Asymptotic behavior of one-dimensional wave equations with set-valued boundary damping

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Sequel to the previous talk by Swann Marx

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Outline

Equivalent discrete-time dynamical system

Solution z	Function g	Sequence $(g_n)_n$
$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \end{cases}$	$g(t) \in S(g(t-2))$	$g_{n+1}(t) \in S(g_n(t))$
$\mathsf{X}_p = W_*^{1,p}(0,1) \times L^p(0,1)$	$L^p_{\mathrm{loc}}(-1,+\infty)$	$\mathsf{Y}_p^{\mathbb{N}} = [L^p(-1,1)]^{\mathbb{N}}$

$W_*^{1,p}(0,1) = \{u \in W^{1,p}(0,1) \mid u(0) = 0\}$ $\mathrm{Graph}(S) = R\Sigma$

$$\|(u,v)\|_{\mathsf{X}_p} = \begin{cases} \dfrac{1}{\sqrt{2}} \left[\displaystyle\int_0^1 \left(|u' + v|^p + |u' - v|^p\right) \mathrm{d}s\right]^{\frac{1}{p}} & p < +\infty \\ \dfrac{1}{\sqrt{2}} \max\left(\|u' + v\|_{L^\infty}, \|u' - v\|_{L^\infty}\right) & p = +\infty \end{cases}$$

$$\|z(t)\|_{\mathsf{X}_p} = \|g(t + \cdot)\|_{L^p(-1,1)} \qquad \|z(2n)\|_{\mathsf{X}_p} = \|g_n\|_{\mathsf{Y}_p}$$

It suffices to consider the sequence $(g_n)_n$ to study the asymptotic behavior of z!

Equivalent discrete-time dynamical system

Summarizing: studying

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma \end{cases}$$

in $X_p = W_*^{1,p} \times L^p$ is the same as studying

$$g_{n+1}(t) \in S(g_n(t)) \qquad t \in [-1,1],\ n \in \mathbb{N}$$

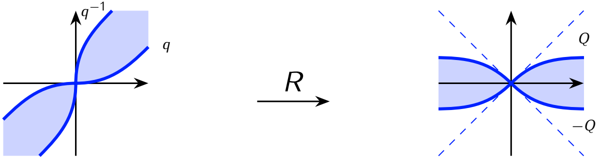
in $Y_p = L^p(-1,1)$ with $\mathrm{Graph}(S) = R\Sigma$

Decay rates

How fast do solutions converge to 0?

Nonlinear sector condition:

- In terms of Σ: $\exists q \in \mathcal{C}^1$ such that $q(0) = 0$, $0 < q(x) < x$, $|q'(x)| < 1$ for $x > 0$ such that
$$q(|x|) \leq |y| \quad \text{and} \quad q(|y|) \leq |x| \qquad \forall (x, y) \in \Sigma$$
- In terms of S: $\exists Q \in \mathcal{C}^1$ and $M > 0$ such that $Q(0) = 0$, $0 < Q(x) < x$, $Q'(x) > 0$ for $x > 0$ such that
$$|y| \leq Q(|x|) \qquad \forall y \in S(x)$$



$$Q(x) = \sqrt{2}(q + \mathrm{Id})^{-1}(\sqrt{2}x) - x$$

Decay rates

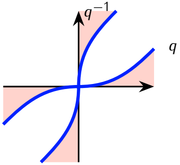
Theorem ([Chitour, Marx, Mazanti])

Assume that Σ satisfies a nonlinear sector condition with functions q and Q as before. Then

$$\|z(t)\|_{X_\infty} \leq Q^{[\lfloor \frac{t}{2} \rfloor]}(\|z(0)\|_{X_\infty})$$

$$Q^{[n]} = \underbrace{Q \circ \cdots \circ Q}_{n}$$

- Similar statement for $p < +\infty$ but with additional terms
- $\|z(t)\|_{X_p} \geq C_1 Q^{[\lfloor \frac{t}{2} \rfloor]}(C_2)$ for non-trivial solutions if



- If Σ is the graph of q or q^{-1}: $\|z(t)\|_{X_p} \sim Q^{[\lfloor \frac{t}{2} \rfloor]}(C)$

Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that Σ satisfies a nonlinear sector condition with functions q and Q as before and let $x_0 > 0$

1. *Assume that $q'(0) = 0$. Then there exists a sequence $(t_n)_n$ with $t_n \sim n$ as $n \to +\infty$ such that*
   $$Q^{[n]}(x_0) = V(t_n) \qquad \forall n \in \mathbb{N}$$
   where V is the solution of $V'(t) = -\sqrt{2}q(\sqrt{2}V(t))$ with $V(0) = x_0$
   If moreover $z \mapsto \frac{V^{-1}(z)q(\sqrt{2}z)}{z}$ is bounded on $(0, x_0]$, then
   $$Q^{[n]}(x_0) \sim V(n) \qquad \text{as } n \to +\infty$$

Same as [Vancostenoble, Martinez; 2000], but for set-valued damping and with the stronger conclusion $Q^{[n]}(x_0) \sim V(n)$ under an additional assumption

Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that Σ *satisfies a nonlinear sector condition with functions* q *and* Q *as before and let* $x_0 > 0$

2. *Assume that* $q'(0) \in (0, 1)$ *and let* $\lambda = 2 \operatorname{artanh}(q'(0))$. *Then*

$$\ln Q^{[n]}(x_0) \sim -\lambda n \qquad \text{as } n \to +\infty$$

If moreover $\sum_{k=0}^{\infty} \psi(e^{-\frac{\lambda}{2} k}) < +\infty$, *where* $\psi(r) = \sup_{s \in (0, r]} \left|\frac{q(s)}{s} - q'(0)\right|$, *then* $\exists C > 1$ *s.t.*

$$C^{-1} e^{-\lambda n} \leq Q^{[n]}(x_0) \leq C e^{-\lambda n} \qquad \forall n \in \mathbb{N}$$

Linear behavior of Σ close to 0 yields exponential decay

Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that Σ satisfies a nonlinear sector condition with functions q and Q as before and let $x_0 > 0$

3. *Assume that* $q'(0) = 1$. *Then*

$$\lim_{n \to +\infty} e^{\lambda n} Q^{[n]}(x_0) = 0 \qquad \text{for every } \lambda > 0$$

If moreover $\exists C > 0, \alpha > 0$ *s.t.* $|q(s) - s| \leq C|s|^{1+\alpha}$ *for* s *small, then* $\exists C_* > 0, \mu_* > 0$ *s.t.*

$$Q^{[n]}(x_0) \leq C_* e^{-\mu_* (1+\alpha)^n} \qquad \forall n \text{ large enough}$$

Faster than any exponential if $q'(0) = 1$

Recall: finite-time convergence for $\sigma(x) = x$

Decay rates

Previous results: (σ is odd, expressions below for $s > 0$ small and t large) e.g. [Vancostenoble, Martinez; 2000] [Alabau-Boussouira; 2012]

$$\sigma(s) = s^p \implies \|z(t)\|_{X_2} \sim t^{-\frac{1}{p-1}} \qquad p > 1$$

$$\sigma(s) = s^p \left(\ln\left(\tfrac{1}{s}\right)\right)^q \implies \|z(t)\|_{X_2} \sim t^{-\frac{1}{p-1}} (\ln t)^{-\frac{q}{p-1}} \qquad p > 1, q > 0$$

$$\sigma(s) = e^{-\frac{1}{s^p}} \implies \|z(t)\|_{X_2} \sim (\ln t)^{-\frac{1}{p}} \qquad p > 0$$

$$\sigma(s) = e^{-e^{1/s}} \implies \|z(t)\|_{X_2} \sim (\ln \ln t)^{-2}$$

$$\sigma(s) = e^{-\left(\ln\left(\frac{1}{s}\right)\right)^p} \implies \|z(t)\|_{X_2} \sim e^{-(\ln t)^{\frac{1}{p}}} \qquad 1 < p < 2$$

$$\sigma(s) = s \left(\ln\left(\tfrac{1}{s}\right)\right)^{-p} \implies \|z(t)\|_{X_2} \lesssim e^{-C t^{\frac{1}{p+1}}} \qquad p > 0$$

- We can retrieve the same behaviors, in any X_r, $r \in [1, +\infty]$
- We can obtain the precise asymptotic behavior in the last case!

Decay rates

Theorem ([Chitour, Marx, Mazanti])

Assume that Σ is the graph of the odd function σ defined for $s > 0$ by $\sigma(s) = s\left(\ln\left(\frac{1}{s}\right)\right)^{-p}$. Then nontrivial solutions z of the wave equation satisfy

$$\|z(t)\|_{X_r} \sim \frac{1}{\sqrt{2}} e^{-\sum_{k=0}^{N} \alpha_k t^{\frac{1-2pk}{p+1}}}$$

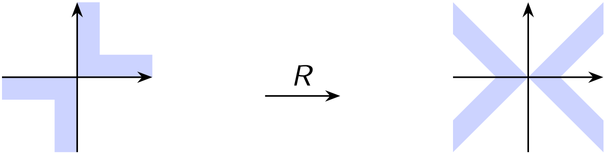
where $N = \left\lfloor \frac{1}{2p} \right\rfloor$, $\alpha_0 = (p+1)^{\frac{1}{p+1}}$, and $\alpha_1, \ldots, \alpha_N$ are real constants.

Arbitrarily slow convergence

Theorem ([Chitour, Marx, Mazanti])

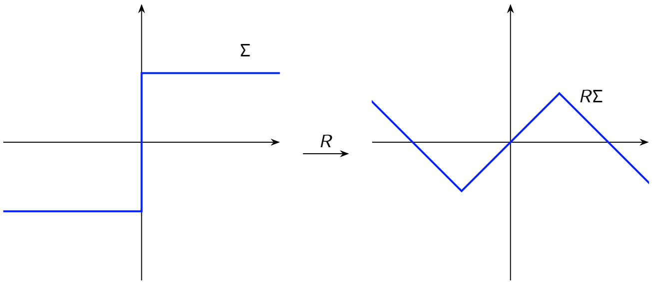
Assume that $\Sigma \subset \{(x, y) \mid |x| \leq C \text{ or } |y| \leq C\}$ *for some* $C > 0$. *Then,* $\forall p \in [1, +\infty)$ $\forall \varphi : [0, +\infty) \to (0, +\infty)$ *decreasing to* 0, $\exists$ *an initial condition in* X_p *s.t.* $\forall$ *solution* z

$$\|z(t)\|_{\mathrm{X}_p} \geq \varphi(t)$$



- Conjectured in [Vancostenoble, Martinez; 2000]
- Assumptions satisfied if σ is saturated
- If $|x|$ is large and $y \in S(x)$, $|y| \geq |x| - \sqrt{2}C$
- Initial conditions with explosions (whence $p < +\infty$)

The case of the sign set-valued map



$$\Sigma \text{ is the graph of } \operatorname{sign}(x) = \begin{cases} \{\sqrt{2}\frac{x}{|x|}\} & \text{if } x \neq 0 \\ [-\sqrt{2}, \sqrt{2}] & \text{if } x = 0 \end{cases}$$

$$S \text{ is single-valued: } S(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ 2 - x & \text{if } x > 1 \\ -2 - x & \text{if } x < -1 \end{cases}$$

The case of the sign set-valued map

From the previous results, $\exists!$ solution

Theorem ([Chitour, Marx, Mazanti])

Let z be a solution of the wave equation and consider the corresponding sequence $(g_n)_{n \in \mathbb{N}}$. Let

$$g_\infty(s) = (-1)^{K(s)} \frac{g_0(s)}{\lvert g_0(s) \rvert} (\lvert g_0(s) \rvert - 2K(s))$$

where $K(s) = \left\lfloor \frac{\lvert g_0(s) \rvert + 1}{2} \right\rfloor$ and z_∞ be the solution of the wave equation whose corresponding sequence starts from g_∞. Then z_∞ is 2-periodic,

$$\lim_{t \to +\infty} \lVert z(t) - z_\infty(t) \rVert_{\mathsf{X}_p} = 0,$$

and the above convergence is in finite time if $p = +\infty$.

The case of the sign set-valued map

With respect to [Cheng-Zhong Xu, Gen Qi Xu; 2019]:

- Existence and uniqueness are straightforward (instead of using semigroup theory in a Hilbertian setting)
- Holds for $p \in [1, +\infty]$ instead of only $p = 2$
- The limit is more explicitly identified (instead of based on a Fourier series expansion)

Input-to-state stability

What if there is a disturbance in the boundary condition?

$$\begin{cases} \partial_{tt}^2 z(t,x) = \partial_{xx}^2 z(t,x) & (t,x) \in \mathbb{R}_+ \times (0,1) \\ z(t,0) = 0 & t \in \mathbb{R}_+ \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma + d(t) & t \in \mathbb{R}_+ \end{cases} \qquad \Sigma \subset \mathbb{R}^2$$

$d : \mathbb{R}_+ \to \mathbb{R}^2$: disturbance

Definition

The system is input-to-state stable (ISS) with respect to X_p and a functional space Z for the disturbance if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ s.t., for every solution and every disturbance,

$$\|z(t)\|_{\mathrm{X}_p} \leq \beta(\|z(0)\|_{\mathrm{X}_p}, t) + \gamma(\|d\|_{\mathrm{Z}})$$

$\mathcal{K}$ function $x \mapsto \gamma(x)$: continuous, increasing, $\gamma(0) = 0$

Input-to-state stability

Theorem ([Chitour, Marx, Mazanti])

1. *If one of the following conditions hold:*
   - $p < +\infty$ *and* $\sum_{n=0}^{\infty} |d(\cdot + 2n + 1)| \in L^p$
   - $d(t) \to 0$ *as* $t \to +\infty$ *and* Σ *is bounded away from the horizontal and vertical axes*

   Then

$$\lim_{t \to +\infty} \|z(t)\|_{\mathsf{X}_p} = 0$$

2. *If* Σ *satisfies a sector condition at infinity, then the wave equation is ISS in* X_p *with respect to disturbances in* L^p*.*