Unique continuation problems and stabilised finite element methods

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Context

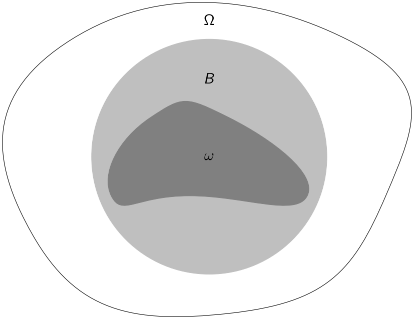
- The unique continuation principle is fundamental in the analysis of partial differential equations.
- Quantitative unique continuation (propagation of smallness) is often used, e.g. in control theory and inverse problems.
- Unique continuation problems are severely ill-posed and solving them numerically requires regularisation.
- There is a need for numerical methods with proven error estimates (convergence rates in terms of the discretisation parameter).

Unique continuation (UC)

Let $\mathcal{L}$ be an elliptic operator and let $\omega \subset B \subset \Omega \subset \mathbb{R}^n$ be open, bounded and connected sets. Given $f \in H^{-1}(\Omega)$, $u_\omega \in H^1(\omega)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = u_\omega & \text{in } \omega. \end{cases} \tag{1}$$

We will focus on finding u in B.



Ill-posedness and the Cauchy problem

Let the linear operator $\mathcal{A}_{uc} : H^1(\Omega) \to H^{-1}(\Omega) \times H^1(\omega)$ given by $\mathcal{A}_{uc} u = (\mathcal{L}u, u|_\omega)$. The range of this operator is not closed and its pseudo-inverse is discontinuous. There is no constant $C > 0$ such that for any $u \in H^1(\Omega)$

$$\|u\|_{H^1(\Omega)} \leq C\Big(\|\mathcal{L}u\|_{H^{-1}(\Omega)} + \|u\|_{H^1(\omega)} \Big).$$

A variation of Hadamard's example can be used to show this. Also, UC is related to the Cauchy problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma, \\ \nabla u \cdot n = g_N & \text{on } \Gamma, \end{cases} \tag{2}$$

where $\Gamma \subset \partial\Omega$ smooth and $f \in H^{-1}(\Omega)$, $g_D \in H^{\frac{1}{2}}(\Gamma)$, $g_N \in H^{-\frac{1}{2}}(\Gamma)$.

Conditional stability estimates (three-ball inequalities)

We prove that a solution $u \in H^1(\Omega)$ to the UC problem satisfies

$$\|u\|_{L^2(B)} \leq C_{st} \left(\|u\|_{L^2(\omega)} + \|\mathcal{L}u\|_{H^{-1}(\Omega)} \right)^{\kappa} \|u\|_{L^2(\Omega)}^{1-\kappa}.$$

for a stability constant $C_{st} > 0$ and a Hölder exponent $\kappa \in (0, 1)$.
We focus on

- convection–diffusion $\mathcal{L}u = -\mu \Delta u + \beta \cdot \nabla u$, $\mu > 0$, $\beta \in W^{1,\infty}(\Omega)^n$.
- Helmholz $\mathcal{L}u = -\Delta u - k^2 u$, $k > 0$.

! We track the dependence of C_{st} on the physical parameters.

Conditional stability estimate. Convection–diffusion

Lemma (Carleman estimate for the Laplacian)

Let $\rho \in C^3(\Omega)$ and $K \subset \Omega$ be a compact set with no critical points of ρ. Let $\alpha, \tau > 0$ and $\phi = e^{\alpha\rho}$. Let $w \in C_0^2(K)$. Then there is $C > 0$ such that

$$\int_K e^{2\tau\phi}(\tau^3 w^2 + \tau|\nabla w|^2)\,\mathrm{d}x \leq C \int_K e^{2\tau\phi}|\Delta w|^2\,\mathrm{d}x,$$

for α large enough and $\tau \geq \tau_0$, where $\tau_0 > 1$ depends only on α and ρ.

We use it to prove a three-ball inequality for convection–diffusion.

- Denote $B_i = B(x_0, r_i)$, $i = 0, 1, 2$. Take $K = \overline{\Omega} \setminus B_0$.
- Take $\rho(x) = -d(x, x_0)$ outside B_0.
- Multiply by μ^2 and insert $\beta \cdot \nabla w$ in the RHS.
- Take $w = \chi u$ for a certain cut-off χ.
- Take $\tau \geq \tau_0 + 2|\beta|^2/\mu^2$.

Three-ball inequality. Convection–diffusion

Corollary

Let $x_0 \in \Omega$ *and* $0 < r_1 < r_2 < d(x_0, \partial\Omega)$*. Then there are* $C > 0$ *and* $\kappa \in (0,1)$ *such that for* $\mu > 0$*,* $\beta \in [L^\infty(\Omega)]^n$ *and* $u \in H^2(\Omega)$ *it holds that*

$$\|u\|_{H^1(B_2)} \leq Ce^{C\tilde{Pe}^2} \left(\|u\|_{H^1(B_1)} + \frac{1}{\mu} \|\mathcal{L}u\|_{L^2(\Omega)} \right)^\kappa \|u\|_{H^1(\Omega)}^{1-\kappa},$$

where $\tilde{Pe} = 1 + |\beta|/\mu$ *and* $|\beta| = \|\beta\|_{L^\infty(\Omega)^n}$*.*

We want to weaken the norms in the RHS for the FEM analysis.

Lemma (Shifted three-ball inequality)

Let $x_0 \in \Omega$ and $0 < r_1 < r_2 < d(x_0, \partial\Omega)$. Then there are $C > 0$ and $\kappa \in (0,1)$ such that for $\mu > 0$, $\beta \in [L^\infty(\Omega)]^n$ and $u \in H^1(\Omega)$ it holds that

$$\|u\|_{L^2(B_2)} \leq \underbrace{Ce^{C\tilde{Pe}^2}}_{C_{st}} \left(\|u\|_{L^2(B_1)} + \frac{1}{\mu} \|\mathcal{L}u\|_{H^{-1}(\Omega)} \right)^{\kappa} \|u\|_{L^2(\Omega)}{}^{1-\kappa},$$

where $\tilde{Pe} = 1 + |\beta|/\mu$ and $|\beta| = \|\beta\|_{[L^\infty(\Omega)]^n}$.

Corollary

For $\mu > 0$, $\beta \in [W^{1,\infty}(\Omega)]^n$ having $\operatorname{ess\,sup}_\Omega \nabla \cdot \beta \leq 0$, and $u \in H^1(\Omega)$ it holds that

$$\|u\|_{H^1(B_2)} \leq C_{st}(\|u\|_{L^2(B_1)} + \frac{1}{\mu} \|\mathcal{L}u\|_{H^{-1}(\Omega)})^{\kappa}(\|u\|_{L^2(\Omega)} + \frac{1}{\mu} \|\mathcal{L}u\|_{H^{-1}(\Omega)})^{1-\kappa}.$$

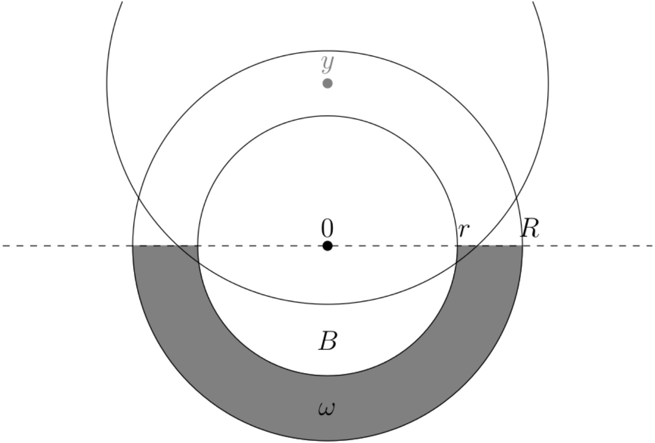
Helmholtz equation. Stability for UC

- Conditional stability estimates with explicit dependence on the wave number k when $\overline{B \setminus \omega} \subset \Omega$, e.g.

$$\|u\|_{H^1(B)} \leq C_{st}(\|u\|_{H^1(\omega)} + \left\|\Delta u + k^2 u\right\|_{L^2(\Omega)})^\alpha \underbrace{\|u\|_{H^1(\Omega)}^{1-\alpha}}_{\text{apriori bound}} .$$

- If there is a line that intersects B but not ω, then $C_{st} = C(k)$ blows up faster than any polynomial in k [Burman–M.N.–Oksanen]. See also [Berge-Mallinikova'20] for exponential blow-up.
- Assuming suitable convexity, the constant C_{st} is independent of k.
- Weaker norms in the RHS appropriate for numerical analysis.

Hemlholtz equation. Convexity assumption



Previous results. Increased stability estimate

In this convex setting, it holds for $F = \|u\|_{H^1(\omega)} + \left\|\Delta u + k^2 u\right\|_{L^2(\Omega)}$ that

$$\|u\|_{L^2(B)} \leq CF + Ck^{-1} F^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha},$$

where the constants C and α are independent of k [Hrycak–Isakov'04].

Previous results. Increased stability estimate

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$$\|u\|_{L^2(B)} \leq CF + Ck^{-1}F^{\alpha}\|u\|^{1-\alpha}_{H^1(\Omega)}, \tag{3}$$

For a plane wave $u(x) = e^{i\boldsymbol{k}\cdot x}$, with $|\boldsymbol{k}| = k$, it holds that

$$\|u\|_{H^1(\omega)} \sim (1+k)\|u\|_{L^2(\omega)}.$$

An analogue of (3) with both sides at the same Sobolev scale could be

$$\|u\|_{L^2(B)} \leq CkE + CE^{\alpha}\|u\|^{1-\alpha}_{L^2(\Omega)},$$

where $E := \|u\|_{L^2(\omega)} + \left\|\Delta u + k^2 u\right\|_{H^{-1}(\Omega)}$.

Shifting in the Sobolev scale

Recall that $E = \|u\|_{L^2(\omega)} + \left\|\Delta u + k^2 u\right\|_{H^{-1}(\Omega)}$. We show a stronger estimate than

$$\|u\|_{L^2(B)} \leq CkE + CE^\alpha \|u\|_{L^2(\Omega)}^{1-\alpha}.$$

Lemma [Burman–M.N.–Oksanen]. For a suitable convex geometry $\omega \subset B \subset \Omega$, there are $C > 0$ and $\alpha \in (0,1)$ such that for all $k \geq 0$,

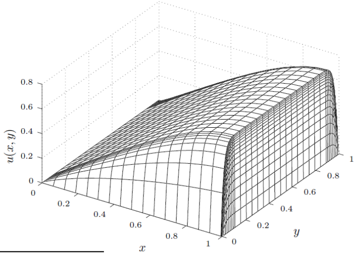
$$\|u\|_{L^2(B)} \leq CE^\alpha \|u\|_{L^2(\Omega)}^{1-\alpha}.$$

Our numerical analysis is based on such estimates.

FEM for well-posed convection–diffusion problems[1,2]

$$-\mu\Delta u + \beta \cdot \nabla u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \mu \ll |\beta|.$$

- Standard methods give spurious oscillations due to sharp layers.
- Hence the need for stabilisation, e.g. streamline diffusion (or SUPG), Galerkin Least Squares, dG, continuous interior penalty, etc.



[1] M. Stynes. "Steady-state convection-diffusion problems". In: *Acta Numerica* 14 (2005), pp. 445–508.

[2] V. John, P. Knobloch, and J. Novo. "Finite elements for scalar convection-dominated equations and incompressible flow problems: a never ending story?" In: *Comput. Vis. Sci.* 19.5 (2018), pp. 47–63.

Continuous Interior Penalty (CIP)[3, 4]

Let $\mathcal{T}_h$ be a triangulation of Ω into elements K with maximal diameter h. Let $V_h := \{v_h \in C(\bar{\Omega}) : v_h|_K \in \mathbb{P}_1(K), K \in \mathcal{T}_h\}$. The PDE weak form

$$a(u_h, v_h) := (\mu \nabla u_h, \nabla v_h)_\Omega + (\beta \cdot \nabla u_h, v_h)_\Omega - \langle \mu \nabla u_h \cdot n, v_h \rangle_{\partial\Omega}.$$

The CIP-FEM reads as follows: Find $u_h \in V_h$ such that

$$a(u_h, v_h) + s_{cip}(u_h, v_h) = f(v_h), \quad \forall v_h \in V_h,$$

where for $\mathcal{F}_i$ denoting the set of internal edges/faces we define

$$s_{cip}(u_h, v_h) := \sum_{F \in \mathcal{F}_i} \int_F h^2 [\![\nabla u_h \cdot n]\!]_F [\![\nabla v_h \cdot n]\!]_F \,\mathrm{d}s.$$

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[3] J. Douglas and T. Dupont. "Interior penalty procedures for elliptic and parabolic Galerkin methods". In: *Computing methods in applied sciences*. Springer, 1976, pp. 207–216.

[4] E. Burman and P. Hansbo. "Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems". In: *Comput. Methods Appl. Mech. Engrg.* 193 (2004), pp. 1437–1453.

Discretise-then-regularise approach for unique continuation

For $(u_h, z_h) \in V_h \times W_h$, consider the discrete regularised Lagrangian

$$L_h(u_h, z_h) := \underbrace{\tfrac{1}{2} s_\omega(u_h - \tilde{U}_\omega, u_h - \tilde{U}_\omega)}_{\text{data term}(L^2 \text{ norm})} + \underbrace{a(u_h, z_h) - (f, z_h)_\Omega}_{\text{PDE constraint}}$$

$$+ \underbrace{\tfrac{1}{2} s(u_h, u_h) - \tfrac{1}{2} s^*(z_h, z_h)}_{\text{stabilisation}}, \quad \tilde{U}_\omega = u|_\omega + \delta u.$$

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The optimality conditions read as: Find $(u_h, z_h) \in V_h \times W_h$ such that

$$\left\{ \begin{array}{rcl} a(u_h, w_h) - s^*(z_h, w_h) & = & (f, w_h)_\Omega \\ s_\omega(u_h, v_h) + a(v_h, z_h) + s(u_h, v_h) & = & s_\omega(\tilde{U}_\omega, v_h) \end{array} \right. \quad \forall (v_h, w_h) \in V_h \times W_h. \tag{4}$$

Heuristics for regularisation

The critical points of L_h satisfy

$$\boxed{\left\{\begin{array}{rcl} a(u_h, w_h) - s^*(z_h, w_h) & = & (f, w_h)_\Omega, \quad \forall w_h \in W_h, \\ s_\omega(u_h, v_h) + a(v_h, z_h) + s(u_h, v_h) & = & s_\omega(\tilde{U}_\omega, v_h), \quad \forall v_h \in V_h. \end{array}\right.}$$

If $z_h \to 0$ and $s(u_h, u_h) \to 0$ as $h \to 0$, then this "converges" to the weak formulation

$$\begin{cases} a(u, w) = (f, w), & \forall w \in H_0^1(\Omega), \\ (u, v)_\omega = (\tilde{U}_\omega, v)_\omega, & \forall v \in H^1(\Omega). \end{cases}$$

So the discrete stabilizer s must be:

- "weak" enough (weakly consistent).
- "strong" enough for the system to have a unique solution.

UC for the Helmholtz equation. Lagrangian

Let $W_h = V_h \cap H_0^1(\Omega)$. For $(u_h, z_h) \in V_h \times W_h$, consider the discrete Lagrangian

$$L_h(u_h, z_h) := \frac{1}{2}\|u_h - \tilde{U}_\omega\|_{L^2(\omega)} + a_h(u_h, z_h) - (f, z_h)_\Omega$$
$$+ \frac{1}{2}s(u_h, u_h) - \frac{1}{2}s^*(z_h, z_h), \quad \tilde{U}_\omega = u|_\omega + \delta.$$

Weak bilinear form of the PDE is $a_h(u_h, z_h) := (\nabla u_h, \nabla z_h)_\Omega - k^2(u_h, z_h)_\Omega$ and the discrete stabilisers are

$$s(v_h, w_h) := \sum_{F \in \mathcal{F}_i} \int_F h[\![\nabla v_h \cdot n]\!]_F [\![\nabla w_h \cdot n]\!]_F \,\mathrm{d}s + h^2 k^4 (v_h, w_h)_\Omega,$$

$$s^*(v_h, w_h) := (\nabla v_h, \nabla w_h)_\Omega.$$

Error estimates

- We use the residual of the PDE and the continuum estimate

$$\|u-u_h\|_{L^2(B)} \le C(\|u-u_h\|_{L^2(\omega)} + \|r\|_{H^{-1}(\Omega)})^{\alpha} \|u-u_h\|_{L^2(\Omega)}^{1-\alpha}.$$

- First we prove that the stabilizing terms and the data fitting term must vanish at an optimal rate for smooth solutions

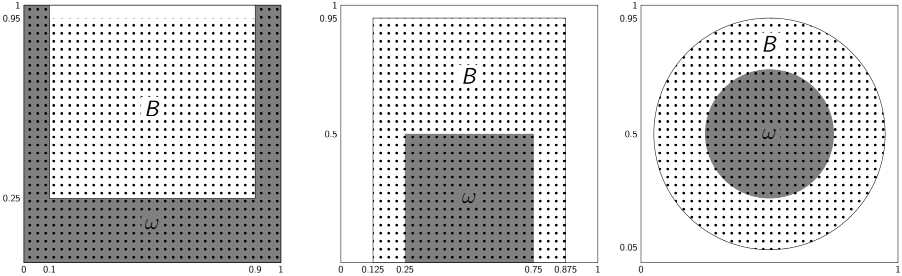
$$|||(u_h - \pi_h u, z_h)||| \le Ch(\|u\|_{H^2(\Omega)} + k^2 \|u\|_{L^2(\Omega)}),$$

  where $|||(u_h, z_h)|||^2 = \|u_h\|_\omega^2 + s(u_h, u_h) + s^*(z_h, z_h)$.
- We then bound the residual by these terms. We finally obtain:

Theorem [Burman–M.N.–Oksanen]. Let $(u_h, z_h) \in V_h \times W_h$ be the unique critical point of the Lagrangian L_h. Then for all $k, h > 0$, satisfying $kh \le 1$, it holds that

$$\|u-u_h\|_{L^2(B)} \le Ch^{\alpha} k^{2\alpha-2} \left(\|u\|_{H^2(\Omega)} + k^2 \|u\|_{L^2(\Omega)} + h^{-1} \|\delta u\|_{L^2(\omega)}\right).$$

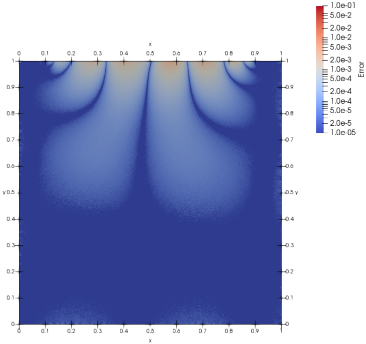
Helmholtz computational domains



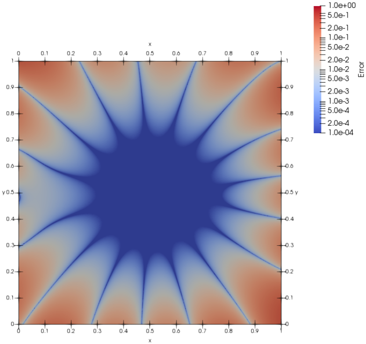
(a) Convex direction. (b) Non-convex direction. (c) Non-convex direction.

Figure: Data set ω (grey) and target region B (dotted).

Convex vs non-convex



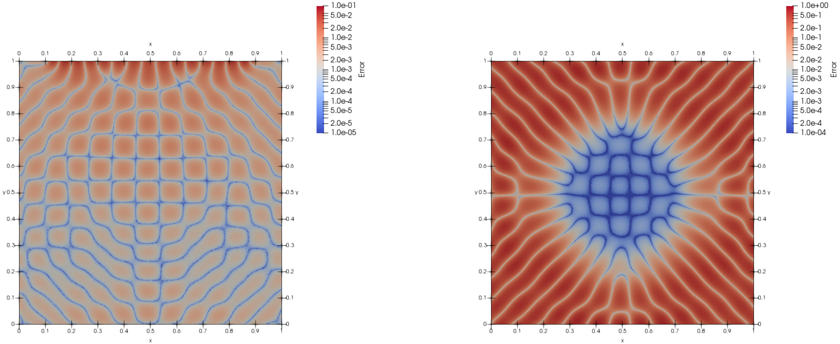
(a) Convex direction.



(b) Non-convex direction disc.

Figure: Absolute errors, exact solution $u(x, y) = \sin \frac{kx}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}}$, $k = 10$. Mesh size $h \approx 0.0025$.

Convex vs non-convex, increased wave number



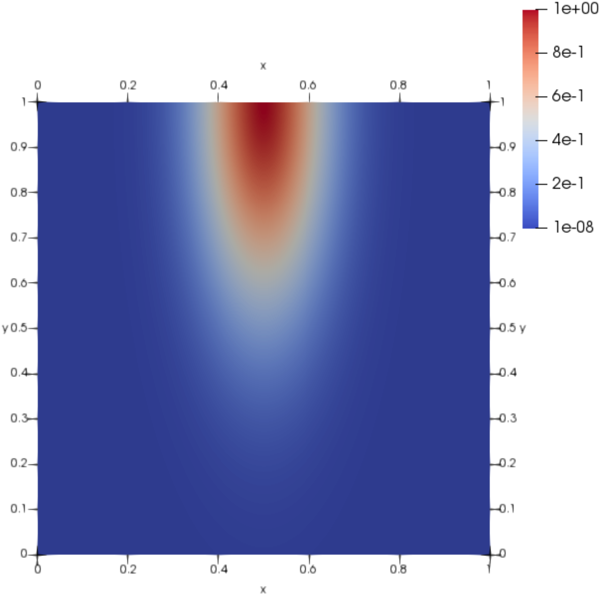
(a) Convex direction.

(b) Non-convex direction disc.

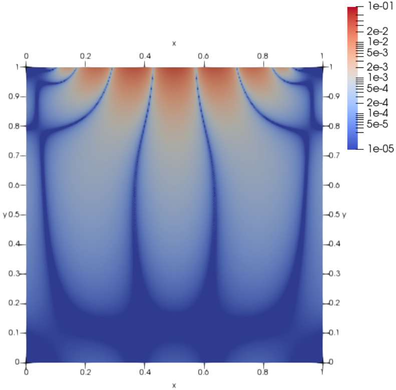
Figure: Absolute errors, exact solution $u(x, y) = \sin \frac{kx}{\sqrt{2}} \cos \frac{ky}{\sqrt{2}}$, $k = 50$. Mesh size $h \approx 0.0025$.

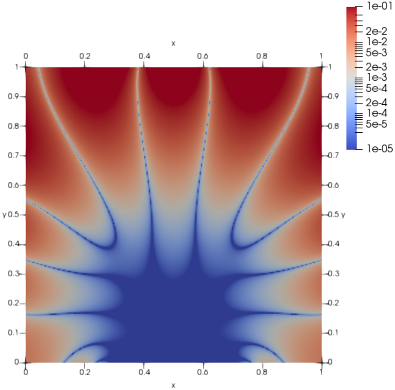
Convex vs non-convex, Gaussian bump

$$u = \exp\left(-\frac{(x-0.5)^2}{2\sigma_x} - \frac{(y-1)^2}{2\sigma_y}\right), \ \sigma_x = 0.01, \ \sigma_y = 0.1, \ k = 10.$$



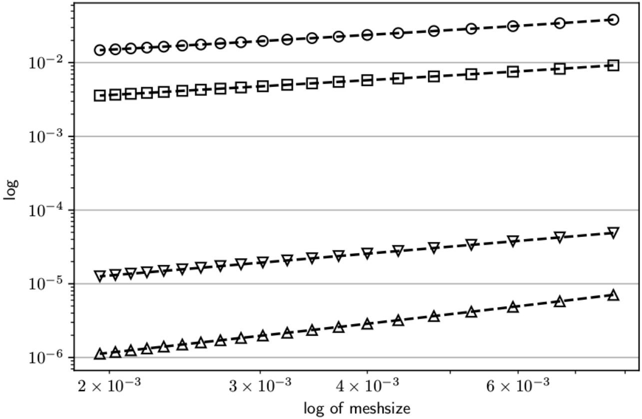
Comparison of the errors for the Gaussian bump





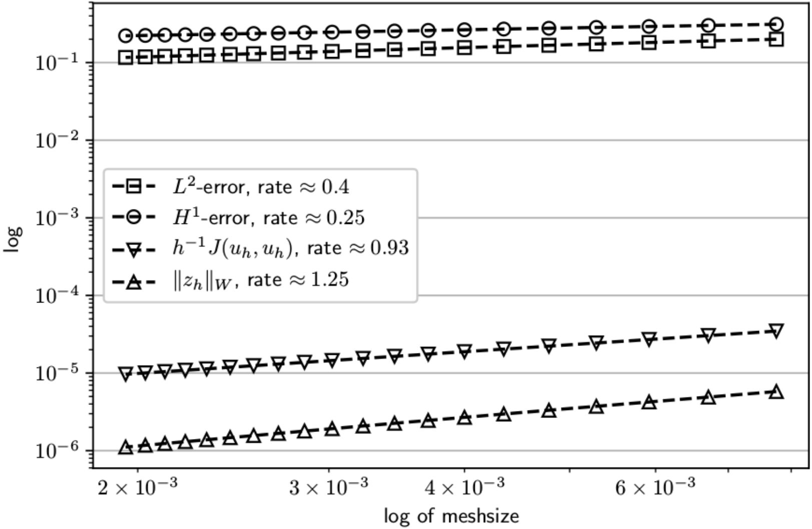
Left. Convex case. *Right.* Non-convex case.

Convergence: a convex case



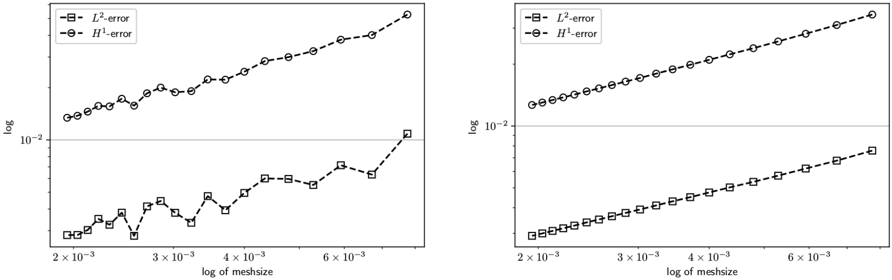
Circles: H^1-error, rate ≈ 0.64. Squares: L^2-error, rate ≈ 0.66. Down triangles: $h^{-1}s(u_h, u_h)$, rate ≈ 1. Up triangles: $\|\nabla z_h\|$, rate ≈ 1.3.

Convergence: a non-convex case



Convergence: the effect of noise in the convex case

$$\|u - u_h\|_{L^2(B)} \leq Ch^{\alpha} k^{2\alpha-2} \left(\|u\|_{H^2(\Omega)} + k^2 \|u\|_{L^2(\Omega)} + h^{-1} \|\delta u\|_{L^2(\omega)} \right).$$



Left. Perturbation $\mathcal{O}(h)$. *Right.* Perturbation $\mathcal{O}(h^2)$.

Convection–diffusion equation. Lagrangian

For $(u_h, z_h) \in [V_h]^2$, consider the discrete Lagrangian functional

$$L_h(u_h, z_h) := \underbrace{\frac{1}{2} s_\omega(u_h - \tilde{U}_\omega, u_h - \tilde{U}_\omega)}_{\text{data term}} + \underbrace{a_h(u_h, z_h) - (f, z_h)_\Omega}_{\text{PDE constraint}}$$

$$+ \underbrace{\frac{\gamma}{2} s(u_h, u_h) - \frac{\gamma^*}{2} s^*(z_h, z_h)}_{\text{stabilisation}}, \quad \tilde{U}_\omega = u|_\omega + \delta u,$$

where $s_\omega(v_h, w_h) := h^{-\zeta}(\mu + |\beta| h)(v_h, w_h)_\omega$, $\zeta \in \{0, 2\}$ and

$$s(v_h, w_h) := \sum_{F \in \mathcal{F}_i} \int_F (\mu + |\beta| h) h [\![\nabla v_h \cdot n]\!]_F [\![\nabla w_h \cdot n]\!]_F \,\mathrm{d}s,$$

$$s^*(v_h, w_h) := (\mu \nabla v_h, \nabla w_h)_\Omega + s_\Omega(v_h, w_h) + \left\langle (|\beta| + \mu h^{-1}) v_h, w_h \right\rangle_{\partial\Omega}.$$

Discrete solution and condition number

Proposition

The discrete system has a unique solution $(u_h, z_h) \in [V_h]^2$ *and the Euclidean condition number* $\mathcal{K}_2$ *of the system matrix satisfies*

$$\mathcal{K}_2 \leq Ch^{-4}.$$

Notice that $\mathcal{K}_2$ does not depend on μ and β.

Diffusion-dominated regime, i.e. $|\beta| h < \mu$

Assume that $|\beta|_{1,\infty} \leq C|\beta|$. Introduce the norm

$$\|(v_h, w_h)\|_s^2 := s_\omega(v_h, v_h) + s(v_h, v_h) + s^*(w_h, w_h).$$

- We use the residual of the PDE and the continuous stability estimate

$$\|u - u_h\|_{L^2(B)} \leq C e^{C \tilde{Pe}^2} \left(\|u - u_h\|_{L^2(\omega)} + \frac{1}{\mu} \|r\|_{H^{-1}(\Omega)} \right)^\kappa \|u - u_h\|_{L^2(\Omega)}^{1-\kappa}.$$

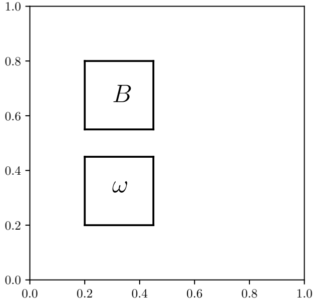
- First we prove that the stabilising terms and the data fitting term must vanish at an optimal rate for smooth solutions

$$\|(\pi_h u - u_h, z_h)\|_s \leq C(\mu^{\frac{1}{2}} h + |\beta|^{\frac{1}{2}} h^{\frac{3}{2}})(|u|_{H^2(\Omega)} + h^{-1} \|\delta\|_\omega).$$

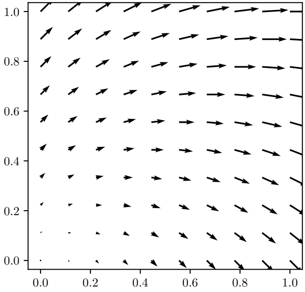
- We then bound the residual by these terms. We finally obtain

$$\boxed{\|u - u_h\|_{L^2(B)} \leq C h^\kappa e^{C \tilde{Pe}^2} (\|u\|_{H^2(\Omega)} + h^{-1} \|\delta u\|_\omega).}$$

Numerical examples when diffusion dominates

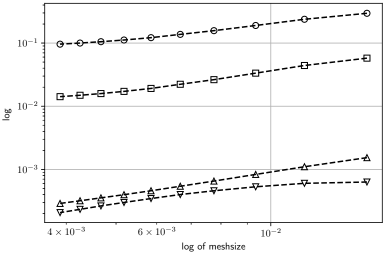


(a) Domains.

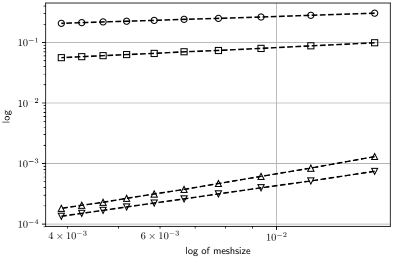


(b) $\beta = 100(x + y, y - x)$,
$\nabla \cdot \beta = 200$.

Convergence rates, $u = 30x(1-x)y(1-y)$, $\mu = 1$



(a) Circles: H^1-error, rate ≈ 0.45; Squares: L^2-error, rate ≈ 0.56; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.1; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.33.



(b) Circles: H^1-error, rate ≈ 0.29; Squares: L^2-error, rate ≈ 0.42; Up triangles: $s(e_h, e_h)^{\frac{1}{2}}$, rate ≈ 1.32; Down triangles: $s^*(z_h, z_h)^{\frac{1}{2}}$, rate ≈ 1.34.

Figure: (L): $\beta = (1, 0)$. (R): $\beta = 100(x + y, y - x)$.

Error contour plot when diffusion dominates

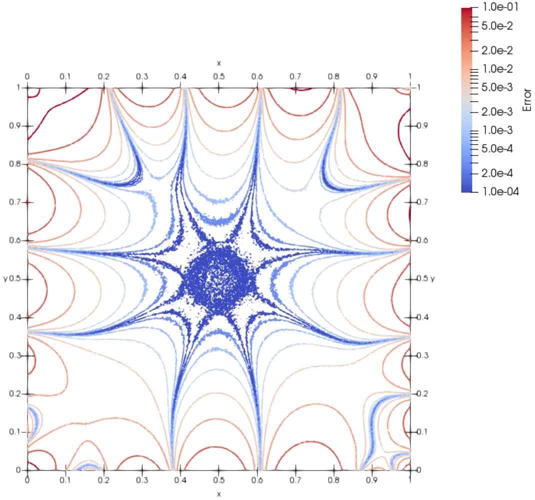
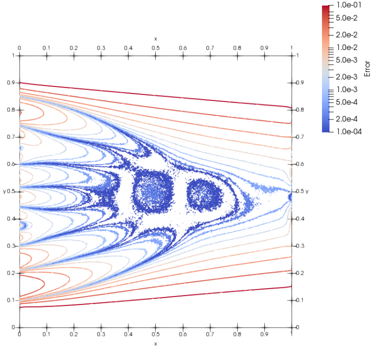
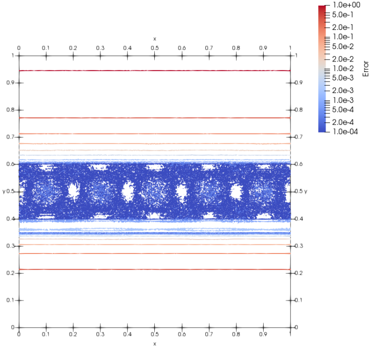


Figure: $\mu = 1$, $\beta = (1, 0)$. Data given in a centred disk of radius 0.1 for the exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

Transition to dominant convection. Error contour plot



(a) $\mu = 10^{-2}$, $\beta = (1, 0)$.



(b) $\mu = 10^{-6}$, $\beta = (1, 0)$.

Figure: Data given in a centred disk of radius 0.1 for the exact solution $u = 2\sin(5\pi x)\sin(5\pi y)$.

Convection–dominated regime, $\mu \lesssim |\beta| h$

The numerical method is essentially the same, the only change consists in a stronger penalty in the data term, from

$$s_\omega(v_h, w_h) := (\mu + |\beta| h)(v_h, w_h)_\omega,$$

to

$$s_\omega(v_h, w_h) := (\mu h^{-2} + |\beta| h^{-1})(v_h, w_h)_\omega.$$

The error analysis is fundamentally different:

- We make no use of the conditional stability estimate.
- We now study local error estimates along the characteristics.

Stability region when $\beta = (\beta_1, 0)$

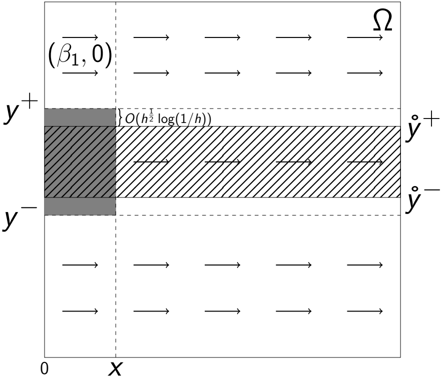


Figure: Data set ω (gray) and the stability region $\mathring{\omega}_\beta$ (hatched).

$$\omega := (0, x) \times (y^-, y^+) \text{ with } x > h \text{ and } y^+ - y^- > h,$$

$$\mathring{\omega}_\beta := \{p \in \omega_\beta : \operatorname{dist}(p, \Omega \setminus \omega_\beta) \geq c_\lambda h^{\frac{1}{2}} \ln(1/h)\}.$$

Downstream case, $\beta_1 > 0$

We consider a Lipschitz weight function

$$\varphi := \psi_1 \psi_2 \in (0,1),$$

that will be used in the weighted norms, where $\psi_1(x,y) := e^{-x}$ and

$$\psi_2 = \begin{cases} 1, & \text{in } \mathring{\omega}_\beta \\ O(h^3), & \text{in } \Omega \setminus \omega_\beta \end{cases}, \quad \beta \cdot \nabla \psi_2 = 0, \quad |\nabla \psi_2| \leq C h^{-\frac{1}{2}}.$$

We take

$$\psi_2(x,y) := \begin{cases} \exp((\mathring{y}^+ - y)/(\lambda h^{\frac{1}{2}})), & y > \mathring{y}^+ \\ 1, & (x,y) \in \mathring{\omega}_\beta \\ \exp((y - \mathring{y}^-)/(\lambda h^{\frac{1}{2}})), & y < \mathring{y}^-. \end{cases}$$

Downstream estimate

For the weighted norm

$$|||v_h|||_{\varphi}^2 := \||\beta|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\Omega}^2 + \|\mu^{\frac{1}{2}} \nabla v_h \varphi^{\frac{1}{2}}\|_{\Omega}^2 + \||\beta \cdot n|^{\frac{1}{2}} v_h \varphi^{\frac{1}{2}}\|_{\partial\Omega^+}^2,$$

we prove the following error estimate

$$|||u - u_h|||_{\varphi} \leq C\big(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta\|_{\omega}\big).$$

This means that in the stability region one has the quasi-optimal estimate

$$\|u - u_h\|_{L^2(\mathring{\omega}_{\beta})} \leq C\big(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta\|_{L^2(\omega)}\big).$$

Upstream case, $\beta_1 < 0$

Now we take $\psi_1(x, y) := -e^{-x}$ and $\varphi := \psi_1\psi_2 \in (-1, 0)$. The weighted norm

$$|||v_h|||_\varphi^2 := \| |\beta|^{\frac{1}{2}} v_h |\varphi|^{\frac{1}{2}} \|_\Omega^2 + \| |\beta \cdot n|^{\frac{1}{2}} v_h |\varphi|^{\frac{1}{2}} \|_{\partial\Omega^-}^2.$$

Denoting the Péclet number by $Pe(h) := |\beta| h/\mu$, we prove the estimates

$$|||u - u_h|||_\varphi \leq C(|\beta|^{\frac{1}{2}} h |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-1} \|\delta\|_\omega), \text{ when } 1 < Pe(h) < h^{-1},$$

and

$$|||u - u_h|||_\varphi \leq C(|\beta|^{\frac{1}{2}} h^{\frac{3}{2}} |u|_{H^2(\Omega)} + |\beta|^{\frac{1}{2}} h^{-\frac{1}{2}} \|\delta\|_\omega), \text{ when } Pe(h) > h^{-1}.$$

Conclusions

- Unique continuation for the Helmholtz and convection–diffusion equations.
- Conditional Hölder stability estimates that are explicit in the physical parameters and with norms suitable for finite element analysis. Carleman estimates represent the starting point, then the norms are shifted.
- Stabilised finite element methods in a discretise-then-regularise approach. Discrete regularisation through continuous interior penalty.
- Error estimates that are explicit in the physical parameters.
- Convergence rates that reflect the continuum stability of the problems.

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