On equilibria for generalized Boussinesq fluid-chemical models with multiplicative controls

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January 21, 2021

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Outline

The Model

Let $T > 0$ be a fixed time horizon, $\Omega \subseteq \mathbb{R}^d$ be a bounded smooth domain, where $d \in \{2,3\}$. We write $Q :=]0,T[\times \Omega$, as well as $\Sigma := [0,T] \times \partial\Omega$. The model we study comprises the evolution of the velocity field $u = u(t,x)$ of a fluid containing a chemical substance, with concentration $\theta = \theta(t,x)$; it is given by

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p = \theta \boldsymbol{g} + v\boldsymbol{u}, & \text{in } Q, \\ \theta_t - \sigma\Delta\theta + \boldsymbol{u}\cdot\nabla\theta = w\theta, & \text{in } Q, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } Q, \\ \boldsymbol{u} = 0 \text{ and } \frac{\partial\theta}{\partial\boldsymbol{\nu}} = 0, & \text{on } \Sigma, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \text{ and } \theta|_{t=0} = \theta_0, & \text{in } \Omega. \end{cases} \tag{1.1}$$

We primarily want to act over the chemical substance. This leads us to consider the functional

$$J_2(v,w) := \frac{\beta}{2}\int_\omega |\theta(T,x)-\theta^T(x)|^2\,dx + \frac{\mu}{2}\int_Q |w(t,x)|^2\,d(t,x).$$

Our action over θ influences the velocity field $\boldsymbol{u}$, and we use the criterion

$$J_1(v,w) := \frac{\alpha}{2}\int_\omega |\boldsymbol{u}(T,x)-\boldsymbol{u}^T(x)|^2\,dx + \frac{\kappa}{2}\int_Q |v(t,x)|^2\,d(t,x),$$

to determine how we should adjust our strategy in accordance to this influence. Our goal is to harmonize the possibly conflicting objective functionals J_1 and J_2. In this direction, we consider Pareto optimal strategies.

Now, we assume there are N agents individually behaving as before. They are acting strategically and competing. Thus, we focus on the analysis of Nash equilibria for the functionals

$$
\begin{aligned}
J_i(\boldsymbol{a}^i;\boldsymbol{a}^{-i}) :=& \frac{1}{2}\int_{\omega_i}\left\{\alpha^i|u(T,x)-u^i_T(x)|^2+\beta^i|\theta(T,x)-\theta^i_T(x)|^2\right\}dx \\
&+\frac{1}{2}\int_Q\left\{\kappa^i|v^i(t,x)|^2+\mu^i|w^i(t,x)|^2\right\}d(t,x).
\end{aligned}
$$

Well-posedness of the main model

We assume $\boldsymbol{u}_0 \in V$, $\theta_0 \in H^1(\Omega)$, $\boldsymbol{g} \in L^\infty(Q)^d$ and $(v, w) \in [L^\infty(Q)]^2$.

Lemma

System (1.1) *admits a weak solution* $(\boldsymbol{u}, p, \theta)$, *in the sense that this triplet belongs to the class*

$$\begin{cases} \boldsymbol{u} \in C_w([0,T]; H) \cap L^2(0,T;V) \cap W^{1,4/3}(0,T;V'), \\ p \in W^{-1,\infty}(0,T;L^2(\Omega)), \\ \theta \in C_w([0,T]; L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap W^{1,4/3}(0,T;H^1(\Omega)'), \end{cases}$$

and satisfies

$$\begin{cases} \langle \boldsymbol{u}_t, \boldsymbol{\xi} \rangle_{V' \times V} + \nu (\boldsymbol{u}, \boldsymbol{\xi})_V + ((\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}, \boldsymbol{\xi})_H = (\theta \boldsymbol{g} + v \boldsymbol{u}, \boldsymbol{\xi})_H, \\ \langle \theta_t, \eta \rangle_{H^1(\Omega)' \times H^1(\Omega)} + \sigma (\nabla \theta, \nabla \eta)_{L^2(\Omega)^d} + (\boldsymbol{u} \cdot \nabla \theta, \eta) = (w\theta, \eta), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \text{ and } \theta(0) = \theta_0, \text{ weakly in } \Omega, \end{cases}$$

for a.e. $t \in [0,T]$ *and every* $(\boldsymbol{\xi}, \eta) \in V \times H^1(\Omega)$.

Corollary

If $d = 2$, *then the weak solution* $(\boldsymbol{u}, p, \theta)$ *of Lemma 1 can be taken to satisfy*

$$\boldsymbol{u}_t \in L^2(0, T; V') \text{ and } \theta_t \in L^2(0, T; H^1(\Omega)'),$$

whence $\boldsymbol{u} \in C([0, T]; H)$ *and* $\theta \in C\left([0, T]; L^2(\Omega)\right)$.

Theorem

There exists $T^* > 0$ *such that, for each* $T \in \,]0, T^*[$*, system* (1.1) *admits a unique strong solution* $(\boldsymbol{u}, p, \theta)$*, i.e., it has the regularity*

$$\begin{cases} \boldsymbol{u} \in L^2(0, T; D(A)) \cap C([0, T]; V) \cap H^1(0, T; H), \\ p \in L^2(0, T; H^1(\Omega)), \int_\Omega p(t, x)\, dx = 0, \ \text{for a.e. } t \in [0, T], \\ \text{and } \theta \in L^2\left(0, T; H^2(\Omega)\right) \cap C\left([0, T]; H^1(\Omega)\right) \cap H^1(0, T; H), \end{cases}$$

and solves system (1.1) *at almost every* $(t, x) \in Q$*. If* $d = 2$*, then the result holds for* $T^* = \infty$*.*

We fix a time horizon $T > 0$ and initial data $(\boldsymbol{u}_0, \theta_0) \in V \times H^1(\Omega)$ such that, for each $(v, w) \in \mathcal{U} \subset L^\infty(Q)^2$, system (1.1) admits a strong solution $(\boldsymbol{u}, p, \theta) \in W$, where

$$W := \Big\{(\boldsymbol{u}, p, \theta) : \ \boldsymbol{u} \in L^2(0, T; D(A)) \cap C([0, T]; H) \cap H^1(0, T; H),$$
$$p \in L^2(0, T; H^1(\Omega)), \int_\Omega p(t, x)\, dx = 0, \ \text{for a.e. } t \in [0, T],$$
$$\theta \in L^2(0, T; H^2(\Omega)) \cap C\left([0, T]; L^2(\Omega)\right) \cap H^1\left(0, T; L^2(\Omega)\right) \Big\}.$$

Definition

We refer to the mapping

$$(v, w) \in \mathcal{U} \mapsto (\boldsymbol{u}(v, w), p(v, w), \theta(v, w)) \in W,$$

where $(\boldsymbol{u}(v, w), p(v, w), \theta(v, w))$ is the solution of (1.1) corresponding to the control (v, w), as the control-to-state mapping.

Linearized System

Given $(v^1, w^1), (v^2, w^2) \in \mathcal{U}$, there exists a weak solution $(\boldsymbol{y}, \pi, \varphi)$ of

$$\begin{cases} \boldsymbol{y}_t - \nu \Delta \boldsymbol{y} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{y} + (\boldsymbol{y} \cdot \nabla) \boldsymbol{u} + \nabla \pi = \varphi \boldsymbol{g} + v^1 \boldsymbol{y} + v^2 \boldsymbol{u}, & \text{in } Q, \\ \varphi_t - \sigma \Delta \varphi + \boldsymbol{u} \cdot \nabla \varphi + \boldsymbol{y} \cdot \nabla \theta = w^1 \varphi + w^2 \theta, & \text{in } Q, \\ \nabla \cdot \boldsymbol{y} = 0, & \text{in } Q, \\ \boldsymbol{y} = 0 \text{ and } \frac{\partial \varphi}{\partial \nu} = 0, & \text{on } \Sigma, \\ \boldsymbol{y}|_{t=0} = 0 \text{ and } \varphi|_{t=0} = 0, & \text{in } \Omega, \end{cases} \tag{2.1}$$

with regularity

$$\begin{cases} \boldsymbol{y} \in C\left([0, T]; H\right) \cap L^2\left(0, T; V\right), \ \boldsymbol{y}_t \in L^2\left(0, T; V'\right), \ \pi \in L^2(Q), \\ \varphi \in C\left([0, T]; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right) \text{ and } \varphi_t \in L^2\left(0, T; H^{-1}(\Omega)\right). \end{cases}$$

Theorem

Given $(v,w),(\widetilde{v},\widetilde{w}) \in \mathcal{U}$, *we write* $(v_\epsilon, w_\epsilon) := (v+\epsilon(\widetilde{v}-v), w+\epsilon(\widetilde{w}-w)) \in \mathcal{U}$, $(\boldsymbol{u}_\epsilon, p_\epsilon, \theta_\epsilon) := (\boldsymbol{u}(v_\epsilon, w_\epsilon), p(v_\epsilon, w_\epsilon), \theta(v_\epsilon, w_\epsilon))$, *and* $(\boldsymbol{u}, p, \theta) := (\boldsymbol{u}(v,w), p(v,w), \theta(v,w))$. *The following limit is valid,*

$$\lim_{\epsilon\to 0}\left(\frac{\boldsymbol{u}_\epsilon-\boldsymbol{u}}{\epsilon}, \frac{p_\epsilon-p}{\epsilon}, \frac{\theta_\epsilon-\theta}{\epsilon}\right) = (\boldsymbol{y}, \pi, \varphi),$$

in the topology of the space

$$\left[L^\infty(0,T;H)\cap L^2(0,T;V)\right]\times L^2(Q)\times\left[L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\right]. \tag{2.2}$$

where $(\boldsymbol{y}, \pi, \varphi)$ *is the solution of (2.1) with* $(v_1, w_1) = (v, w)$ *and* $(v_2, w_2) = (\widetilde{v}-v, \widetilde{w}-w)$.

Pareto Equilibrium

Definition

Given performance criteria $\{J_i : \mathcal{U} \to \mathbb{R}\}_{i=1}^{N}$, we call $(v^*, w^*) \in \mathcal{U}$ a Pareto equilibrium for the system if there does not exist $(\widetilde{v}, \widetilde{w}) \in \mathcal{U}$ with the properties:

(i) For all $i \in \{1, ..., N\}$,

$$J_i(\widetilde{v}, \widetilde{w}) \leqslant J_i(v^*, w^*), \text{ for all } i \in \{1, ..., N\};$$

(ii) For some $i \in \{1, ..., N\}$, strict inequality in (i) holds.

Intuitively, a Pareto equilibrium is a setting in which a particular functional can only attain a smaller value if some other increases.

First order optimality conditions

In the present setting, we focus on the following set of functionals.

Definition

Let $\boldsymbol{u}^T \in L^2(\Omega)^d$ and $\theta^T \in L^2(\Omega)$ be targets, the constants $\alpha, \beta, \kappa, \mu > 0$, $i \in \{1, ..., N\}$, be the preferences of the agent, and ω be a nonempty open subset of Ω. The performance criteria $\{J_i : \mathcal{U} \to \mathbb{R}\}_{i=1}^2$ are defined as

$$J_1(v, w) := \frac{\alpha}{2} \int_\omega |\boldsymbol{u}(T, x) - \boldsymbol{u}^T(x)|^2 \, dx + \frac{\kappa}{2} \int_Q |v(t, x)|^2 \, d(t, x), \quad (3.1)$$

and

$$J_2(v, w) := \frac{\beta}{2} \int_\omega |\theta(T, x) - \theta^T(x)|^2 \, dx + \frac{\mu}{2} \int_Q |w(t, x)|^2 \, d(t, x). \quad (3.2)$$

The following result gives us sufficient conditions to find a Pareto equilibria.

Lemma

Let the performance criteria $\{J_i : \mathcal{U} \to \mathbb{R}\}_{i=1}^N$ *be given. If there exist* $\lambda_1, ..., \lambda_N > 0$*, with* $\lambda_1 + \cdots + \lambda_N = 1$*, such that a control* $(v^*, w^*) \in \mathcal{U}$ *satisfies*

$$(v^*, w^*) \in \text{argmin}_{(v,w)\in\mathcal{U}} \left(\sum_{i=1}^{N} \lambda_i J_i(v, w) \right),$$

then it is a Pareto equilibria for these criteria.

Adjoint System

Given $(v^1, w^1) \in \mathcal{U}$, alongside with $(\boldsymbol{z}^T, \psi^T) \in L^2(\Omega)^2 \times L^2(\Omega)$, there exists a weak solution $(\boldsymbol{z}, \Pi, \psi)$ of the system

$$\begin{cases} -\boldsymbol{z}_t - \nu\Delta\boldsymbol{z} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{z} + (\nabla\boldsymbol{u})^\intercal\,\boldsymbol{z} + \nabla\Pi = v\boldsymbol{z} - \psi\nabla\theta, & \text{in } Q,\\ -\psi_t - \sigma\Delta\psi - \boldsymbol{u}\cdot\nabla\psi = w\psi + \boldsymbol{g}\cdot\boldsymbol{z}, & \text{in } Q,\\ \nabla\cdot\boldsymbol{z} = 0, & \text{in } Q,\\ \boldsymbol{z} = 0 \text{ and } \frac{\partial\psi}{\partial\nu} = 0, & \text{on } \Sigma,\\ \boldsymbol{z}|_{t=T} = \boldsymbol{z}^T \text{ and } \psi|_{t=T} = \psi^T, & \text{in } \Omega, \end{cases} \tag{3.3}$$

meaning that the triplet $(\boldsymbol{z}, \Pi, \psi)$ has regularity

$$\begin{cases} \boldsymbol{z} \in C\left([0,T]; H\right) \cap L^2\left(0,T; V\right),\ \boldsymbol{z}_t \in L^2\left(0,T;V'\right),\ \Pi \in L^2(Q),\\ \psi \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H^1(\Omega)\right) \text{ and } \psi_t \in L^2\left(0,T; H^{-1}(\Omega)\right). \end{cases}$$

Let us consider $\lambda \in]0,1[$ and $J := \lambda J_1 + (1-\lambda)J_2$. If $(v,w),(\widetilde{v},\widetilde{w}) \in \mathcal{U}$, then the Gâteaux derivative of J at (v,w) in the direction of $(v-\widetilde{v}, w-\widetilde{w})$ is

$$
\begin{aligned}
&\left\langle J'(v,w),(\widetilde{v}-v,\widetilde{w}-w)\right\rangle \\
&= \int_\omega \lambda\alpha(\boldsymbol{u}(T,x)-\boldsymbol{u}^T(x))\boldsymbol{y}(T,x) + (1-\lambda)\beta(\theta(T,x)-\theta^T(x))\varphi(T,x)\,dx \\
&+ \int_Q \{\lambda\kappa v(\widetilde{v}-v) + (1-\lambda)\mu w(\widetilde{w}-w)\}\ d(t,x) \\
&= \int_Q \{[\boldsymbol{z}\cdot\boldsymbol{u} + \kappa\lambda v]\,(\widetilde{v}-v) + [\psi\theta + \mu(1-\lambda)w]\,(\widetilde{w}-w)\}\ d(t,x).
\end{aligned}
$$

Theorem

If (v^*, w^*) *is a minimizer of* $J = \lambda J_1 + (1-\lambda)J_2$, *then*

$$(v^*, w^*) = P_{\mathcal{U}}\left(-\frac{1}{\kappa\lambda}\boldsymbol{z}\cdot\boldsymbol{u}, -\frac{1}{\mu(1-\lambda)}\psi\theta\right), \tag{3.4}$$

where $(\boldsymbol{u}, p, \theta) = (\boldsymbol{u}(v^*, w^*), p(v^*, w^*), \theta(v^*, w^*))$, $(\boldsymbol{z}, \pi, \psi)$ *solves* (3.3) *with the control* $(v, w) = (v^*, w^*)$ *and the terminal condition* $(\boldsymbol{z}, \psi)|_{t=T} = \big(\lambda\alpha(\boldsymbol{u}|_{t=T} - \boldsymbol{u}^T), (1-\lambda)\mu(\theta|_{t=T} - \theta^T)\big)$, *and* $P_{\mathcal{U}} : L^2(Q)^2 \to \mathcal{U}$ *is the projection onto* $\mathcal{U}$.

Second order conditions

$$\begin{cases} -\boldsymbol{h}_t - \nu\Delta\boldsymbol{h} - (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{h} - (\boldsymbol{y}\cdot\nabla)\,\boldsymbol{z} + (\nabla\boldsymbol{u})^{\intercal}\,\boldsymbol{h} + (\nabla\boldsymbol{y})^{\intercal}\,\boldsymbol{z} + \nabla\widetilde{\Pi} & \\ = v^1\boldsymbol{h} + v^2\boldsymbol{z} - \psi\nabla\varphi - \eta\nabla\theta, & \text{in } Q, \\ -\eta_t - \sigma\Delta\eta - \boldsymbol{u}\cdot\nabla\eta - \boldsymbol{y}\cdot\nabla\psi = w^1\eta + w^2\psi + \boldsymbol{g}\cdot\boldsymbol{h}, & \text{in } Q, \\ \nabla\cdot\boldsymbol{h} = 0, & \text{in } Q, \\ \boldsymbol{h} = 0 \text{ and } \frac{\partial\eta}{\partial\nu} = 0, & \text{on } \Sigma, \\ \boldsymbol{h}|_{t=T} = \boldsymbol{h}^T \text{ and } \eta|_{t=T} = \eta^T, & \text{in } \Omega, \end{cases} \tag{3.5}$$

and satisfies

$$\sup_{[0,T]}\left(\|\boldsymbol{h}(t)\|_H + \|\eta(t)\|\right) + \|\boldsymbol{h}\|_{L^2(0,T;V)} + \|\boldsymbol{h}_t\|_{L^2(0,T;V')} + \|\Pi\|_{L^2(Q)}$$
$$+ \|\eta\|_{L^2(0,T;H^1(\Omega))} + \|\eta_t\|_{L^2(0,T;H^{-1}(\Omega))}$$
$$\leqslant [(\alpha+\beta+1)e^{C(\alpha+\beta+1)}](\|v^2\|_{L^2(Q)} + \|w^2\|_{L^2(Q)}).$$

Calculating the second order derivative of J, we obtain

$$
\begin{aligned}
&\left\langle J''(v,w);(\widetilde{v}-v,\widetilde{w}-w),(\widetilde{v}-v,\widetilde{w}-w)\right\rangle \\
&:= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left\langle J'\left(v+\epsilon\left(\widetilde{v}-v\right),w+\epsilon\left(\widetilde{w}-w\right)\right)-J'(v,w),(\widetilde{v}-v,\widetilde{w}-w)\right\rangle \\
&= \int_Q \{(\boldsymbol{h}\cdot\boldsymbol{u}+\boldsymbol{z}\cdot\boldsymbol{y}+\kappa\lambda(\widetilde{v}-v))(\widetilde{v}-v) \\
&+ (\eta\theta+\psi\varphi+\mu(1-\lambda)(\widetilde{w}-w))(\widetilde{w}-w)\}\,d(t,x)
\end{aligned}
$$

$$
= I_1+I_2+I_3+I_4+\kappa\lambda\int_Q|\widetilde{v}-v|^2\,d(t,x)+\mu(1-\lambda)\int_Q|\widetilde{w}-w|^2\,d(t,x).
$$

Then, we estimate

$$
\begin{aligned}
|I_1| &\leqslant \int_Q |\boldsymbol{h}||\boldsymbol{u}||\widetilde{v} - v|\, d(t,x) \\
&\leqslant \|\widetilde{v} - v\|_{L^2(Q)} \|\boldsymbol{h}\|_{L^3(Q)^d} \|\boldsymbol{u}\|_{L^6(Q)^d} \\
&\leqslant C\|\widetilde{v} - v\|_{L^2(Q)} \|\boldsymbol{h}\|^{1/2}_{L^\infty(0,T;H)} \|\boldsymbol{h}\|^{1/2}_{L^2(0,T;V)} \|\boldsymbol{u}\|_{L^\infty(0,T;V)} \\
&\leqslant (\alpha + \beta + 1)e^{C(\alpha+\beta+1)} \left(\|\widetilde{v} - v\|^2_{L^2(Q)} + \|\widetilde{w} - w\|^2_{L^2(Q)} \right).
\end{aligned}
$$

Finally, we obtain

$$
\langle J''(v,w), (\widetilde{v}-v, \widetilde{w}-w)\rangle \geqslant (\mu+\kappa-(\alpha+\beta+1)e^{C_0(\alpha+\beta+1)})\|(\widetilde{v}-v, \widetilde{w}-w)\|^2_{L^2(Q)}
$$

Corollary

Let us assume that $\mu + \kappa - (\alpha + \beta + 1)e^{C_0(\alpha+\beta+1)} > 0$. *Then, a control* $(v, w) \in \mathcal{U}$ *is a Pareto equilibrium for* $\{J_1, J_2\}$ *if, and only if, it is the unique minimizer of* $J := \lambda J_1 + (1 - \lambda)J_2$, *for some* $\lambda \in]0, 1[$.

Nash Equilibrium

Now, we develop the study of equilibria of the system from a competitive multiplayer perspective. Thus, we consider $N > 1$ agents, and proceed to investigate Nash equilibria of the system they act upon. For each $i \in \{1, ..., N\}$, let us consider a set $\mathcal{U}_i \subseteq L^\infty(Q)^2$, convex and closed in $L^2(Q)^2$, comprising the controls that are admissible to player $i \in \{1, ..., N\}$. For a strategy profile $\left((v^i, w^i)\right)_{i=1}^N \in \mathcal{U} := \Pi_{i=1}^N \mathcal{U}_i$, system (1.1) reads as

$$\begin{cases} \boldsymbol{u}_t - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla p = \theta\boldsymbol{g} + \left(\sum_{i=1}^N v^i\right)\boldsymbol{u}, & \text{in } Q, \\ \theta_t - \sigma\Delta\theta + (\boldsymbol{u}\cdot\nabla)\,\theta = \left(\sum_{i=1}^N w^i\right)\theta, & \text{in } Q, \\ \nabla\cdot\boldsymbol{u} = 0, & \text{in } Q, \\ \boldsymbol{u} = 0 \text{ and } \frac{\partial\theta}{\partial\nu} = 0, & \text{on } \Sigma, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \text{ and } \theta|_{t=0} = \theta_0, & \text{in } \Omega. \end{cases} \tag{4.1}$$

We now fix some notations. Firstly, $\mathcal{U} := \Pi_{j=1}^{N} \mathcal{U}_j$. Secondly, for $i \in \mathcal{N} := \{1, ..., N\}$, set $\mathcal{U}_{-i} := \Pi_{j \neq i} \mathcal{U}_j$, and for $\boldsymbol{a} \in \mathcal{U}$ we write

$$\boldsymbol{a}^{-i} = \left(\boldsymbol{a}^1, ..., \boldsymbol{a}^{i-1}, \boldsymbol{a}^{i+1}, ..., \boldsymbol{a}^N\right) \in \mathcal{U}_{-i},$$

as well as

$$\boldsymbol{a} = (\boldsymbol{a}^i; \boldsymbol{a}^{-i}).$$

Definition

Let the set of criteria $\{J_i : \mathcal{U} \to \mathbb{R}\}_{i=1}^N$ be given.

(a) We say that $\boldsymbol{a}^* \in \mathcal{U}$ is a Nash equilibrium under these criteria if, for each $i \in \{1, ..., N\}$,

$$\boldsymbol{a}^{*i} = \text{argmin}_{\boldsymbol{a}^i \in \mathcal{U}_i} J_i(\boldsymbol{a}^i, \boldsymbol{a}^{*-i}).$$

(b) Assume that, for each $\boldsymbol{a}^{-i} \in \mathcal{U}_{-i}$, the functional $\boldsymbol{a}^i \mapsto J_i(\boldsymbol{a}^i, \boldsymbol{a}^{-i})$ is Gâteaux differentiable (at each point $\boldsymbol{a}^i \in \mathcal{U}_i$, in every admissible direction). We say $\boldsymbol{a}^* \in \mathcal{U}$ is a Nash quasi-equilibrium for the criteria $\{J_i\}_{i=1}^N$ if, for each $i \in \{1, ..., N\}$,

$$\left\langle D_i J_i(\boldsymbol{a}^{*i}; \boldsymbol{a}^{*-i}), \boldsymbol{a}^i - \boldsymbol{a}^{*i} \right\rangle := \lim_{\epsilon \to 0} \frac{J_i(\boldsymbol{a}^i + \epsilon(\widetilde{\boldsymbol{a}}^i - \boldsymbol{a}^i); \boldsymbol{a}^{-i}) - J_i(\boldsymbol{a}^i; \boldsymbol{a}^{-i})}{\epsilon} \geqslant 0,$$

for every $\boldsymbol{a}^i \in \mathcal{U}_i$.

Definition

For each i, we consider targets $u_T^i \in L^2(\Omega)^d$, $\theta_T^i \in L^2(\Omega)$, and constants $\alpha^i, \beta^i, \kappa^i, \mu^i \geqslant 0$. Given $\boldsymbol{a} \in \mathcal{U}$, $\boldsymbol{a} = \left((v^1, w^1), ..., (v^N, w^N)\right)$, player i has the following performance criterion for strategy (v^i, w^i), assuming the others utilize (v^j, w^j), $j \neq i$,

$$
\begin{aligned}
J_i(\boldsymbol{a}^i; \boldsymbol{a}^{-i}) :=& \frac{1}{2} \int_{\omega_i} \left\{ \alpha^i |u(T,x) - u_T^i(x)|^2 + \beta^i |\theta(T,x) - \theta_T^i(x)|^2 \right\} dx \\
&+ \frac{1}{2} \int_Q \left\{ \kappa^i |v^i(t,x)|^2 + \mu^i |w^i(t,x)|^2 \right\} d(t,x).
\end{aligned}
\tag{4.2}
$$

Lemma

(a) *For each* $i \in \mathcal{N}$, *each* $\boldsymbol{a}^{-i} \in \mathcal{U}_{-i}$, *and each pair of elements* $\boldsymbol{a}^i = (v^i, w^i)$ *and* $\widetilde{\boldsymbol{a}}^i = (\widetilde{v}^i, \widetilde{w}^i)$ *of* $\mathcal{U}_i$, *the functional* J_i *admits a Gâteaux derivative*

$$\left\langle D_i J_i(\boldsymbol{a}^i; \boldsymbol{a}^{-i}), \widetilde{\boldsymbol{a}}^i \right\rangle = \int_Q \left\{ (\boldsymbol{z}^i \cdot \boldsymbol{u} + \kappa^i v^i)(\widetilde{v}^i - v^i) + (\psi^i \theta + \mu^i w^i)(\widetilde{w}^i - w^i) \right\} d(t,x), \tag{4.3}$$

where $(\boldsymbol{u}, p, \theta) = \left(\boldsymbol{u}\left(\sum_{j=1}^N \boldsymbol{a}^j\right), p\left(\sum_{j=1}^N \boldsymbol{a}^j\right), \theta\left(\sum_{j=1}^N \boldsymbol{a}^j\right)\right)$ *and* $\left(\boldsymbol{z}^i, \Pi^i, \psi^i\right)$ *is the solution of* (3.3) *corresponding to the control* $(v, w) = \sum_{j=1}^N \boldsymbol{a}^j$ *and terminal condition* $(\boldsymbol{z}^i, \psi^i)|_{t=T} = \left(\alpha^i(\boldsymbol{u}|_{t=T} - \boldsymbol{u}_T^i)\mathbb{I}_{\omega_i}, \beta^i(\theta|_{t=T} - \theta_T^i)\mathbb{I}_{\omega_i}\right)$.

Lemma

(b) *For each* $i \in \mathcal{N}$*, there exists a constant* $C_i > 0$*, independent of* $\kappa^i, \mu^i, \alpha^i$ *and* β^i*, such that, for each* $\boldsymbol{a}^{-i} \in \mathcal{U}_{-i}$*, if*

$$\kappa^i + \mu^i - \left(\alpha^i + \beta^i + 1\right) e^{C_i\left(\alpha^i+\beta^i+1\right)} > 0, \tag{4.4}$$

then the functional $\boldsymbol{a}^i \in \mathcal{U}_i \mapsto J_i\left(\boldsymbol{a}^i; \boldsymbol{a}^{-i}\right) \in \mathbb{R}$ *is strictly convex.*

(c) *For* $i \in \mathcal{N}$*, let us assume conditions* (4.4) *hold, and that* $\boldsymbol{a}^{-i} \in \mathcal{U}_{-i}$*. Then, the functional* $\boldsymbol{a}^i \in \mathcal{U}_i \mapsto J_i(\boldsymbol{a}^i; \boldsymbol{a}^{-i}) \in \mathbb{R}$ *admits a unique minimizer* $\boldsymbol{a}^{*i}$*, which must solve the fixed point equation*

$$\boldsymbol{a}^{*i} = P_{\mathcal{U}_i}\left(-\frac{1}{\kappa^i}\boldsymbol{z}^{*i} \cdot \boldsymbol{u}^*, -\frac{1}{\mu^i}\psi^{*i}\theta^*\right), \tag{4.5}$$

where
$(\boldsymbol{u}^*, p^*, \theta^*) = (\boldsymbol{u}(\boldsymbol{a}^{*i} + \sum_{j\neq i} \boldsymbol{a}^j), p(\boldsymbol{a}^{*i} + \sum_{j\neq i} \boldsymbol{a}^j), \theta(\boldsymbol{a}^{*i} + \sum_{j\neq i} \boldsymbol{a}^j))$ *and* $\left(\boldsymbol{z}^{*i}, \Pi^{*i}, \psi^{*i}\right)$ *is the solution of* (3.3) *corresponding to the control* $(v, w) = \boldsymbol{a}^{*i} + \sum_{j\neq i} \boldsymbol{a}^j$ *and terminal condition* $(\boldsymbol{z}^{*i}, \psi^{*i})|_{t=T} = \left(\alpha^i(\boldsymbol{u}^*|_{t=T} - \boldsymbol{u}^i_T)\mathbb{I}_{\omega_i}, \beta^i(\theta^*|_{t=T} - \theta^i_T)\mathbb{I}_{\omega_i}\right).$

We observe that, if $\left\{\left(v^{*i}, w^{*i}\right)\right\}_{i=1}^{N}$ is a Nash equilibrium, then Lemma 13 (c) implies that (v^{*i}, w^{*i}) is given by (4.5), for each $i \in \mathcal{N}$, where the state vector $\left\{\boldsymbol{u}^*, p^*, \theta^*, \left(\left(\boldsymbol{z}^{*i}, \Pi^{*i}, \psi^{*i}\right)\right)_{i=1}^{N}\right\}$ solves the optimality system

$$\begin{cases}
\boldsymbol{u}_t^* - \nu\Delta\boldsymbol{u}^* + (\boldsymbol{u}^*\cdot\nabla)\,\boldsymbol{u}^* = \theta^*\boldsymbol{g} + \left[\sum_{j=1}^{N} P_1 P_{\mathcal{U}_j}\left(-\frac{1}{\kappa^j}\boldsymbol{z}^{*j}\cdot\boldsymbol{u}^*, -\frac{1}{\mu^j}\psi^{*j}\theta^*\right)\right]\boldsymbol{u}^*, & \text{in } Q,\\
\theta_t^* - \sigma\Delta\theta^* + \boldsymbol{u}^*\cdot\nabla\theta^* = \left[\sum_{j=1}^{N} P_2 P_{\mathcal{U}_j}\left(-\frac{1}{\kappa^j}\boldsymbol{z}^{*j}\cdot\boldsymbol{u}^*, -\frac{1}{\mu^j}\psi^{*j}\theta^*\right)\right]\theta^*, & \text{in } Q,\\
-\boldsymbol{z}_t^{*i} - \nu\Delta\boldsymbol{z}^{*i} - (\boldsymbol{u}^*\cdot\nabla)\,\boldsymbol{z}^{*i} + (\nabla\boldsymbol{u}^*)^{\intercal}\,\boldsymbol{z}^{*i} + \nabla\Pi^{*i} & \\
= \left[\sum_{j=1}^{N} P_1 P_{\mathcal{U}_j}\left(-\frac{1}{\kappa^j}\boldsymbol{z}^{*j}\cdot\boldsymbol{u}^*, -\frac{1}{\mu^j}\psi^{*j}\theta^*\right)\right]\boldsymbol{z}^{*i} - \psi^{*i}\nabla\theta^*, & \text{in } Q,\\
-\psi_t^{*i} - \sigma\Delta\psi^{*i} - \boldsymbol{u}^*\cdot\nabla\psi^{*i} = \left[\sum_{j=1}^{N} P_2 P_{\mathcal{U}_j}\left(-\frac{1}{\kappa^j}\boldsymbol{z}^{*j}\cdot\boldsymbol{u}^*, -\frac{1}{\mu^j}\psi^{*j}\theta^*\right)\right]\psi^{*i} + \boldsymbol{g}\cdot\boldsymbol{z}^{*i}, & \text{in } Q,\\
\nabla\cdot\boldsymbol{u}^* = 0 \text{ and } \nabla\cdot\boldsymbol{z}^{*i} = 0, & \text{in } Q,\\
\boldsymbol{u}^* = 0,\ \frac{\partial\theta^*}{\partial\boldsymbol{\nu}} = 0,\ \boldsymbol{z}^{*i} = 0 \text{ and } \frac{\partial\psi^{*i}}{\partial\boldsymbol{\nu}} = 0, & \text{on } \Sigma,\\
\boldsymbol{u}^*|_{t=0} = \boldsymbol{u}_0 \text{ and } \theta^*|_{t=0} = \theta_0, & \text{in } \Omega,\\
\boldsymbol{z}^{*i}|_{t=T} = \alpha^i\left(\boldsymbol{u}^*|_{t=T} - u_T^i\right)\mathbb{I}_{\omega_i} \text{ and } \psi^{*i}|_{t=T} = \beta^i\left(\theta|_{t=T} - \theta_T^i\right)\mathbb{I}_{\omega_i}, & \text{in } \Omega.
\end{cases} \tag{4.6}$$

By our assumptions, the set $\mathcal{U}$ is convex, contained in $L^\infty(Q)^{2N}$ and closed in $L^2(Q)^{2N}$. Henceforth, we further assume:

(A1) There exists $R > 0$ such that, for each $i \in \mathcal{N}$,
$\mathcal{U}_i \subseteq \{(v, w) : |v| \leqslant R \text{ and } |w| \leqslant R\}$;

(A2) If $\boldsymbol{a}^n, \boldsymbol{a} \in \mathcal{U}$, for $n \geqslant 1$, and $\boldsymbol{a}^n \to \boldsymbol{a}$ a.e. in Q, as $n \to \infty$, then

$$P_{\mathcal{U}}(\boldsymbol{a}^n) \to P_{\mathcal{U}}(\boldsymbol{a}) \text{ a.e. in } Q,$$

as $n \to \infty$.

Theorem

The mapping $\Lambda : \mathcal{U} \to \mathcal{U}$ *defined as*

$$\Lambda(\boldsymbol{a}) := \left(P_{\mathcal{U}_i}(-\frac{1}{\kappa^i}\boldsymbol{z}^i \cdot \boldsymbol{u}, -\frac{1}{\mu^i}\psi^i\theta) \right)_{i=1}^{N} \qquad (\boldsymbol{a} = ((v^1, w^1), ..., (v^N, w^N))),$$

where $(\boldsymbol{u}, p, \theta) =$ $\left(\boldsymbol{u}\left(\sum_{j=1}^N v^i, \sum_{j=1}^N w^i\right), p\left(\sum_{j=1}^N v^i, \sum_{j=1}^N w^i\right), \theta\left(\sum_{j=1}^N v^i, \sum_{j=1}^N w^i\right)\right),$ *and* $\left(\boldsymbol{z}^i, p^i, \psi^i\right)$ *is the solution of* (3.3) *corresponding to* $(v, w) = \left(\sum_{j=1}^N v^j, \sum_{j=1}^N w^j\right)$ *and the terminal condition* $(\boldsymbol{z}^i, \psi^i)|_{t=T} = \left(\alpha_i(\boldsymbol{u}|_{t=T} - \boldsymbol{u}_T^i)\mathbb{I}_{\omega_i}, \beta_i(\theta|_{t=T} - \theta_T^i)\mathbb{I}_{\omega_i}\right)$, *admits a fixed point* $\boldsymbol{a}^* = ((v^{i*}, w^{i*}))_{i=1}^N$.

We begin by proving compactness of Λ. Let us consider a sequence $\{\boldsymbol{a}^n\} = \left\{ \left(v^{i,n}, w^{i,n} \right)_{i=1}^{N} \right\} \subset \mathcal{U}$. By the estimates derived in Theorem 3, we can employ the Aubin-Lions Theorem and, if necessary, pass to a subsequence to deduce the convergences

$$\begin{cases} \boldsymbol{u}^n \to \boldsymbol{u}, \text{ strongly in } C\left([0,T]; H\right) \cap L^2(0,T;V), \\ \theta^n \to \theta, \text{ strongly in } C\left([0,T]; L^2(\Omega)\right) \cap L^2(0,T;H^1(\Omega)), \\ \boldsymbol{z}^{i,n} \to \boldsymbol{z}^i \text{ and } \psi^{i,n} \to \psi^i, \text{ strongly in } L^2(Q)^d \text{ and } L^2(Q), \text{ respectively,} \end{cases} \tag{4.7}$$

from where we obtain, possibly by refining the subsequences,

$$\begin{cases} \boldsymbol{u}^n \cdot \boldsymbol{z}^{i,n} \to \boldsymbol{u} \cdot \boldsymbol{z}^i, \text{ strongly in } L^1(Q) \text{ and a.e. in } Q, \\ \theta^n \psi^{i,n} \to \theta \psi^i, \text{ strongly in } L^1(Q) \text{ and a.e. in } Q, \end{cases} \tag{4.8}$$

Utilizing assumption (**A2**), alongside with (4.8), we infer

$$\Lambda(\boldsymbol{a}^n)_i = P_{\mathcal{U}_i}(-\frac{1}{\kappa^i}\boldsymbol{u}^n \cdot \boldsymbol{z}^{i,n}, -\frac{1}{\mu^i}\theta^n \psi^{i,n}) \to P_{\mathcal{U}_i}(-\frac{1}{\kappa^i}\boldsymbol{u} \cdot \boldsymbol{z}^i, -\frac{1}{\mu^i}\theta \psi^i) =: \boldsymbol{b}, \text{ a.e. in } Q, \tag{4.9}$$

for $i \in \mathcal{N}$.

We notice that assumption (**A1**) implies

$$\|\Lambda(\boldsymbol{a}^n) - \boldsymbol{b}\|_{L^\infty(Q)^{2N}} \leqslant 2R. \tag{4.10}$$

Putting (4.9) and (4.10) together, we can apply the Dominated Convergence Theorem to infer the convergence

$$\Lambda(\boldsymbol{a}^n) \to \boldsymbol{b}, \text{ strongly in } L^2(Q)^{2N}, \text{ as } n \to \infty.$$

Therefore Λ is compact. Since it is continuous, and by our assumption on the admissible control set $\mathcal{U}$, we can apply the Schauder fixed point theorem to conclude the desired result. This finishes the proof.

Thank You!

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