On the reachable space of the heat equation

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Joint work with Andreas Hartmann

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Control in Time of Crisis
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Initial Control Problem

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- We consider the one-dimensional heat equation :

$$(HE): \begin{cases} \dfrac{\partial y}{\partial t}(t,x) - \dfrac{\partial^2 y}{\partial x^2}(t,x) = 0 & t \geqslant 0, \ x \in (0,\pi), \\ y(t,0) = u_0(t), \quad y(t,\pi) = u_\pi(t) & t \geqslant 0, \\ y(0,x) = f(x) & x \in (0,\pi). \end{cases}$$

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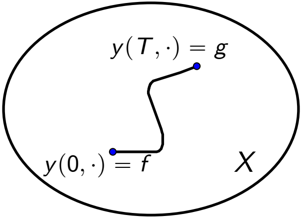
- For any control $u := (u_0, u_\pi) \in L^2((0,+\infty), \mathbb{C}^2)$ and any initial condition $f \in X := W^{-1,2}(0,\pi)$, this equation admits a unique solution $y \in C([0,+\infty), X)$.

Initial Control Problem



Definition

A function $g \in X$ is said to be **reachable** from $f \in X$ in time $T > 0$, if there exists $u = (u_0, u_\pi) \in L^2\big([0,\ T], \mathbb{C}^2\big)$ such that the solution of (HE) satisfies $y(T,\cdot) = g$.

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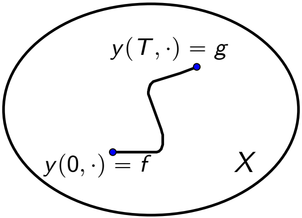

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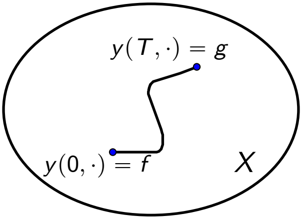

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We call reachable set, the set $\mathcal{R}_T^f$ of functions which are reachable from f in time T.

Question : Can we describe $\mathcal{R}_T^f$?

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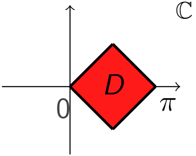
- $\mathcal{R} \subset \mathrm{Hol}(D)$ (Martin, Rosier, Rouchon, 2016) where D is the square :



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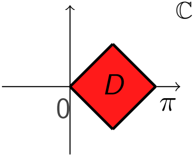

- Conversely, for all open neighbourhood D_ε of D, we have $\mathrm{Hol}(D_\varepsilon) \subset \mathcal{R}$ (Dardé, Ervedoza 2016).

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Function spaces

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Let $1 \leqslant p < +\infty$. Let $\Omega \subset \mathbb{C}$ be a regular open set.

- **Bergman Space :**

$$A^p(\Omega) = \left\{ f \in Hol(\Omega) \;\middle|\; \int_\Omega |f(x+iy)|^p dxdy < +\infty \right\}.$$

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  Example : if $\Omega = \mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$ then $E^p(\mathbb{D}) = H^p(\mathbb{D})$.

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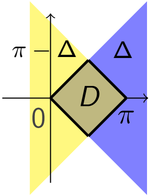

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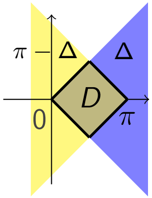

Theorem : O. (2019), Kellay-Normand-Tucsnak (2019)

We have $\mathcal{R} = A^2(\Delta) + A^2(\pi - \Delta)$.

Sketch of the proof

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$$y(T,x) = 2\int_0^T \sum_{m\in\mathbb{Z}} \frac{\partial K}{\partial x}(T-\sigma, x-2m\pi)u_0(\sigma)\,d\sigma$$
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where $K(\sigma,x) = -\sqrt{\frac{1}{4\pi\sigma}}e^{-\frac{x^2}{4\sigma}},\ \sigma > 0,\ x \in (0,\ \pi)$.

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Hence we have $y(T,\cdot) = \tilde{\Phi}_{T,0}u_0 + \tilde{\Phi}_{T,\pi}u_\pi + R_1u_0 + R_2u_\pi$ where $\tilde{\Phi}_{T,0}u_0$ and $\tilde{\Phi}_{T,\pi}u_\pi$ are Laplace type transforms with $\left[\tilde{\Phi}_{T,\pi}v\right](z) = -\left[\tilde{\Phi}_{T,0}v\right](\pi - z)$.

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$$y(T,x) = 2\int_0^T \sum_{m\in\mathbb{Z}} \frac{\partial K}{\partial x}(T-\sigma, x-2m\pi)u_0(\sigma)\,d\sigma - 2\int_0^T \sum_{m\in\mathbb{Z}} \frac{\partial K}{\partial x}(T-\sigma, x-(2m+1)\pi)u_\pi(\sigma)\,d\sigma$$

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Paley-Wiener type theorem : $\mathcal{L} : L^2\left(\mathbb{R}_+, \frac{dt}{t}\right) \to A^2(\mathbb{C}_+)$ is an unitary operator where $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

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Separation of singularities problem

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Let Ω_1 and Ω_2 be two open sets in $\mathbb{C}$ such that $\Omega_1 \cap \Omega_2 \neq \emptyset$.

- **Question** : If $f \in \text{Hol}(\Omega_1 \cap \Omega_2)$, can we find two functions $f_1 \in \text{Hol}(\Omega_1)$ and $f_2 \in \text{Hol}(\Omega_2)$ such that $f = f_1 + f_2$ on $\Omega_1 \cap \Omega_2$?

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- The question also holds in Banach spaces $\mathcal{B}$ of analytic functions : Is the equality $\mathcal{B}(\Omega_1 \cap \Omega_2) = \mathcal{B}(\Omega_1) + \mathcal{B}(\Omega_2)$ true ?
  - H^∞ : Havin, Nersessian, Ortega-Cerdà
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  - H^∞ : Havin, Nersessian, Ortega-Cerdà
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- In this talk, we are interested in the Bergman space A^p.

Results

Theorem 1 : (Hartmann, O.)

Let $1 < p < \infty$. Let Ω_1 and Ω_2 be open sets of $\mathbb{C}$ such that $\Omega_1 \cap \Omega_2$ is non-empty and bounded. If $\text{dist}(\Omega_1 \backslash \Omega_2,\ \Omega_2 \backslash \Omega_1) > 0$, then we have

$$A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) + A^p(\Omega_2).$$

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Theorem 2 : (Hartmann, O.)

Let $1 < p < \infty$. Let H_1, H_2 be two half-planes such that $\Sigma := H_1 \cap H_2 \neq \varnothing$ is a sector. Then $A^p(\Sigma) = A^p(H_1) + A^p(H_2)$.

Main tools : reproducing kernels, Bergman projection, conformal mapping.

Results (2)

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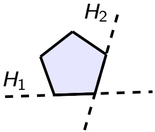
Theorem 3 : (Hartmann, O.)

Let $1 < p < \infty$. Let $H_1, H_2, \ldots, H_n$ be half-planes such that $\mathcal{P} := \bigcap_{k=1}^{n} H_k \neq \varnothing$ is a convex polygon. Then $A^p(\mathcal{P}) = \sum_{k=1}^{n} A^p(H_k)$.



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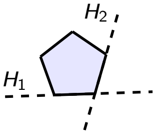
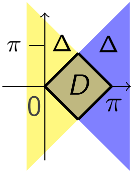
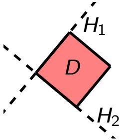


Corollary :

We have $A^2(D) = A^2(\Delta) + A^2(\pi - \Delta)$ $(= \mathcal{R})$





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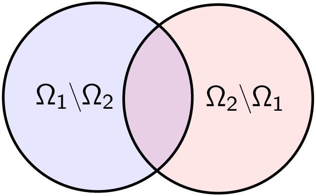
Let $\tilde{\partial} E$ be the boundary of a one-dimensional manifold E.

Theorem 4 : (Hartmann, O.)

Let $1 < p < \infty$. Let Ω_1 and Ω_2 be two open **convex** sets in $\mathbb{C}$ such that

1. $\Omega_1 \cap \Omega_2$ is non-empty and bounded,
2. The set $\tilde{\partial}\left(\partial\Omega_1 \cap \partial\Omega_2\right)$ is finite.

Then $A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) + A^p(\Omega_2)$.



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$$\mathrm{Hol}(B) \subset \mathcal{R}^{\mathrm{HH}}$$

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Conversly,

Theorem : (Hartmann, O.)

We have $\mathcal{R}^{\mathrm{HH}} \subset A^2(D)$.

Some perspectives

- Reachable space of the Hermite heat equation
- Trace of the Bergman space on the real line
- More inputs : L^∞-control, C^0-control, Gevrey type, ...
- Internal control
- In higher dimensions, in some particular geometries
- Applications to non linear equations

Thank you for your attention !

Strohmaier-Waters result

$$\mathcal{E}(\Omega) = \{z = x + iy \in \mathbb{C}^d \mid x \in \Omega,\ |y| < \mathrm{dist}(x, \partial\Omega)\}$$

Theorem : (Strohmaier-Waters, 2020)

$$\mathrm{Hol}(\overline{\mathcal{E}(\Omega)}) \subset \mathcal{R} \subset \mathrm{Hol}(\mathcal{E}(\Omega))$$

Example