Control of a partial differential equation system of dispersive type

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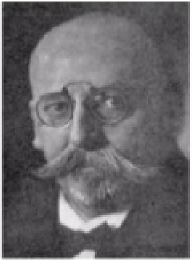
Control en Tiempos de Crisis

Outline

Diederik Korteweg
(1848-1941)



Gustav de Vries
(1866-1934)

- **The Korteweg-de Vries equation :**

$$v_t + vv_x + v_{xxx} = 0, \quad x \in [0, L], \, t \geq 0.$$

Motivation

- In last years, a lot of works have studied controllability properties for the Korteweg-de Vries equation and the Kuramoto-Sivashinsky equation.
- The control of systems of partial differential equations is interesting because it appears in many physical models and there are very challenging problems.
- Questions :

Can you control one system if you have N coupled PDE ?

What is the minimum number of controls ?

Controllability of the Korteweg-de Vries equation on a star-shaped network

Example of network configuration

In the case $N = 3$, the KdV system on a star-shaped network is shown in the following figure

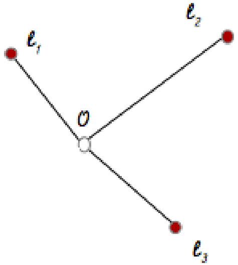


FIGURE – Star-Shaped Network for $N = 3$.

In general, we consider the following N Korteweg-de Vries (KdV) control system

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x u_j u_j + \partial_x^3 u_j)(t,x) = 0, & x \in (0, l_j), \forall t > 0, \\ u_j(t,0) = u_k(t,0), & t > 0, j, k = 1, \cdots N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2 + f_0(t), & t > 0, j = 1, \cdots N, \\ u_j(t, l_j) = 0 & t > 0, j = 1, \cdots N, \\ \partial_x u_j(t, l_j) = g_j(t), & t > 0, j = 1, \cdots N, \\ u_j(0,x) = u_j^0(x), & x \in (0, l_j), j = 1, \cdots N. \end{cases}$$

Where $\alpha > N/2$ with $N \in \mathbb{N}^*$.

Previous Result

The Korteweg-de Vries equation on a finite star-shaped network with the condition $\alpha > N/2$ is exactly controllable with $N+1$ controls : N controls at the ends of the network and one control in the center of the network.

Ammari, K. & Crépeau, E. (2018) Feedback Stabilization and Boundary Controllability of the Korteweg–de Vries Equation on a Star-Shaped Network, *SIAM J. Control Optim*, **56**, 1620–1639.

Goal

We want to prove that the system remains controllable without the control acting in the center of the network considering also the case $\alpha = N/2$.

The controllability problem is described by the following system :

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x u_j u_j + \partial_x^3 u_j)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\ u_j(t,0) = u_k(t,0), & \forall t > 0, j,k = 1, \cdots N, \\ \sum_{j=1}^{N} \partial_x^2 u_j(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & \forall t > 0, j = 1, \cdots N, \\ u_j(t,l_j) = 0, & \forall t > 0, j = 1, \cdots N, \\ \partial_x u_j(t,l_j) = g_j(t), & \forall t > 0, j = 1, \cdots N, \\ u_j(0,x) = u_j^0(x), & \forall x \in (0,l_j), j = 1, \cdots N. \end{cases} \tag{1}$$

Where $\alpha \geq N/2$.

We define the spaces

$$\mathbb{L}^2(\mathcal{T}) = \prod_{j=1}^{N} L^2(0, l_j), \quad L^1(0, T; \mathbb{L}^2(\mathcal{T})) = \prod_{j=1}^{N} L^1(0, T; L^2(0, l_j)),$$

$$H_r^s(0, l_j) = \left\{ v \in H^s(0, l_j) \Big/ \left(\frac{d}{dx}\right)^{i-1} v(l_j) = 0 \text{ for any } 1 \leq i \leq s \right\},$$

$$\mathbb{H}_e^s(\mathcal{T}) = \left\{ u = (u_1, \cdots, u_N) \in \prod_{j=1}^{N} H_r^s(0, l_j) \Big/ u_j(0) = u_k(0), \forall j, k = 1, \cdots, N \right\},$$

and

$$\mathbb{B} := C([0, T], \mathbb{L}^2(\mathcal{T})) \cap L^2(0, T; \mathbb{H}_e^1(\mathcal{T})).$$

Theorem 1 [Cerpa, E. Crépeau, E. Moreno, C. (2020). IMA J. Math. Control Inform.]

There exist $\varepsilon > 0$ and $C > 0$ such that for $u^0 = (u_1^0, \cdots, u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $(g_1, \cdots, g_N) \in L^2(0, T)^N$ with

$$\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + \|(g_1, \cdots, g_N)\|_{L^2(0,T)^N} \leq \varepsilon,$$

there exists a unique solution $u = (u_1, \cdots, u_N) \in \mathbb{B}$ of the nonlinear system (1) which satisfies

$$||u||_{\mathbb{B}} \leq C\Big(\|u^0\|_{\mathbb{L}^2(\mathcal{T})} + ||(g_1, \cdots, g_N)||_{L^2(0,T)^N}\Big).$$

Controllability of the linear system

Linear system

The linear system is described by the following equation :

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x^3 u_j)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\ u_j(t,0) = u_k(t,0), & \forall t > 0, j,k = 1, \cdots N, \\ \sum_{j=1}^N \partial_x^2 u_j(t,0) = -\alpha u_1(t,0), & \forall t > 0, j = 1, \cdots N, \\ u_j(t,l_j) = 0, & \forall t > 0, j = 1, \cdots N, \\ \partial_x u_j(t,l_j) = g_j(t), & \forall t > 0, j = 1, \cdots N, \\ u_j(0,x) = u_j^0(x), & \forall x \in (0,l_j), j = 1, \cdots N. \end{cases} \tag{2}$$

Where $\alpha \geq N/2$. The exact controllability of (2) is equivalent to the surjectivity of the operator

$$\Lambda : (g_1, \cdots, g_N) \in L^2(0,T)^N \mapsto (u_1(T,\cdot), \cdots u_N(T,\cdot)) \in \mathbb{L}^2(\mathcal{T}),$$

where $u = (u_1, \cdots, u_N)$ is the solution of (2).

Adjoint system

$$\begin{cases} (\partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j)(t,x) = 0, & \forall x \in (0, l_j), \forall t > 0 \\ \varphi_j(t,0) = \varphi_k(t,0), & \forall t > 0, j,k = 1, \cdots N \\ \sum_{j=1}^N \partial_x^2 \varphi_j(t,0) = (\alpha - N)\varphi_1(t,0), & \forall t > 0, j = 1, \cdots N \\ \varphi_j(t, l_j) = 0, & \forall t > 0, j = 1, ..N \\ \partial_x \varphi_j(t, l_j) = 0, & \forall t > 0, j = 1, ..N \\ \varphi_j(T,x) = \varphi_j^T(x), & \forall x \in (0, l_j), j = 1, \cdots N. \end{cases} \quad (3)$$

Theorem 2 [Cerpa, E. Crépeau, E. Moreno, C. (2020)]

Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geq N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min},$$

then we have

$$\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C \sum_{j=1}^N \|\partial_x \varphi_j(t, l_j)\|^2_{L^2(0,T)}, \quad \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}),$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the solution of (3) with final condition $\varphi^T = (\varphi_1^T, \cdots, \varphi_N^T)$ and C is a positive constant.

Adjoint system

$$\begin{cases} (\partial_t \varphi_j + \partial_x \varphi_j + \partial_x^3 \varphi_j)(t,x) = 0, & \forall x \in (0, l_j), \forall t > 0 \\ \varphi_j(t,0) = \varphi_k(t,0), & \forall t > 0, j, k = 1, \cdots N \\ \sum_{j=1}^N \partial_x^2 \varphi_j(t,0) = (\alpha - N)\varphi_1(t,0), & \forall t > 0, j = 1, \cdots N \\ \varphi_j(t, l_j) = 0, & \forall t > 0, j = 1, ..N \\ \partial_x \varphi_j(t, l_j) = 0, & \forall t > 0, j = 1, ..N \\ \varphi_j(T,x) = \varphi_j^T(x), & \forall x \in (0, l_j), j = 1, \cdots N. \end{cases} \quad (3)$$

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Let $(l_j)_{j=1,\cdots,N} \in (0, +\infty)^N$ and $\alpha \geq N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min},$$

then we have

$$\|\varphi^T\|^2_{\mathbb{L}^2(\mathcal{T})} \leq C \sum_{j=1}^N \|\partial_x \varphi_j(t, l_j)\|^2_{L^2(0,T)}, \quad \forall \varphi^T \in \mathbb{L}^2(\mathcal{T}),$$

where $\varphi = (\varphi_1, \cdots, \varphi_N)$ is the solution of (3) with final condition $\varphi^T = (\varphi_1^T, \cdots, \varphi_N^T)$ and C is a positive constant.

Idea of the proof

Step 1. By multilplying (3) by $q_j\varphi_j$ Integrating by parts we get after some computations :

$$\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(T,x)|^2 q_j(T,x)dx-\sum_{j=1}^{N}\int_0^{l_j}|\varphi_j(s,x)|^2 q_j(s,x)dx=$$

$$\int_s^T\sum_{j=1}^{N}\int_0^{l_j}(\partial_t q_j+\partial_x q_j+\partial_x^3 q_j)|\varphi_j|^2 dxdt-3\int_s^T\sum_{j=1}^{N}\int_0^{l_j}|\partial_x\varphi_j|^2\partial_x q_j dxdt$$

$$+\int_s^T\sum_{j=1}^{N}|\varphi_j(t,0)|^2\partial_x^2 q_j(t,0)dt+\int_s^T\sum_{j=1}^{N}|\varphi_j(t,0)|^2 q_j(t,0)dt$$

$$+\int_s^T\sum_{j=1}^{N}|\partial_x\varphi_j(t,l)|^2 q_j(t,l)dt+2\int_s^T\sum_{j=1}^{N}q_j(t,0)\partial_x^2\varphi_j(t,0)\varphi_j(t,0)dt. \quad (4)$$

By choosing $q_j(t,x) = t$ and $s = 0$, from (4) that we deduce :

$$\sum_{j=1}^{N} \int_0^{l_j} T|\varphi_j(T,x)|^2 dx \leq \int_0^T \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(t,x)|^2 dxdt +$$

$$T \int_0^T \sum_{j=1}^{N} |\partial_x \varphi_j(t,l)|^2 dt + T(2\alpha - N) \int_0^T |\varphi_1(t,0)|^2 dt.$$

By choosing $q_j(t,x) = 1$, we get that

$$\int_0^T \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(s,x)|^2 dxdt \leq T \sum_{j=1}^{N} \int_0^{l_j} |\varphi_j(T,x)|^2 dx.$$

By choosing $q_j(t,x) = \frac{(2l_j - x)(l_j - x)}{2l_j^2}$ and $s = 0$, for this function we have

- $1 \geq q_j(t,x) \geq 0$,
- $-\frac{-3}{2l_j} < \partial_x q_j(t,x) < \frac{-1}{2l_j}$,
- $\partial_x^2 q_j(t,x) = \frac{1}{l_j^2} > 0$.

Step 2. For $L = \max l_j$ and $l = \min l_j$ and the Poincairé inequality we deduce :

$$\|\varphi(T,x)\|^2_{L^2(\mathcal{T})} \leq T/M \sum_{j=1}^{N} \|\partial_x \varphi_j(t,l_j)\|^2_{L^2(0,T)},$$

where

$$M = T\left(1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\right) - T^2 \frac{\frac{3}{2l}(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}} - \frac{2L^3}{3\pi^2}.$$

We want to study the sign of discriminant given by

$$\triangle = (1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}})^2 - 4\frac{\frac{3}{2l}(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}(\frac{2L^3}{3\pi^2}).$$

The discriminant is positive if and only if :

$$\Big(1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\Big)^2 > 4\frac{(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\Big(\frac{L^3}{l\pi^2}\Big),$$

We obtain the observability inequality with this direct proof if L is small enough and $T_{\min} < T < T_{\max}$ where $a = T_{\min}$ and $b = T_{\max}$ are the roots given by the following equation :

$$\Big(1 - \frac{L^3}{l\pi^2} - \frac{2\alpha - N}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\Big)\Big(\frac{\frac{3}{l}(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\Big)^{-1} \pm \Delta^{1/2}\Big(\frac{\frac{3}{l}(2\alpha - N)}{(2\alpha - N) + \sum_{j=1}^{N} \frac{1}{l_j^2}}\Big)^{-1}.$$

Quadratic Equation

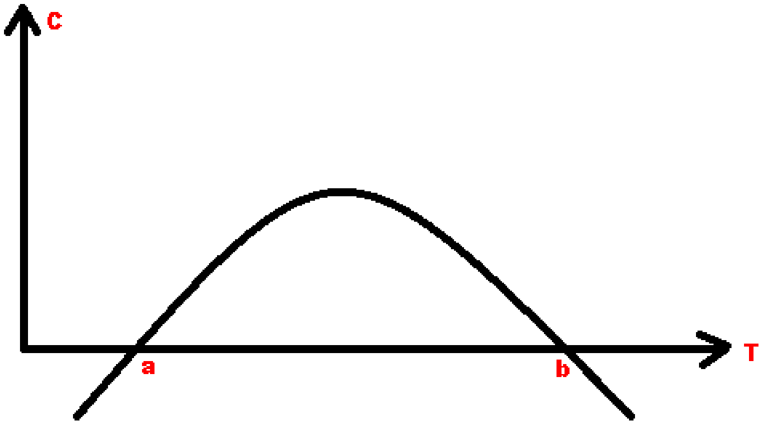

FIGURE – Roots of the Quadratic Equation

Limit case $\alpha = N/2$

Observability inequality

$$\left\{T\left(1-\frac{L^3}{l\pi^2}\right)-\frac{2L^3}{3\pi^2}\right\}\|\varphi(T,x)\|^2_{\mathbb{L}^2(\mathcal{T})} \leq T\sum_{j=1}^{N}\|\partial_x\varphi_j(t,l_j)\|^2_{L^2(0,T)}$$

under the conditions

$$\frac{L^3}{l\pi^2} < 1 \quad \text{and} \quad T > \frac{\frac{2L^3}{3\pi^2}}{1-\frac{L^3}{l\pi^2}}.$$

Duality

Theorem 3 [Cerpa, E. Crépeau, E. Moreno, C. (2020)]

Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geq N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min},$$

then the linear control system (2) is exactly controllable. This means that for any states $u^0 = (u_1^0, \cdots, u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $u^T = (u_1^T, \cdots, u_N^T) \in \mathbb{L}^2(\mathcal{T})$, there exist some controls $g = (g_1, \cdots, g_N) \in L^2(0,T)^N$ such that the solution $u \in \mathbb{B}$ of (2) satisfies

$$u_1(T,\cdot) = u_1^T, \quad u_2(T,\cdot) = u_2^T, \ \cdots, u_N(T,\cdot) = u_N^T.$$

We study the local exact controllability for the nonlinear system :

$$\begin{cases} (\partial_t u_j + \partial_x u_j + \partial_x u_j u_j + \partial_{xxx} u_j)(t,x) = 0, & \forall x \in (0,l_j), \forall t > 0, \\ u_j(t,0) = \partial_k u_j(t,0), & \forall t > 0, j,k = 1, \cdots N \\ \sum_{j=1}^{N} \partial_{xx} u(t,0) = -\alpha u_1(t,0) - \frac{N}{3}(u_1(t,0))^2, & \forall t > 0, j = 1, \cdots N \\ u(t,l_j) = 0, & \forall t > 0, j = 1, \cdots N \\ \partial_x u(t,l_j) = g_j(t), & \forall t > 0, j = 1, ..N \\ u_j(0,x) = u_j^0(x), & \forall x \in (0,l_j), j = 1, \cdots N. \end{cases} \tag{5}$$

Theorem 4 [Cerpa, E. Crépeau, E. Moreno, C. (2020)]

Let $(l_j)_{j=1,\cdots,N} \in (0,+\infty)^N$ and $\alpha \geq N/2$. There exist $L_0, T_{\min} > 0$ such that if

$$L = \max_{j=1,..,N} l_j < L_0 \quad \text{and} \quad T > T_{\min},$$

then the nonlinear control system (5) is locally exactly controllable. This means that there exists $\varepsilon > 0$ such that for any states $u^0 = (u_1^0, \cdots, u_N^0) \in \mathbb{L}^2(\mathcal{T})$ and $u^T = (u_1^T, \cdots, u_N^T) \in \mathbb{L}^2(\mathcal{T})$ with

$$\|u^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon \quad \text{and} \quad \|u^T\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$$

there exist some controls $g = (g_1, \cdots, g_N) \in L^2(0,T)^N$ such that the solution $u = (u_1, \cdots, u_N) \in \mathbb{B}$ of (5) satisfies

$$u_1(T,\cdot) = u_1^T, \quad u_2(T,\cdot) = u_2^T, \ \cdots, u_N(T,\cdot) = u_N^T.$$

Idea of the Proof

Step 1. Let $u^0, u^T \in \mathbb{L}^2(\mathcal{T})$ such that $\|u^0\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$ and $\|u^T\|_{\mathbb{L}^2(\mathcal{T})} < \varepsilon$ for some $\varepsilon > 0$ to be chosen later. We consider the map

$$\Pi : v \in \mathbb{B} \longrightarrow u^1 + u^2 + u^3 \in \mathbb{B},$$

where u^1, u^2, u^3 are the solutions of

$$\begin{cases} (\partial_t u_j^1 + \partial_x u_j^1 + \partial_{xxx} u_j^1)(t,x) = 0, & \forall x \in (0, l_j), \forall t > 0, \\ u_j^1(t,0) = \partial_k u_j^1(t,0), & \forall t > 0, j, k = 1, \cdots N \\ \sum_{j=1}^N \partial_{xx} u_j^1(t,0) = -\alpha u_1^1(t,0), & \forall t > 0, j = 1, \cdots N \\ u_j^1(t, l_j) = 0, & \forall t > 0, j = 1, \cdots N \\ \partial_x u_j^1(t, l_j) = 0, & \forall t > 0, j = 1, \cdots N \\ u_j^1(0,x) = u_j^0(x), & \forall x \in (0, l_j), j = 1, \cdots, N. \end{cases}$$

$$\begin{cases} (\partial_t u_j^2 + \partial_x u_j^2 + \partial_x^3 u_j^2)(t,x) = -v_j\partial_x v_j, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j^2(t,0) = u_k^2(t,0), & j,k=1,\cdots,N,\ t>0,\\ \sum_{j=1}^N \partial_x^2 u_j^2(t,0) = -\alpha u_1^2(t,0) - \frac{N}{3}(v_1(t,0))^2, & t>0,\\ u_j^2(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ \partial_x u_j^2(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ u_j^2(0,x)=0, & j=1,\cdots,N,\ x\in(0,l_j), \end{cases}$$

and

$$\begin{cases} (\partial_t u_j^3 + \partial_x u_j^3 + \partial_x^3 u_j^3)(t,x) = 0, & j=1,\cdots,N,\ x\in(0,l_j),\ t>0,\\ u_j^3(t,0) = u_k^3(t,0), & j,k=1,\cdots,N,\ t>0,\\ \sum_{j=1}^N \partial_x^2 u_j^3(t,0) = -\alpha u_1^3(t,0), & t>0,\\ u_j^3(t,l_j)=0, & j=1,\cdots,N,\ t>0,\\ \partial_x u_j^3(t,l_j)=g_j(t), & j=1,\cdots,N,\ t>0,\\ u_j^3(0,x)=0, & j=1,\cdots,N,\ x\in(0,l_j), \end{cases}$$

With $g\in L^2(0,T)^N$ the control such that

$$u^3(T,\cdot) = u^T - u^1(T,\cdot) - u^2(T,\cdot).$$

This control exists for the exact controllability of the linear system. The proof ends if we are able to find a fixed point $u\in\mathbb{B}$ of the operator Π.

$$\begin{cases} (\partial_t u_j^2 + \partial_x u_j^2 + \partial_x^3 u_j^2)(t,x) = -v_j \partial_x v_j, & j = 1, \cdots, N,\ x \in (0, l_j),\ t > 0, \\ u_j^2(t,0) = u_k^2(t,0), & j,k = 1, \cdots, N,\ t > 0, \\ \sum_{j=1}^N \partial_x^2 u_j^2(t,0) = -\alpha u_1^2(t,0) - \frac{N}{3}(v_1(t,0))^2, & t > 0, \\ u_j^2(t, l_j) = 0, & j = 1, \cdots, N,\ t > 0, \\ \partial_x u_j^2(t, l_j) = 0, & j = 1, \cdots, N,\ t > 0, \\ u_j^2(0,x) = 0, & j = 1, \cdots, N,\ x \in (0, l_j), \end{cases}$$

and

$$\begin{cases} (\partial_t u_j^3 + \partial_x u_j^3 + \partial_x^3 u_j^3)(t,x) = 0, & j = 1, \cdots, N,\ x \in (0, l_j),\ t > 0, \\ u_j^3(t,0) = u_k^3(t,0), & j,k = 1, \cdots, N,\ t > 0, \\ \sum_{j=1}^N \partial_x^2 u_j^3(t,0) = -\alpha u_1^3(t,0), & t > 0, \\ u_j^3(t, l_j) = 0, & j = 1, \cdots, N,\ t > 0, \\ \partial_x u_j^3(t, l_j) = g_j(t), & j = 1, \cdots, N,\ t > 0, \\ u_j^3(0,x) = 0, & j = 1, \cdots, N,\ x \in (0, l_j), \end{cases}$$

With $g \in L^2(0,T)^N$ the control such that

$$u^3(T,\cdot) = u^T - u^1(T,\cdot) - u^2(T,\cdot).$$

This control exists for the exact controllability of the linear system. The proof ends if we are able to find a fixed point $u \in \mathbb{B}$ of the operator Π.

Step 2. Let $R > 0$ and define

$$B(0,R) = \left\{ u \in L^2(0,T,\mathbb{H}^1_e(\mathcal{T})) \Big/ \|u\|_{L^2(0,T,\mathbb{H}^1_e(\mathcal{T}))} \leq R \right\}.$$

with R and ε small enough, we get that

$$\Pi(B(0,R)) \subset B(0,R)$$

Furthermore, $\forall u, v \in B(0,R)$,

$$\begin{aligned} \|\Pi(u) - \Pi(v)\|_{\mathbb{B}} &\leq C_4(\|uu_x - vv_x\|_{L^1(0,T;\mathbb{L}^2\mathcal{T})} + \|u_1(.,0) - v_1(.,0)\|_{L^2(0,T)}) \\ &\leq C_4' R \|u - v\|_{L^2(0,T,\mathbb{H}^1_e(\mathcal{T}))}, \end{aligned}$$

where R is small enough and $C_4' R \in (0,1)$, thus, we obtain that Π is a contraction in $B(0,R) \subset \mathbb{B}$, which ends the proof. □

Boundary exact controllability of the Hirota-Satsuma system

The Hirota-Satsuma system is given by the following equations :

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} & = 3uu_x - 6vv_x + 3w_x, \\ v_t + \frac{1}{2}v_{xxx} & = -3vv_x, \\ w_t + \frac{1}{2}w_{xxx} & = -3uw_x, \end{cases} \tag{6}$$

with the boundary and initial conditions :

$$\begin{cases} u(t,0) = 0,\ u(t,L) = 0,\ u_x(t,0) = h_1(t), \\ v(t,0) = 0,\ v(t,L) = 0,\ v_x(t,L) = h_2(t), \\ w(t,0) = 0,\ w(t,L) = 0,\ w_x(t,L) = h_3(t), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x),\ w(0,x) = w_0(x). \end{cases} \tag{7}$$

Our main result is the following :

Teorema 1 [Moreno, C.]

Let $L, T > 0$. Then the system (6)-(7) is locally exactly controllable. This means that there exists $r > 0$ such that for any initial data $(u_0, v_0, w_0) \in L^2(0, L)^3$ and for all $(u_1, v_1, w_1) \in L^2(0, L)^3$ verifying

$$\|(u_0, v_0, w_0)\|_{L^2(0,L)^3} < r \quad \textit{and} \quad \|(u_1, v_1, w_1)\|_{L^2(0,L)^3} < r,$$

there exist three controls, $(h_1, h_2, h_3) \in L^2(0, T)^3$ such that the solution (u, v, w) of (6)-(7) satisfies

$$u(T, \cdot) = u^1, \;\; v(T, \cdot) = v^1, \;\; w(T, \cdot) = w^1.$$

Control problem for a fourth-order parabolic system

Let $T > 0$, $L > 0$. The parabolic system is described by the following equation :

$$\begin{cases} u_t + u_{xxxx} + b(x)u_{xx} = v_x + g_1(x)v + f_1(x)u_x + g_2(x)u + 1_\omega h, & \text{in } Q_T, \\ v_t + v_{xxxx} + d(x)v_{xx} = u_x + f_2(x)v_x + g_4(x)v, & \text{in } Q_T,, \\ u(t,0) = u_x(t,0) = 0, \quad u(t,L) = u_x(t,L) = 0, & \text{in } (0,T), \\ v(t,0) = v_x(t,0) = 0, \quad v(t,L) = v_x(t,L) = 0, & \text{in } (0,T), \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & \text{in } (0,L), \end{cases} \tag{8}$$

where $Q_T = (0,T) \times (0,L)$, $\omega \subset (0,L)$, $h \in L^2(Q_T)$, b, d, f_1, $f_2 \in C^\infty(0,L)$, g_1, $g_2, g_4 \in C^\infty(0,L)$ and $(u_0, v_0) \in L^2(0,L)^2$.

Goal

We study the null controllability of the fourth-order parabolic system.

Conjecture 1

Let $L > 0$ and $T > 0$, for any $(u_0, v_0) \in L^2(0,L)^2$ there exists $h \in L^2(Q_T)$ such that the unique solution $(u, v) \in C([0,T]; L^2(0,L)^2)$ of (8) satisfies $u(T,\cdot) = v(T,\cdot) = 0$. Moreover, the corresponding control h satisfies

$$\| h \|_{L^2(Q_T)} \leq \| (u_0, v_0) \|_{L^2(0,L)^2} .$$

Let us consider the systems

$$\begin{cases} u_t + u_{xxxx} = v_x + \hat{\eta}_1, & \text{in } Q_T, \\ v_t + v_{xxxx} = u_x + \hat{\eta}_2, & \text{in } Q_T, \\ u(t,0) = u_x(t,0) = 0, \quad u(t,L) = u_x(t,L) = 0, & \text{in } (0,T), \\ v(t,0) = v_x(t,0) = 0, \quad v(t,L) = v_x(t,L) = 0, & \text{in } (0,L), \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & \text{in } (0,L), \end{cases}$$

and

$$\begin{cases} u_t + u_{xxxx} + b(x)u_{xx} = v_x + g_1(x)v + f_1(x)u_x + g_2(x)u + \eta_1, & \text{in } Q_T, \\ v_t + v_{xxxx} + d(x)v_{xx} = u_x + f_2(x)v_x + g_4(x)v + \eta_2, & \text{in } Q_T, \\ u(t,0) = u_x(t,0) = 0, \quad u(t,L) = u_x(t,L) = 0, & \text{in } (0,T), \\ v(t,0) = v_x(t,0) = 0, \quad v(t,L) = v_x(t,L) = 0, & \text{in } (0,L), \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & \text{in } (0,L). \end{cases}$$

We consider an open subset $\omega_0 \subset \omega$, $\theta \in C^\infty([0,L])$ such that

$$\begin{cases} \mathrm{Supp}(\theta) \subseteq \omega, & \\ \theta = 1, & \text{in } \omega_0, \\ 0 \leq \theta \leq 1, & \text{in } (0,L), \end{cases}$$

and the system with two enough regular controls.

$$\begin{cases} \hat{u}_t + \hat{u}_{xxxx} + b(x)\hat{u}_{xx} = \hat{v}_x + g_1(x)\hat{v} + f_1(x)\hat{u}_x + g_2(x)\hat{u} + \theta\hat{h}_1 & \text{in } Q_T, \\ \hat{v}_t + \hat{v}_{xxxx} + d(x)\hat{v}_{xx} = \hat{u}_x + f_2(x)\hat{v}_x + g_4(x)\hat{v} + \theta\hat{h}_2 & \text{in } Q_T, \\ \hat{u}(t,0) = \hat{u}_x(t,0) = 0, \quad \hat{u}(t,L) = \hat{u}_x(t,L) = 0, & \text{in } (0,T), \\ \hat{v}(t,0) = \hat{v}_x(t,0) = 0, \quad \hat{v}(t,L) = \hat{v}_x(t,L) = 0, & \text{in } (0,L), \\ \hat{u}(0,x) = u_0(x), \quad \hat{v}(0,x) = v_0(x), & \text{in } (0,L). \end{cases} \tag{9}$$

Null controllability with two controls

Theorem 1 [Cerpa, E. Crépeau, E. Moreno, C.]

For every $(u_0, v_0) \in L^2(0,L)^2$ there exists $(\hat{h}_1, \hat{h}_2) \in L^2(0,T,L^2(0,L)^2)$ such that the unique solution $(\hat{u}, \hat{v}) \in C([0,T]; L^2(0,L)^2) \cap L^2(0,T; H_0^2(0,L)^2)$ of equation (9) satisfies

$$\hat{u}(T,\cdot) = \hat{v}(T,\cdot) = (0,0).$$

Duality

Observability inequality :

There exists $C_{obs} > 0$ such that

$$\int_0^L (|\varphi(0,\cdot)|^2 + |\psi(0,\cdot)|^2)dx \leq C_{obs} \int_0^T \int_\omega e^{-2s\alpha}(|\varphi|^2 + |\psi|^2)dxdt, \quad (10)$$

where (φ, ψ) is solution of

$$\begin{cases} -\varphi_t + \varphi_{xxxx} + (b\varphi)_{xx} = -\psi_x - (f_1\varphi)_x + g_2\varphi & \text{in}\ , Q_T, \\ -\psi_t + \psi_{xxxx} + (d\psi)_{xx} = -\varphi_x + g_1\varphi - (f_2\psi)_x + g_4\psi & \text{in}\ Q_T, \\ \varphi(t,0) = \varphi_x(t,0) = 0, \quad \varphi(t,L) = \varphi_x(t,L) = 0, & \text{in}\ (0,T), \\ \psi(t,0) = \psi_x(t,0) = 0, \quad \psi(t,L) = \psi_x(t,L) = 0, & \text{in}\ (0,T), \\ \varphi(T,x) = \varphi_T(x), \quad \psi(T,x) = \psi_T(x), & \text{in}\ (0,L). \end{cases} \quad (11)$$

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Algebraic solvability method

- Algebraic Problem
- Find a differential operator
- Find the adjoint differential operator
- Algebraic solution
- Regularity of control
- Null controllability with one control

We are working on obtaining a Carleman inequality with non-homogeneous conditions which implies the Conjecture 1.

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Thank you for your attention.