The fixed angle scattering problem with a first order perturbation

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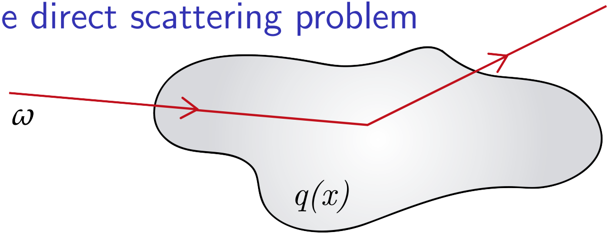
The direct scattering problem



We consider a potential $q \in L^r(\mathbb{R}^n)$ with $r > n/2$, and the stationary Schrödinger equation:

$$\begin{cases} (-\Delta - \lambda^2 + q)u = 0 & \text{in} \quad \mathbb{R}^n \\ u(\lambda, x, \omega) = e^{i\lambda\omega\cdot x} + u_s(\lambda, x, \omega) & \text{in} \quad \mathbb{R}^n \end{cases}$$

$u_s(\lambda, x, \omega)$ is the scattering solution. Satisfies as $|x| \to \infty$

$$u_s(\lambda, x, \omega) = C\lambda^{\frac{n-3}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} a_q\left(\lambda, \frac{x}{|x|}, \omega\right) + o(|x|^{-\frac{n-1}{2}}),$$

The Inverse Problem

$a_q(\lambda, \theta; \omega)$ is called the far field pattern. Is the angular profile of the scattering solution very far away.

The fixed angle scattering problem consists in recovering q from the knowledge of $a_q(\lambda, \theta; \omega)$ for all $\lambda > \lambda_0$, $\theta \in S^{n-1}$ for a fixed $\omega \in S^{n-1}$. We consider this as one measurement in the sense that only one incident wave has sent.

What is known:

- Uniqueness potential zero: $a_q(\lambda, \theta; \omega) = 0 \implies q = 0$. A. Bayliss, Y. Li and C. Morawetz 1989.
- Generic uniqueness ($n = 3$) Stefanov 1992.
- Uniqueness and reconstruction small potentials: J.A Barceló, C. Castro, T. Luque, M.C. Vilela 2018 and 2019 (low regularity results).
- Recovery of singularities. A. Ruiz 2001, C. M. 2018

Rakesh and M. Salo results 2019,2020:

- If one knows $a_q(\lambda, \theta; e_1)$ and $a_q(\lambda, \theta; -e_1)$, then this uniquely determines $q \in C_c^\infty(\mathbb{R}^n)$ (two measurements).
- If the potential is symmetric with respect to e_1: $q(x_1, x_2, \dots, x_n) = \pm q(-x_1, x_2, \dots, x_n)$ or close to a symmetric potential, then only one measurement is necessary.

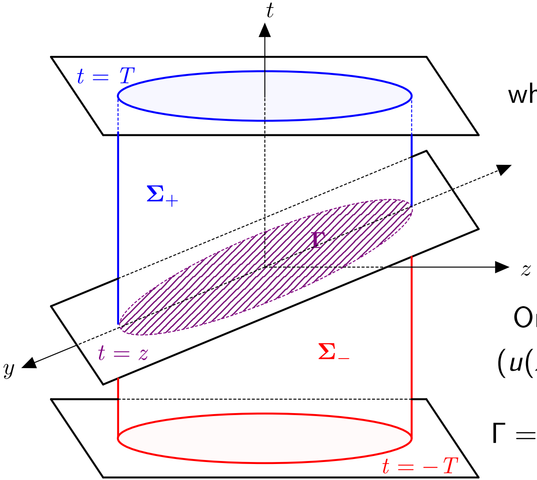
1. Reduction to a time domain inverse scattering problem for the wave equation: The direct problem is now

$$(\partial_t^2 - \Delta + q) U_q(x, t; \omega) = 0 \text{ in } \mathbb{R}^{n+1}, \qquad U_q|_{\{t<-1\}} = \delta(t - x \cdot \omega),$$

so that $U_q(x, t; \omega) = u(x, t; \omega) H(t - x \cdot \omega) + \delta(t - x \cdot \omega)$

$$\begin{aligned} (\partial_t^2 - \Delta + q) u = 0 & \qquad \text{in } \{t > x \cdot \omega\}, \\ u(x, x \cdot \omega; \omega) = -\frac{1}{2} \int_{-\infty}^0 q(x + s\omega)\, ds. & \qquad \text{in } \{t = x \cdot \omega\}. \end{aligned}$$

Measurements in time domain



where $z = x \cdot \omega$ and $y \in \mathbb{R}^{n-1}$

$$\Sigma = \partial B \times (-T, T)$$
$$\Sigma_+ = \Sigma \cap \{t \geq x \cdot \omega\}$$

One measures $u(x, t; \omega)$ at Σ.
($u(x, t; \omega)$ is supported in Σ_+).

$$\Gamma = B \times (-T, T) \cap \{t = x \cdot \omega\}$$

Carleman Estimate

We consider $q_1, q_2 \in C_c^\infty$. If ϕ is strongly pseudoconvex weight, rapidly decreasing in $(t - z)$ one has the Carleman estimate

$$\|e^{\sigma\phi}(\partial_t + \partial_z)w_+\|^2_{L^2(\Gamma)} + \sigma \int_\Gamma e^{2\sigma\phi} F^j(x, \sigma w_+, \nabla w_+)\nu_j \, dS$$
$$\lesssim \|e^{\sigma\phi}(\Box + q_1)w_+\|^2_{L^2(Q_+)} + \sigma^3 \|e^{\sigma\phi} w_+\|^2_{H^1(\Sigma_+)}.$$

We take $w_+(x,t) = u_1(x,t,\omega) - u_2(x,t,\omega)$. One needs the integral in time of $e^{\sigma\phi}$ to go to zero with σ.

$$\begin{aligned} |(\Box + q_1)w_+| &\le |(q_2 - q_1)u_2| \lesssim |q_1 - q_2| \quad && \text{in} \quad Q_+, \\ |(\partial_t + \partial_z)w_+| &= |q_1 - q_2| && \text{in} \quad \Gamma, \end{aligned}$$

$$\|e^{\sigma\phi}(q_1 - q_2)\|^2_{L^2(B)} + \sigma \int_\Gamma e^{2\sigma\phi} F^j(x, \sigma w_+, \nabla w_+)\nu_j \, dS \lesssim \sigma^3 \|e^{\sigma\phi} w_+\|^2_{H^1(\Sigma_+)}.$$

We define $w_-(x,t) = -u_1(x,-t,-\omega) + u_2(x,-t,-\omega)$ so that

$$\|e^{\sigma\phi}(q_1-q_2)\|^2_{L^2(B)} - \sigma \int_\Gamma e^{2\sigma\phi} F^j(x,\sigma w_-,\nabla w_-)\nu_j \, dS \lesssim \sigma^3 \|e^{\sigma\phi} w_-\|^2_{H^1(\Sigma_-)}.$$

Adding the Carleman estimates one gets

$$\|e^{\sigma\phi}(q_1-q_2)\|^2_{L^2(B)} \lesssim \sigma^3 \|e^{\sigma\phi}(w_+-w_-)\|^2_{H^1(\Gamma)} + \sigma^3 \sum_{\pm} \|e^{\sigma\phi} w_\pm\|^2_{H^1(\Sigma_\pm)}.$$

We can use now that

$$\begin{aligned} 2(w_+-w_-)|_\Gamma(x) &= \int_{-\infty}^0 (q_1-q_2)(x+s\omega)\, ds + \int_{-\infty}^0 (q_1-q_2)(x-s\omega)\, ds \\ &= \int_{-\infty}^{\infty} (q_1-q_2)(x+s\omega)\, ds \quad (=0). \end{aligned}$$

Magnetic case

The direct problem becomes

$$\begin{cases} (H_{\mathbf{A},q} - \lambda^2)u = 0 & \text{in} \quad \mathbb{R}^n \\ u(\lambda, x, \omega) = e^{i\lambda\omega\cdot x} + u_s(\lambda, x, \omega) & \text{in} \quad \mathbb{R}^n \end{cases}$$

where the Hamiltonian is given by

$$H_{\mathbf{A},q}u = (i\nabla - \mathbf{A})^2 u + qu = -\Delta u - i\mathbf{A}\cdot\nabla u + (-i\nabla\cdot\mathbf{A} + \mathbf{A}^2 + q)u$$

- $a_{\mathbf{A},q}(\lambda, \theta; \omega)$ is defined as before.
- Gauge invariance: if $H_{\mathbf{A},q}u = v$ for some functions u and v, then $H_{\mathbf{A}+\nabla f,q}(e^{-if}u) = e^{-if}v$.

Analogously, the time domain direct problem is

$$(\partial_t^2 + H_{\mathbf{A},q})U = 0 \text{ in } \mathbb{R}^{n+1}, \quad U|_{\{t<-1\}} = \delta(t - x\cdot\omega),$$

Theorem (C. M, L. Potenciano, M. Salo)

Let $e_1, \dots, e_n$ be any orthonormal basis in $\mathbb{R}^n$. Assume $\mathbf{A}_1, \mathbf{A}_2 \in C_c^\infty(B)$ and $q_1, q_2 \in C_c^\infty(B)$ and that

$$(C) \quad \int_{-\infty}^{\infty} e_n \cdot \mathbf{A}_k(x_1, \dots, x_{n-1}, s)\, ds = 0 \quad \text{for } x_1, \dots, x_{n-1} \in \mathbb{R}.$$

If for all $\theta \in S^{n-1}$ and $\lambda \geq \lambda_0$ we have

$$a_{\mathbf{A}_1, q_1}(\lambda, \theta; \pm e_j) = a_{\mathbf{A}_2, q_2}(\lambda, \theta; \pm e_j) \quad \text{for all } j = 1, \dots n,$$

then $d\mathbf{A}_1 = d\mathbf{A}_2$ and $q_1 = q_2$.

- Gauge invariance implies that one recovers at best only $d\mathbf{A}$. We use $2n$ measurements to recover n functions.
- Main difficulty: How to decouple information from q from information on $\mathbf{A}$.

- If we fix the zero order term, gauge invariance breaks, and one recovers completely $\mathbf{A}_1 = \mathbf{A}_2$ without assuming (C).
- The numbers of measurements are reduced if the potentials have certain symmetries.

Theorem (C. M, L. Potenciano, M. Salo)

Let $e_1, \dots, e_n$ be any orthonormal basis in $\mathbb{R}^n$. Assume $\mathbf{A}_1, \mathbf{A}_2 \in C_c^\infty(B)$ and $q_1, q_2 \in C_c^\infty(B)$ and that

$$(C) \quad \int_{-\infty}^{\infty} e_n \cdot \mathbf{A}_k(x_1, \dots, x_{n-1}, s)\, ds = 0 \quad \text{for } x_1, \dots, x_{n-1} \in \mathbb{R}.$$

Also, let $1 \leq j \leq n$ and consider the $2n$ solutions $U_{k,\pm j}(x,t)$ of

$$(\partial_t^2 + H_{\mathbf{A}_k, q_k})U_{k,\pm j} = 0 \ \text{ in } \mathbb{R}^{n+1}, \quad U_{k,\pm j}|_{\{t<-1\}} = \delta(t - \pm x_j).$$

If for each $1 \leq j \leq n$ one has $U_{1,\pm j} = U_{2,\pm j}$ on the surface $\Sigma \cap \{t \geq \pm x_j\}$, then $d\mathbf{A}_1 = d\mathbf{A}_2$ and $q_1 = q_2$.

$$(\star) \quad (\partial_t^2 + H_{\mathbf{A},q})U_\delta = 0 \text{ in } \mathbb{R}^{n+1}, \quad U_\delta|_{\{t<-1\}} = \delta(t - x\cdot\omega),$$

Then there is a unique solution U_δ of $(\star)$ and

$$U_\delta(x,t;\omega) = v(x,t)H(t - x\cdot\omega) + e^{\psi(x)}\delta(t - x\cdot\omega),$$

where $\psi(x) = -\int_{-\infty}^{0} \omega\cdot i\mathbf{A}(x + s\omega)\,ds$ and

$$(\partial_t^2 + H_{\mathbf{A},q})v = 0 \qquad \{t > x\cdot\omega\}$$

and

$$v|_\Gamma(x) = \frac{1}{2}e^{\psi(x)}\int_{-\infty}^{0}\left[\nabla\cdot(i\mathbf{A} + \nabla\psi) + (i\mathbf{A} + \nabla\psi)^2 - q\right](x+s\omega)\,ds.$$

We substitute $(\star)$ by

$$(\partial_t^2 + H_{\mathbf{A},q})U_H = 0 \text{ in } \mathbb{R}^{n+1}, \quad U_H|_{\{t<-1\}} = H(t - x\cdot\omega),$$

Equivalence of information between H-waves and δ-waves:

$$U_H(x,t;\omega) = u(x,t)H(t - x\cdot\omega),$$

where u satisfies

$$\begin{aligned} (\partial_t^2 + H_{\mathbf{A},q})u = 0 \qquad &\text{in} \quad \{t > x\cdot\omega\}\,, \\ u|_\Gamma(x) = e^{\psi} = e^{-\int_{-\infty}^0 \omega\cdot i\mathbf{A}(x+s\omega)\,ds} \end{aligned} \tag{1}$$

Main steps in the proof:

1. Prove the equivalence of H-waves and δ-waves.

2. The boundary data allows us to make a change of gauge where $(\mathbf{A}_1 - \mathbf{A}_2)\cdot e_n = 0$. In this gauge $d\mathbf{A}_1 = d\mathbf{A}_2 \iff \mathbf{A}_1 = \mathbf{A}_2$.
3. Construction of a pseudoconvex weight $\phi(x,t;\omega)$ such that $\phi(x, x\cdot\omega;\omega) = \phi_0(x)$ independent of ω (and all previous requirements).
4. Obtain the Carleman estimate with a first order perturbation

$$\|e^{\sigma\phi}(\partial_t + \partial_z + i\mathbf{A}_1\cdot\omega)w_\pm\|^2_{L^2(\Gamma)} \pm \sigma \int_\Gamma e^{2\sigma\phi} F^j(x, \sigma w_\pm, \nabla w_\pm)\nu_j \, dS$$
$$\lesssim \|e^{\sigma\phi}(\partial_t^2 + H_{\mathbf{A}_1,q_1})w_\pm\|^2_{L^2(Q_\pm)} + \sigma^3\|e^{\sigma\phi}w_\pm\|^2_{H^1(\Sigma_\pm)}.$$

If $w_\pm$ is a difference of solutions of the wave equation for potentials $\mathbf{A}_1, q_1$ and $\mathbf{A}_2, q_2$ one has in any case that

$$\|e^{\sigma\phi}(\partial_t + \partial_z + iA_1 \cdot \omega)w_\pm\|^2_{L^2(\Gamma)} \pm \sigma \int_\Gamma e^{2\sigma\phi} F^j(x, \sigma w_\pm, \nabla w_\pm)\nu_j \, dS$$
$$\lesssim \kappa(\sigma)(\|e^{\sigma\phi}(\mathbf{A}_1 - \mathbf{A}_2)\|^2_{L^2(B)} + \|e^{\sigma\phi}(q_1 - q_2)\|^2_{L^2(B)}) + \sigma^3\|e^{\sigma\phi} w_\pm\|^2_{H^1(\Sigma_\pm)}.$$

5. Apply this with H-waves and $\omega = \pm e_1, \pm e_2, \ldots, \pm e_{n-1}$

$$w_+ = u_1(x, t; \omega) - u_2(x, t; \omega) \quad w_- = e^\mu(u_1(x, -t; -\omega) - u_2(x, -t; -\omega))$$

so that

$$\|e^{\sigma\phi}\omega \cdot (\mathbf{A}_1 - \mathbf{A}_2)\|^2_{L^2(B)} \lesssim \kappa(\sigma)(\|e^{\sigma\phi}(\mathbf{A}_1 - \mathbf{A}_2)\|^2_{L^2(B)} + \|e^{\sigma\phi}(q_1 - q_2)\|^2_{L^2(B)})$$

6. Apply the Carleman with δ-waves. from directions $\pm e_n$.

$$\|e^{\sigma\phi}(q_1 - q_2)\|^2_{L^2(B)} \lesssim \kappa(\sigma)\|e^{\sigma\phi}(\mathbf{A}_1 - \mathbf{A}_2)\|^2_{L^2(B)}$$

Thank you!

¡Gracias!