text{Constitutive model and energy} \\
\quad \text{interpolation scheme} \\
\text{Laplacian extension into the volume} \\
0 \leq \psi_{\mathcal{E}^i}(\boldsymbol{X}) \leq 1; \quad \psi_{\mathcal{E}^i}(\boldsymbol{X}) \in H^1(\mathcal{E}^i)
\end{cases}
\end{cases}
$$



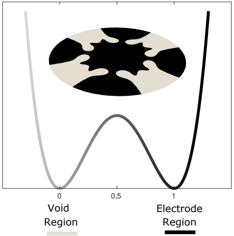
$$
\partial_\tau \psi_{\mathcal{E}^i} = \kappa \boldsymbol{\nabla}^2_{\mathcal{E}_i} \psi_{\mathcal{E}_i} - \partial_{\psi_{\mathcal{E}^i}} \left(\frac{1}{4} \Phi(\psi_{\mathcal{E}^i}) + \eta \, g_{\mathcal{E}^i}(\boldsymbol{\xi}) h(\psi_{\mathcal{E}^i}) \right)
$$

$$
0 = \boldsymbol{\nabla}_{\mathcal{E}^i} \psi_{\mathcal{E}^i} \cdot \boldsymbol{N}|_{\partial \mathcal{E}^i}
$$

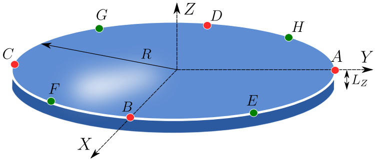
$$
\psi_{\mathcal{E}^i}\big|_{\tau_0} = \psi^0_{\mathcal{E}^i}
$$

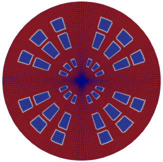
$$
\Phi(\psi_{\mathcal{E}^i}) = \psi^2_{\mathcal{E}^i}(1 - \psi_{\mathcal{E}^i})^2
$$

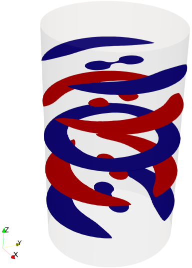
$$
h(\psi_{\mathcal{E}^i}) = \psi^3_{\mathcal{E}^i}(6\psi^2_{\mathcal{E}^i} - 15\psi_{\mathcal{E}^i} + 10)
$$

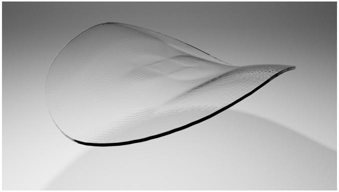


Numerical example I



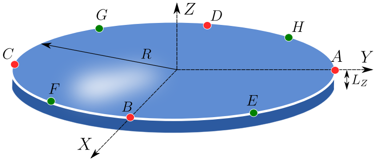


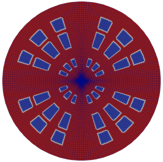


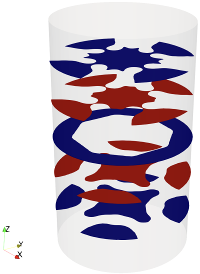


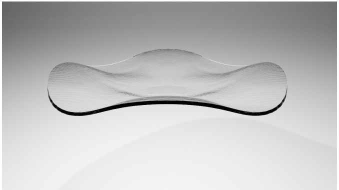
J: Martínez-Frutos, R. Ortigosa and A. J: Gil *In-silico design of electrode meso-arquitecture for shape morphing dielectric elastomers*, **IJMPS**, *submitted*.

Numerical example II









J: Martínez-Frutos, R. Ortigosa and A. J: Gil *In-silico design of electrode meso-arquitecture for shape morphing dielectric elastomers*, **IJMPS**, *submitted*.