BOUNDARY STABILIZATION OF THE 1D WAVE EQUATION IN NON-CYLINDRICAL DOMAINS

Control in Times of Crisis

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04/03/2021

Outline

Outline

1. Preliminaries
2. Main results
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Problem statement

Let α and β be two smooth real functions defined on $\mathbb{R}_+$ and Q be

$$Q = \left\{ (t,x) \in \mathbb{R}^2,\ x \in (\alpha(t), \beta(t)),\ \alpha(t) < \beta(t),\ t \in (0,\infty) \right\},$$

with $\alpha(0) = 0$ and $\beta(0) = 1$. We consider the system

$$(S) : \begin{cases} y_{tt}(t,x) = y_{xx}(t,x), & \text{in} \quad Q, \\ y_x(t,\alpha(t)) = f(t) y_t(t,\alpha(t)),\ y(t,\beta(t)) = 0, & \text{in} \quad (0,\infty), \\ y(0,x) = y_0(x),\ y_t(0,x) = y_1(x), & \text{in} \quad (0,1). \end{cases}$$

Previous results

- M. Gugat (2007): $\beta \equiv 1$, $\|\alpha'\|_{L^\infty(\mathbb{R}_+)} < 1$, $f = 1 \implies$ **Finite time stability with optimal time.**

- K. Ammari, A. Bchatnia, and K. El Mufti (2018): $\beta \equiv 1$, α is 1-periodic with $\|\alpha'\|_{L^\infty(\mathbb{R}_+)} < 1$, particular class of $f \implies$ **Exponential decay rate**.

- L. Lu, Y. Feng (2020): $\beta \equiv 1$, $\alpha(t) = rt$, $r \in (-1,0) \implies$ **Polynomial decay rate.**

The problem

Problem

- *How does the nature of the domain affect the asymptotic behavior of the solution to system* (S) *for general* $\alpha, \beta?.$
- *What happens when* $f = f(t)?.$

Notation

- For any function z, the functions $z^{\pm}$ will represent the quantities

$$z^{\pm}(t) = t \pm z(t).$$

- $H^1_{(\beta(t))}(\alpha(t), \beta(t))$ will denote the space

$$H^1_{(\beta(t))}(\alpha(t), \beta(t)) = \left\{ v \in H^1(\alpha(t), \beta(t)),\ v(\beta(t)) = 0,\ \forall t \geq 0 \right\}.$$

Outline

Assumptions

In the sequel, we assume that:

1) α, β are C^1 with $\alpha(t) < \beta(t)$, $f(t) \neq -1$, $\forall t > 0$.

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2) $\|\alpha'\|_{L^\infty(\mathbb{R}_+)}$, $\|\beta'\|_{L^\infty(\mathbb{R}_+)} < 1$.

Assumptions

In the sequel, we assume that:

1) α, β are C^1 with $\alpha(t) < \beta(t)$, $f(t) \neq -1$, $\forall t > 0$.
2) $\|\alpha'\|_{L^\infty(\mathbb{R}_+)}$, $\|\beta'\|_{L^\infty(\mathbb{R}_+)} < 1$.

$\Longrightarrow$ The functions

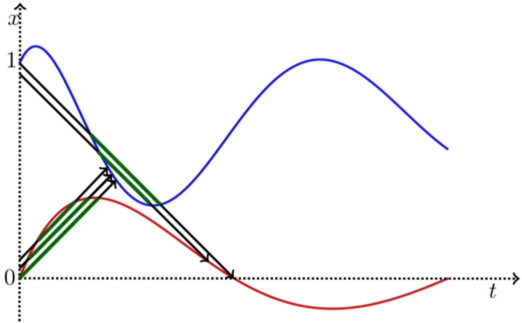
$$\alpha^{\pm}(t) = t \pm \alpha(t) : [0,\infty) \to [0,\infty),$$

$$\beta^{\pm}(t) = t \pm \beta(t) : [0,\infty) \to [\pm 1,\infty),$$

are invertible.

Assumptions

Assumption **(2)** is necessary for the existence.



Existence result

Under the assumptions **(1)**-**(2)**,

Theorem

For any $(y_0, y_1, f) \in H^1_{(1)}(0,1) \times L^2(0,1) \times C([0,\infty);\mathbb{R})$, *there exists a unique solution to systems* (S) *satisfying*

$$y \in C\left([0,t]; H^1_{(\beta(t))}(\alpha(t),\beta(t))\right) \cap C^1\left([0,t]; L^2(\alpha(t),\beta(t))\right), \ t \geq 0.$$

Extra-notations

Define the function $\phi := \phi(\alpha, \beta)$ by

$$\phi := \alpha^- \circ \left(\alpha^+\right)^{-1} \circ \beta^+ \circ \left(\beta^-\right)^{-1}.$$

The function $\phi : [-1, \infty) \to [\alpha^- \circ (\alpha^+)^{-1}(1), \infty)$ is well defined and increasing function as composition of increasing functions, and hence invertible.

Extra-notations

Let $(\psi_n(\cdot))_{n\geq 0}$ be a sequence of functions

$$\begin{array}{rcl} \psi_n(\cdot) & : & [0,\phi(0)) \to [0,\infty) \\ \tau & \mapsto & \psi_n(\tau) = \prod_{i=0}^{n} \left| F\left(\left(\alpha^-\right)^{-1} \circ \phi^{[i]}(\tau) \right) \right|, \end{array}$$

where $F = \frac{1-f}{1+f}$. The notation $\phi^{[n]}$ refers to the n^{th} composed of ϕ i.e.

$$\phi^{[n]} = \underbrace{\phi \circ \cdots \circ \phi}_{n \text{ times}},$$

with the convention $\phi^{[0]} = I$.

Extra-assumptions

Assume that the assumptions **(1)-(2)** are satisfied, in addition, assume that:

3) $\phi(\tau) < \cdots < \phi^{[n]}(\tau) < \phi^{[n+1]}(\tau) \underset{n\to\infty}{\longrightarrow} \infty$, $\forall \tau \in [0, \phi(0))$.

$$\Updownarrow$$

$$t \rightsquigarrow \infty$$

Extra-notations

Let $E(\cdot)$ the usual energy functional

$$\begin{aligned} E(t) \quad : & = \|(y, y_t)\|^2_{H^1_{(\beta(t))}(\alpha(t),\beta(t))\times L^2(\alpha(t),\beta(t))} \\ & = \int_{\alpha(t)}^{\beta(t)} \left(y_t^2(t,x) + y_x^2(t,x) \right) dx. \end{aligned}$$

Then we have:

Main results

Theorem

Let $(y_0, y_1, f) \in H^1_{(1)}(0,1) \times L^2(0,1) \times C([0,\infty);\mathbb{R})$. *Assume that* **(1)-(3)** *are satisfied. Then:*

$$E(t) \underset{t\to\infty}{\longrightarrow} 0 \Leftrightarrow \psi_n(\tau) \underset{n\to\infty}{\longrightarrow} 0, \ \forall \tau \in [0, \phi(0)).$$

Further, if there exists $g \in C(\mathbb{R}, (0,\infty))$ *such that*

$$\psi_n(\tau) \underset{n\to\infty}{\sim} Cg\left(\phi^{[n]}(\tau)\right), \ \forall \tau \in [0, \phi(0)),$$

for some positive constant $C > 0$, *then*

$$E(t) \leq Cg(t)E(0), \ \forall t > 0.$$

Main results

In particular, if we let $g(t) = e^{\omega t}$ in the previous theorem we obtain:

Corollary

Let $(y_0, y_1, f) \in H^1_{(1)}(0,1) \times L^2(0,1) \times C([0,\infty);\mathbb{R})$. *Assume that* **(1)-(3)** *are satisfied. The exists* $C > 0$ *and* $\omega \in \mathbb{R}$ *such that*

$$E(t) \leq Ce^{t\omega} E(0), \ \forall t \geq 0,$$

if, and only if

$$\sup_{\tau \in [0, \phi(0))} \lim_{n \to \infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \omega < \infty.$$

Remark: If $\omega = -\infty$, the solution is called super-stable.

Main results

Theorem

Let $(y_0, y_1, f) \in H^1_{(1)}(0,1) \times L^2(0,1) \times C([0,\infty);\mathbb{R})$. *Assume that* **(1)-(3)** *are satisfied. The energy* $E(\cdot)$ *vanishes in finite time* T, *i.e.*

$$E(T) = 0,$$

if, and only if $f \equiv 1$ *and* $T \geq T^* = (\alpha^+)^{-1} \circ \beta^+ \circ (\beta^-)^{-1}(0)$.

Main results

Conversely, we have:

Corollary

Let $(y_0, y_1) \in H^1_{(1)}(0,1) \times L^2(0,1)$. *For any fixed decay rate* g, *there exists a feedback function* f *given by*

$$\frac{g(\alpha^-(t)) - g(\phi \circ \alpha^-(t))}{g(\alpha^-(t)) + g(\phi \circ \alpha^-(t))} = f_g(t),$$

such that

$$E(t) \leq Cg(t)E(0), \ \forall t \geq 0.$$

Set

$$F(t) = \frac{g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t))}, \ \forall t \geq 0,$$

with $g(t) > 0$, for all $t \geq 0$, we obtain

$$\begin{aligned} \psi_n(\tau) &= \prod_{i=0}^{n} \left| F\left(\left(\alpha^{-}\right)^{-1} \circ \phi^{[i]}(\tau) \right) \right| = \prod_{i=0}^{n} \left| \frac{g(\phi^{[i+1]}(\tau))}{g(\phi^{[i]}(\tau)} \right| \\ &= \left| \frac{g(\phi^{[n+1]}(\tau))}{g(\phi^{[0]}(\tau))} \right|. \end{aligned}$$

In this case, the existence of g is straightforward and since $F = \frac{1-f}{1+f}$, we obtain

$$\frac{g(\alpha^{-}(t)) - g(\phi \circ \alpha^{-}(t))}{g(\alpha^{-}(t)) + g(\phi \circ \alpha^{-}(t))} = f_g(t), \ \forall t \geq 0.$$

Outline

Example: Cylindrical domains

If $\alpha \equiv 0$, $\beta \equiv 1$, by the previous corollary we obtain

$$\frac{g(t) - g(t+2)}{g(t) + g(t+2)} = f_g(t), \ \forall t \geq 0.$$

- Exponential decay:

$$f(t) = \frac{e^{-st} - e^{-s(t+2)}}{e^{-st} + e^{-s(t+2)}} \Longrightarrow g(t) = e^{-st}, s > 0.$$

- Polynomial decay:

$$f(t) = \frac{(t+1)^{-s} - (t+3)^{-s}}{(t+1)^{-s} + (t+3)^{-s}} \Longrightarrow g(t) = (t+1)^{-s}, s > 0.$$

- Logarithmic decay:

$$f(t) = \frac{\log^{-s}(t+1) - \log^{-s}(t+3)}{\log^{-s}(t+1) + \log^{-s}(t+3)} \Longrightarrow g(t) = \log^{-s}(t+1), s > 0.$$

Example: Cylindrical domains

- Now, let $f(t) = \frac{t}{2+t} \Longrightarrow F(t) = \frac{1}{t+1}$, consequently, we obtain

$$\psi_n(\tau) = \prod_{i=0}^{n} \frac{1}{\tau + 2i + 1} = \frac{1}{(\tau+1)\, 2^n n!} \prod_{i=1}^{n} \left(\frac{\tau+1}{2i} + 1 \right)^{-1}.$$

A simple computation yields

$$\psi_n(\tau) \underset{n\to\infty}{\sim} \frac{C(\tau)}{(\tau+1)\, n^{\frac{\tau+1}{2}} 2^n n!},$$

Growth bound? In cylindrical domain case, $\phi^{[n]}(\tau) = 2n + \tau$.

$$\omega = \lim_{n\to\infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = -\lim_{n\to\infty} \frac{\ln n!}{2n} = -\infty.$$

$\Longrightarrow$**Super stability.**

Example: **Non**-Cylindrical domains

Let $\alpha(t) = rt$, $\beta(t) = kt + 1$, $r, k \in (-1, 1)$. Assume that $k \geq r \Longrightarrow \alpha(t) < \beta(t), \forall t \geq 0$. Then:

$$\phi(\tau) = \underbrace{\frac{(1+k)\,(1-r)}{(1-k)\,(1+r)}}_{a}\tau + \underbrace{\frac{2\,(1-r)}{(1-k)\,(1+r)}}_{b} = a\tau + b,$$

consequently,

$$\phi^{[n]}(\tau) = \begin{cases} a^n\left(\tau - \frac{b}{(1-a)}\right) + \frac{b}{(1-a)}, & r < k, \\ \tau + \frac{2n}{(1+r)}, & r = k. \end{cases}$$

Assumption **(3)** is satisfied if, and only if $a > 1$.

Example: **Non**-Cylindrical domains

$$\left(\alpha^{-}\right)^{-1} \circ \phi^{[n]}(\tau)$$

$$= \begin{cases} \underbrace{a^n\left(\frac{\tau}{1-r} - \frac{b}{(1-a)(1-r)}\right)}_{s(\tau)} + \underbrace{\frac{b}{(1-a)(1-r)}}_{z}, & r < k, \\ \frac{\tau}{1-r} + \frac{2n}{(1+r)(1-r)}, & r = k. \end{cases}$$

Example: **Non**-Cylindrical domains (**Constant feedback**)

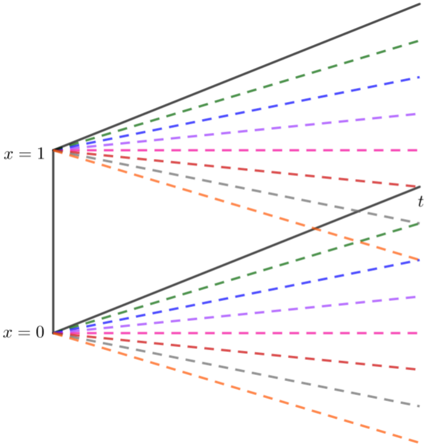


Figure: r=k

Example: **Non**-Cylindrical domains

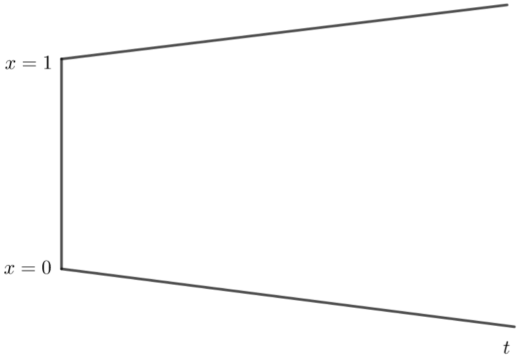


Figure: r<k

Example: **Non**-Cylindrical domains

Let $f \in \mathbb{R}\backslash\{-1,1\}$, then

$$\psi_n = F^{n+1} = \left|\frac{f-1}{f+1}\right|^{n+1}.$$

- If $r < k$:
  Growth bound?

$$\omega = \lim_{n\to\infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n\to\infty} \frac{(n+1)\ln\left|\frac{f-1}{f+1}\right|}{a^n} = 0.$$

$\Longrightarrow$**Weaker than exponential!**

Example: **Non**-Cylindrical domains (**Constant feedback**)

Let $g(t) = t^{-s}$. if $a^{-s} = \frac{f-1}{f+1}$, $s > 0$, then

$$\lim_{n\to\infty} \frac{\psi_n(\tau)}{g\left(\phi^{[n]}(\tau)\right)} = \lim_{n\to\infty} \frac{\left|\frac{f-1}{f+1}\right|^{n+1}}{\left(a^n s(\tau) + z\right)^{-s}} = C(\tau, r, k), \forall \tau \in [0, b).$$

$\Longrightarrow$**Polynomial decay rate**: t^{-s}, $s = -\ln^{-1}|a| \ln\left|\frac{f-1}{f+1}\right|$.

- If $r = k$:
  Growth bound?

$$\begin{aligned} \omega &= \lim_{n\to\infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n\to\infty} \frac{(n+1)\ln\left|\frac{f-1}{f+1}\right|}{\frac{2n}{(1+r)(1-r)}} \\ &= \frac{(1+r)(1-r)}{2} \ln\left|\frac{f-1}{f+1}\right|. \end{aligned}$$

$\Longrightarrow$**Exponential decay rate.**

Example: **Non**-Cylindrical domains (**Time-variant feedback**)

Take $f(t) = \frac{t}{2+t} \Longrightarrow F(t) = \frac{1}{t+1}$.

$\mathbf{Recall\ that\ f(t) =} \frac{t}{2+t} \Longrightarrow \mathbf{super - stability\ in\ th\ cylindrical\ case}.$

- If $r < k$:

$$\psi_n(\tau) = \prod_{i=0}^{n} \frac{1}{|a^i s(\tau) + z|} = \frac{1}{a^{\frac{n(n+1)}{2}} s^{n+1}(\tau)} \prod_{i=0}^{n} \left| 1 + \frac{z}{a^i s(\tau)} \right|^{-1}.$$

Since $a > 1$

$$\psi_n(\tau) \underset{n\to\infty}{\sim} C(r,k,\tau) a^{-\frac{n(n+1)}{2}} s^{-n-1}(\tau),\ \forall \tau \in [0,b).$$

Growth bound?

$$\omega = \lim_{n\to\infty} \frac{\ln \psi_n(\tau)}{\phi^{[n]}(\tau)} = -\lim_{n\to\infty} \frac{n^2}{2a^n} \ln|a| = 0.$$

$\Longrightarrow$**Weaker than exponential!**

Example: **Non**-Cylindrical domains (**Time-variant feedback**)

What is the decay rate in this case?.

The leading term of $\psi_n(\tau)$ is $a^{-\frac{n^2}{2}}$. Letting $g(t) = e^{-\frac{1}{2}\log_a^2(t)}$ yields

$$a^{-\frac{n^2}{2}} \underset{n\to\infty}{\sim} Cg(a^n s(\tau)+z),\ \forall \tau \in [0,b)\,.$$

$\Longrightarrow$**Decay rate with $g(t) = e^{-\frac{1}{2}\log_a^2(t)}$.**

- If $k = r$:

$$\psi_n(\tau) \underset{n\to\infty}{\sim} C(r,\tau)\frac{(1+r)^n\,(1-r)^{n+1}}{2^n n!\,(\tau+1-r)\,n^{\frac{(1+r)(\tau+1-r)}{2}}}.$$

Growth bound?

$$\omega = \lim_{n\to\infty}\frac{\ln\psi_n(\tau)}{\phi^{[n]}(\tau)} = \lim_{n\to\infty} -\frac{\ln n!}{2n} = -\infty.$$

$\Longrightarrow$**Super stable!**

Outline

Sketch of the proof

Introduce the Riemann invariants

$$\begin{cases} p = & y_t - y_x, \\ q = & y_t + y_x. \end{cases}$$

An elementary computation shows that system (S) transforms into

$$(P) : \begin{cases} p_t + p_x = 0, & \text{in} \quad Q, \\ q_t - q_x = 0, & \text{in} \quad Q, \\ (p + F(t)q)\,(t, \alpha(t)) = 0, \;\; (p+q)\,(t, \beta(t)) = 0, & \text{in} \quad (0, \infty), \\ p(0,x) = \widetilde{p}(x)\ ,\ q(0,x) = \widetilde{q}(x), & \text{in} \quad (0,1), \end{cases}$$

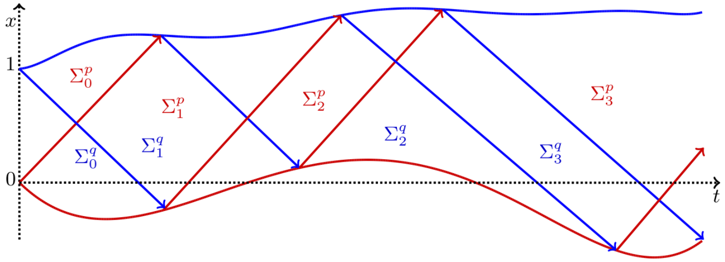
where $F(t) = \frac{1-f(t)}{1+f(t)}$.

$$\|p(t)\|^2_{L^2(\alpha(t),\beta(t))} + \|q(t)\|^2_{L^2(\alpha(t),\beta(t))} = 4E(t).$$

Sketch of the proof

Let us start by splitting Q into an infinite number of parts. Namely

$$Q = \cup_{n\geq 0}\Sigma_n^p = \cup_{n\geq 0}\Sigma_n^q, \quad \Sigma_i^p \cap \Sigma_j^p, \Sigma_i^q \cap \Sigma_j^q = \varnothing, \; i \neq j,$$



Sketch of the proof

Denote by p_n, q_n the restriction of p, q on Σ_n^p and Σ_n^q respectively. The solution p_n, q_n, $n \geq 0$ to system (P) are given by

$$p_0(t,x) = \widetilde{p}(x-t),$$

$$q_0(t,x) = \widetilde{q}(x+t),$$

$$p_1(t,x) = -F\left(\left(\alpha^-\right)^{-1}(t-x)\right)\widetilde{q}\left(\alpha^+ \circ \left(\alpha^-\right)^{-1}(t-x)\right),$$

$$q_1(t,x) = -\widetilde{p}\left(-\beta^- \circ \left(\beta^+\right)^{-1}(x+t)\right),$$

Sketch of the proof

$$p_2(t,x) = F\left(\left(\alpha^-\right)^{-1}(t-x)\right)\widetilde{p}\left(-\phi^{-1}(t-x)\right),$$

$$q_2(t,x) = F\left(\left(\alpha^-\right)^{-1}\circ\beta^-\circ\left(\beta^+\right)^{-1}(x+t)\right)\widetilde{q}\left(\xi^{-1}(x+t)\right),$$

where

$$\xi := \beta^+\circ\left(\beta^-\right)^{-1}\circ\alpha^-\circ\left(\alpha^+\right)^{-1}$$

If $f\equiv 1$ then $F\equiv 0 \Rightarrow q$ is identically zero from the time that q_2 will be zero, that is $t\geq T^* = \left(\alpha^+\right)^{-1}\circ\beta^+\circ\left(\beta^-\right)^{-1}(0)$.

Sketch of the proof

$$p_{2n+1}(t,x) = -\prod_{k=0}^{n} F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[k]}(t-x)\right) \times \widetilde{q}\left(\left(\xi^{-1}\right)^{[n]} \circ \alpha^{+} \circ \left(\alpha^{-}\right)^{-1}(t-x)\right),$$

$$p_{2n+2}(t,x) = \prod_{k=0}^{n} F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[k]}(t-x)\right) \times \widetilde{p}\left(-\left(\phi^{-1}\right)^{[n+1]} \circ (t-x)\right),$$

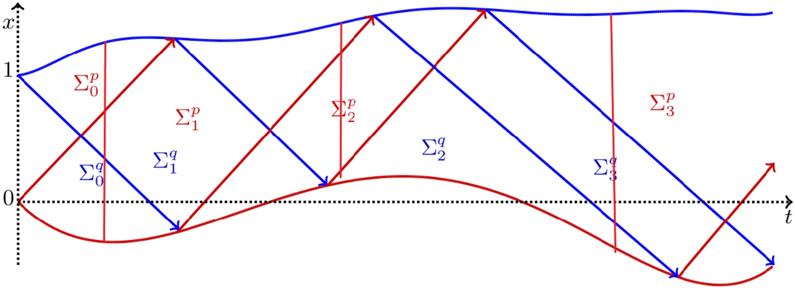
Sketch of the proof

$$q_{2n+1}(t,x) = -\prod_{k=0}^{n-1} F\left(\left(\alpha^-\right)^{-1} \circ \left(\phi^{-1}\right)^{[k]} \circ \beta^- \circ \left(\beta^+\right)^{-1}(x+t)\right) \times$$
$$\widetilde{p}\left(-\left(\phi^{-1}\right)^{[n]} \circ \beta^- \circ \left(\beta^+\right)^{-1}(x+t)\right)$$

$$q_{2n+2}(t,x) = \widetilde{q}\left(\left(\xi^{-1}\right)^{[n+1]}(x+t)\right) \times$$
$$\prod_{k=0}^{n} F\left(\left(\alpha^-\right)^{-1} \circ \left(\phi^{-1}\right)^{[k]} \circ \beta^- \circ \left(\beta^+\right)^{-1}(x+t)\right).$$

Sketch of the proof

At time $t > 0$, the components $p(t)$ and $q(t)$ might involve at most three values of the restrictive solutions $p_n(t)$ and $q_n(t)$ respectively.



Sketch of the proof

If for instance $p(t)$ is expressed in function of $p_{2n-1}(t), p_{2n}(t), p_{2n+1}(t)$, it can be expressed as

$$p(t,x) = \begin{cases} p_{2n-1}(t,x), & (t,x) \in \Sigma^p_{2n-1}, \\ p_{2n}(t,x), & (t,x) \in \Sigma^p_{2n}, \\ p_{2n+1}(t,x), & (t,x) \in \Sigma^p_{2n+1}, \end{cases}$$

therefore

$$\|p(t)\|^2_{L^2(\alpha(t),\beta(t))} = \sum_{k=1}^{3} \int_{(t,x)\in\Sigma^p_{2n+k-2}} |p_{2n+k-2}(t,x)|^2 \, dx,$$

Sketch of the proof

By using the exact solution formulas we obtain for $k = 1, 2, 3$

$$\begin{aligned} & \|p(t)\|_{L^2(\alpha(t),\beta(t))} \\ \leq \quad & \|(\widetilde{p}, \widetilde{q})\|_{[L^2(0,1)]^2} \times \\ & \sum_{k=1}^{3} \sup_{x,(t,x)\in\Sigma^p_{2n+k-2}} \prod_{i=0}^{n-1+\left[\frac{k-1}{2}\right]} \left| F\left(\left(\alpha^-\right)^{-1} \circ \left(\phi^{-1}\right)^{[i]} (t-x) \right) \right| . \end{aligned}$$

By definition of the regions Σ^p_n, $n \geq 0$ we have

$$\begin{aligned} (t,x) \quad \in \quad \Sigma^p_{2n} \quad & \Leftrightarrow t - x \in \left[\phi^{[n-1]} \circ \alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi^{[n]}(0)\right) \\ & \Leftrightarrow \quad t - x \in \phi^{[n-1]}\left(\left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right)\right), \end{aligned}$$

$$\begin{aligned} (t,x) \quad \in \quad \Sigma^p_{2n+1} & \Leftrightarrow t - x \in \left[\phi^{[n]}(0), \phi^{[n]} \circ \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right) \\ & \Leftrightarrow \quad t - x \in \phi^{[n]}\left(\left[0, \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right)\right), \end{aligned}$$

Sketch of the proof

Therefore, there exist a sequences $\tau_1^n(t,x) \in \left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right)$ and $\tau_2^n(t,x) \in \left[0, \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right)$ such that

$$\begin{array}{rcl} (t,x) & \in & \Sigma_{2n}^p \quad \Leftrightarrow t - x = \phi^{[n-1]}\left(\tau_1^n(t,x)\right), \\ (t,x) & \in & \Sigma_{2n+1}^p \Leftrightarrow t - x = \phi^{[n]}(\tau_2^n(t,x)). \end{array}$$

Observe that when (t,x) runs Σ_{2n}^p (resp. Σ_{2n+1}^p), the bounded sequence $\tau_1^n(t,x)$ (resp. $\tau_2^n(t,x)$) rises $\left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right)$ (resp. $\left[0, \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right)$). These sequences will play the role of two parameters $\tau_1 \in \left[\alpha^- \circ \left(\alpha^+\right)^{-1}(1), \phi(0)\right)$ and $\tau_2 \in \left[0, \alpha^- \circ \left(\alpha^+\right)^{-1}(1)\right)$. With these notations, we have

Sketch of the proof

$$\sum_{k=1}^{3} \sup_{x,(t,x)\in\Sigma^{p}_{2n+k-2}} \prod_{i=0}^{n+k-2} \left| F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[i]} (t-x) \right) \right|$$

$$\leq \sup_{\tau_2\in\left[0,\alpha^{-}\circ(\alpha^{+})^{-1}(1)\right)} \prod_{i=0}^{n-1} \left| F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[i]} \circ \phi^{[n-1]}(\tau_2) \right) \right|$$

$$+ \sup_{\tau_1\in\left[\alpha^{-}\circ(\alpha^{+})^{-1}(1),\phi(0)\right)} \prod_{i=0}^{n-1} \left| F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[i]} \circ \phi^{[n-1]}(\tau_1) \right) \right|$$

$$+ \sup_{\tau_2\in\left[0,\alpha^{-}\circ(\alpha^{+})^{-1}(1)\right)} \prod_{i=0}^{n} \left| F\left(\left(\alpha^{-}\right)^{-1} \circ \left(\phi^{-1}\right)^{[i]} \circ \phi^{[n]}(\tau_2) \right) \right|$$

Sketch of the proof

$$= \sup_{\tau\in\left[0,\alpha^-\circ(\alpha^+)^{-1}(1)\right)} \psi_{n-1}(\tau) + \sup_{\tau\in\left[\alpha^-\circ(\alpha^+)^{-1}(1),\phi(0)\right)} \psi_{n-1}(\tau)$$
$$+ \sup_{\tau\in\left[0,\alpha^-\circ(\alpha^+)^{-1}(1)\right)} \psi_n(\tau).$$

So,

$$\|p(t)\|_{L^2(\alpha(t),\beta(t))} \leq C\,\|(\widetilde{p},\widetilde{q})\|_{[L^2(0,1)]^2} \sup_{\tau\in[0,\phi(0))} \psi_n(\tau)$$

We deal with q in the same way.

Sketch of the proof

From what preceed, we deduce that

$$\sup_{\tau\in[0,\phi(0))} \psi_n(\tau) \underset{n\to\infty}{\longrightarrow} 0 \Longrightarrow E(t) \underset{t\to\infty}{\longrightarrow} 0.$$

If there exists a positive function g such that

$$Cg\left(\phi^{[n]}(\tau)\right) \underset{n\to\infty}{\sim} \psi_n(\tau),\ \forall\tau\in[0,\phi(0)),$$

then obviously the solution decays like g.

Sketch of the proof

The proof of the necessary part is straightforward, we have

$$
\begin{aligned}
&\int_{(t,x)\in\Sigma^{p}_{2n+1}} |p_{2n+1}(t,x)|^2\,dx + \int_{(t,x)\in\Sigma^{p}_{2n+1}} |p_{2n+1}(t,x)|^2\,dx \\
\geq\ & C\,\|(\widetilde{p},\widetilde{q})\|^2_{L^2(0,1)\times L^2(0,1)} \times \\
& \left[\inf_{\tau\in\left[0,\alpha^-\circ(\alpha^+)^{-1}(1)\right)} \psi_n^2(\tau) + \inf_{\tau\in\left[\alpha^-\circ(\alpha^+)^{-1}(1),\phi(0)\right)} \psi_n^2(\tau)\right].
\end{aligned}
$$

Without loss of generality, if $\inf\limits_{\tau\in[0,\phi(0))} \psi_n(\tau)$ doesn't tend to zero there is no decay.

Thank You!