Asymptotic behavior of one-dimensional wave equations with set-valued boundary damping

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Joint work with Yacine Chitour and Swann Marx
Sequel to the previous talk by Swann Marx

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Outline

1. Equivalent discrete-time dynamical system
2. Decay rates
3. Arbitrarily slow convergence
4. The case of the sign set-valued map
5. Input-to-state stability

Asymptotic behavior of one-dimensional wave equations with set-valued boundary damping

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### Equivalent discrete-time dynamical system

<table>
<thead>
<tr>
<th>Solution $z$</th>
<th>Function $g$</th>
<th>Sequence $(g_n)_n$</th>
</tr>
</thead>
</table>
| $\begin{align*}
\partial_{tt}^2 z(t,x) &= \partial_{xx}^2 z(t,x) \\
z(t,0) &= 0 \\
(\partial_t z(t,1), -\partial_x z(t,1)) &\in \Sigma
\end{align*}$ | $g(t) \in S(g(t-2))$ | $g_{n+1}(t) \in S(g_n(t))$ |
| $X_p = W_{*}^{1,p}(0,1) \times L^p(0,1)$ | $L^p_{\text{loc}}(-1, +\infty)$ | $Y^\mathbb{N}_p = [L^p(-1, 1)]^\mathbb{N}$ |

For $p \neq +\infty$,

$$
\| (u, v) \|_{X_p} = \begin{cases} 
\frac{1}{\sqrt{2}} \left[ \int_0^1 \left( |u'|^p + |v|^p + |u'|^p - |v|^p \right) ds \right]^{\frac{1}{p}} & p < +\infty \\
\frac{1}{\sqrt{2}} \max \left( \| u' + v \|_{L^\infty}, \| u' - v \|_{L^\infty} \right) & p = +\infty
\end{cases}
$$

For $p = +\infty$,

$$
\| z(t) \|_{X_p} = \| g(t + \cdot) \|_{L^p(-1,1)}
$$

It suffices to consider the sequence $(g_n)_n$ to study the asymptotic behavior of $z$!
Equivalent discrete-time dynamical system

Summarizing: studying

\[
\begin{align*}
\partial_{tt}^2 z(t, x) &= \partial_{xx}^2 z(t, x) \\
z(t, 0) &= 0 \\
(\partial_t z(t, 1), -\partial_x z(t, 1)) &\in \Sigma
\end{align*}
\]

in \(X_p = W^{1,p}_\ast \times L^p\) is the same as studying

\[
g_{n+1}(t) \in S(g_n(t)) \quad t \in [-1, 1], \ n \in \mathbb{N}
\]

in \(Y_p = L^p(-1, 1)\) with \(\text{Graph}(S) = R\Sigma\)
Decay rates

How fast do solutions converge to 0?

**Nonlinear sector condition:**

- In terms of $\Sigma$: $\exists q \in C^1$ such that $q(0) = 0$, $0 < q(x) < x$, $|q'(x)| < 1$ for $x > 0$ such that $q(|x|) \leq |y|$ and $q(|y|) \leq |x|$ for all $(x, y) \in \Sigma$.

- In terms of $S$: $\exists Q \in C^1$ and $M > 0$ such that $Q(0) = 0$, $0 < Q(x) < x$, $Q'(x) > 0$ for $x > 0$ such that $|y| \leq Q(|x|)$ for all $y \in S(x)$.

$$Q(x) = \sqrt{2}(q + Id)^{-1}(\sqrt{2}x) - x$$
Assume that $\Sigma$ satisfies a nonlinear sector condition with functions $q$ and $Q$ as before. Then

$$\|z(t)\|_{X_\infty} \leq Q\lfloor \frac{t}{2} \rfloor (\|z(0)\|_{X_\infty})$$

$$Q^n = Q \circ \cdots \circ Q$$

- Similar statement for $p < +\infty$ but with additional terms

- $\|z(t)\|_{X_p} \geq C_1 Q\lfloor \frac{t}{2} \rfloor (C_2)$ for non-trivial solutions if

- If $\Sigma$ is the graph of $q$ or $q^{-1}$: $\|z(t)\|_{X_p} \sim Q\lfloor \frac{t}{2} \rfloor (C)$
Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that $\Sigma$ satisfies a nonlinear sector condition with functions $q$ and $Q$ as before and let $x_0 > 0$

1. Assume that $q'(0) = 0$. Then there exists a sequence $(t_n)_n$
    with $t_n \sim n$ as $n \to +\infty$ such that
    $$Q^n(x_0) = V(t_n) \quad \forall n \in \mathbb{N}$$
    where $V$ is the solution of $V'(t) = -\sqrt{2}q(\sqrt{2}V(t))$ with
    $V(0) = x_0$

If moreover $z \mapsto \frac{V^{-1}(z)q(\sqrt{2}z)}{z}$ is bounded on $(0, x_0]$, then
    $$Q^n(x_0) \sim V(n) \quad \text{as } n \to +\infty$$

Same as [Vancostenoble, Martinez; 2000], but for set-valued damping and with the stronger conclusion $Q^n(x_0) \sim V(n)$ under an additional assumption
Theorem ([Chitour, Marx, Mazanti])

Assume that $\Sigma$ satisfies a nonlinear sector condition with functions $q$ and $Q$ as before and let $x_0 > 0$

1. Assume that $q'(0) \in (0, 1)$ and let $\lambda = 2 \text{artanh}(q'(0))$. Then
$$\ln Q^{[n]}(x_0) \sim -\lambda n$$ as $n \to +\infty$

If moreover $\sum_{k=0}^{\infty} \psi(e^{-\lambda k}) < +\infty$, where
$$\psi(r) = \sup_{s \in (0, r]} \left| \frac{q(s)}{s} - q'(0) \right|,$$
then $\exists C > 1$ s.t.
$$C^{-1} e^{-\lambda n} \leq Q^{[n]}(x_0) \leq C e^{-\lambda n} \quad \forall n \in \mathbb{N}$$

Linear behavior of $\Sigma$ close to $0$ yields exponential decay
Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that \( \Sigma \) satisfies a nonlinear sector condition with functions \( q \) and \( Q \) as before and let \( x_0 > 0 \)

Assume that \( q'(0) = 1 \). Then

\[
\lim_{n \to +\infty} e^{\lambda n} Q^n(x_0) = 0 \quad \text{for every } \lambda > 0
\]

If moreover \( \exists C > 0, \alpha > 0 \) s.t. \( |q(s) - s| \leq C |s|^{1+\alpha} \) for small, then \( \exists C_* > 0, \mu_* > 0 \) s.t.

\[
Q[n](x_0) \leq C_* e^{-\mu_*(1+\alpha)^n} \quad \forall n \text{ large enough}
\]

Faster than any exponential if \( q'(0) = 1 \)

Recall: finite-time convergence for \( \sigma(x) = x \)
Previous results: (σ is odd, expressions below for s > 0 small and t large) e.g. [Vancostenoble, Martinez; 2000] [Alabau-Boussouira; 2012]

\[
\sigma(s) = s^p \quad \implies \quad \|z(t)\|_{X_2} \sim t^{-\frac{1}{p-1}} \quad p > 1
\]

\[
\sigma(s) = s^p \left( \ln \left( \frac{1}{s} \right) \right)^q \quad \implies \quad \|z(t)\|_{X_2} \sim t^{-\frac{1}{p-1}} \left( \ln t \right)^{-\frac{q}{p-1}} \quad p > 1, q > 0
\]

\[
\sigma(s) = e^{-\frac{1}{sp}} \quad \implies \quad \|z(t)\|_{X_2} \sim (\ln t)^{-\frac{1}{p}} \quad p > 0
\]

\[
\sigma(s) = e^{-e^{1/s}} \quad \implies \quad \|z(t)\|_{X_2} \sim (\ln \ln t)^{-2}
\]

\[
\sigma(s) = e^{-\left( \ln \left( \frac{1}{s} \right) \right)^p} \quad \implies \quad \|z(t)\|_{X_2} \sim e^{-\left( \ln t \right)^{\frac{1}{p}}} \quad 1 < p < 2
\]

\[
\sigma(s) = s \left( \ln \left( \frac{1}{s} \right) \right)^{-p} \quad \implies \quad \|z(t)\|_{X_2} \lesssim e^{-C t^{\frac{1}{p+1}}} \quad p > 0
\]

- We can retrieve the same behaviors, in any \( X_r, r \in [1, +\infty] \)
- We can obtain the precise asymptotic behavior in the last case!
Theorem ([Chitour, Marx, Mazanti])

Assume that $\Sigma$ is the graph of the odd function $\sigma$ defined for $s > 0$ by $\sigma(s) = s \left( \ln \left( \frac{1}{s} \right) \right)^{-p}$. Then nontrivial solutions $z$ of the wave equation satisfy

$$
\|z(t)\|_{\mathcal{X}} \sim \frac{1}{\sqrt{2}} e^{-\sum_{k=0}^{N} \alpha_k t} \frac{1-2pk}{p+1}
$$

where $N = \left\lfloor \frac{1}{2p} \right\rfloor$, $\alpha_0 = (p + 1)^{\frac{1}{p+1}}$, and $\alpha_1, \ldots, \alpha_N$ are real constants.
Arbitrarily slow convergence

Theorem ([Chitour, Marx, Mazanti])

Assume that \( \Sigma \subset \{(x, y) \mid |x| \leq C \text{ or } |y| \leq C\} \) for some \( C > 0 \).
Then, \( \forall p \in [1, +\infty) \forall \varphi : [0, +\infty) \to (0, +\infty) \) decreasing to 0, \( \exists \) an initial condition in \( X_p \) s.t. \( \forall \) solution \( z \)
\[
\|z(t)\|_{X_p} \geq \varphi(t)
\]

- Conjectured in [Vancostenoble, Martinez; 2000]
- Assumptions satisfied if \( \sigma \) is saturated
- If \( |x| \) is large and \( y \in S(x), |y| \geq |x| - \sqrt{2}C \)
- Initial conditions with explosions (whence \( p < +\infty \)
The case of the sign set-valued map

\[ \Sigma \text{ is the graph of } \text{sign}(x) = \begin{cases} \{ \sqrt{2} \frac{x}{|x|} \} & \text{if } x \neq 0 \\ [-\sqrt{2}, \sqrt{2}] & \text{if } x = 0 \end{cases} \]

\[ S \text{ is single-valued: } S(x) = \begin{cases} \frac{x}{|x|} & \text{if } |x| \leq 1 \\ 2 - x & \text{if } x > 1 \\ -2 - x & \text{if } x < -1 \end{cases} \]
From the previous results, $\exists$ solution

**Theorem ([Chitour, Marx, Mazanti])**

Let $z$ be a solution of the wave equation and consider the corresponding sequence $(g_n)_{n \in \mathbb{N}}$. Let

$$g_\infty(s) = (-1)^{K(s)} \frac{g_0(s)}{|g_0(s)|} (|g_0(s)| - 2K(s))$$

where $K(s) = \left\lfloor \frac{|g_0(s)|+1}{2} \right\rfloor$ and $z_\infty$ be the solution of the wave equation whose corresponding sequence starts from $g_\infty$. Then $z_\infty$ is 2-periodic,

$$\lim_{t \to +\infty} \|z(t) - z_\infty(t)\|_{X_p} = 0,$$

and the above convergence is in finite time if $p = +\infty$. 
The case of the sign set-valued map

With respect to [Cheng-Zhong Xu, Gen Qi Xu; 2019]:

- Existence and uniqueness are straightforward (instead of using semigroup theory in a Hilbertian setting)
- Holds for $p \in [1, +\infty]$ instead of only $p = 2$
- The limit is more explicitly identified (instead of based on a Fourier series expansion)
Input-to-state stability

What if there is a disturbance in the boundary condition?

\[
\begin{cases}
\partial_{tt}z(t, x) = \partial_{xx}z(t, x) \\
z(t, 0) = 0 \\
(\partial_tz(t, 1), -\partial_xz(t, 1)) \in \Sigma + d(t) \\
t \in \mathbb{R}_+ \\
\Sigma \subset \mathbb{R}^2
\end{cases}
\]

\(d : \mathbb{R}_+ \rightarrow \mathbb{R}^2\): disturbance

Definition

The system is input-to-state stable (ISS) with respect to \(X_p\) and a functional space \(Z\) for the disturbance if there exist \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}\) s.t., for every solution and every disturbance,

\[
\|z(t)\|_{X_p} \leq \beta(\|z(0)\|_{X_p}, t) + \gamma(\|d\|_Z)
\]

\(\mathcal{K}\) function \(x \mapsto \gamma(x)\): continuous, increasing, \(\gamma(0) = 0\)
Input-to-state stability

Theorem ([Chitour, Marx, Mazanti])

1. If one of the following conditions hold:
   - $p < +\infty$ and $\sum_{n=0}^{\infty} |d(\cdot + 2n + 1)| \in L^p$
   - $d(t) \to 0$ as $t \to +\infty$ and $\Sigma$ is bounded away from the horizontal and vertical axes

   Then
   \[
   \lim_{t \to +\infty} \|z(t)\|_{X^p} = 0
   \]

2. If $\Sigma$ satisfies a sector condition at infinity, then the wave equation is ISS in $X^p$ with respect to disturbances in $L^p$. 
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