Asymptotic stability of a one-dimensional wave equation with set-valued damping

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System under study

Wave equation with set-valued damping

$$\begin{cases} \partial_{tt} z(t,x) = \partial_{xx} z(t,x), \\ z(t,0) = 0 \\ (\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma, \\ z(0,x) = z_0(x), z_t(0,x) = z_1(x). \end{cases}$$

Applications : Σ is the graph of a function $\Rightarrow$ **saturations**. Σ might be also the graph of a **set-valued** map $\Rightarrow$ **sign** function.

Questions

Q1. Well-posedness ?

Q2. Asymptotic stability ?

Existing literature

Wave equation with nonlinear damping

$$\begin{cases} \partial_{tt}z(t,x) = \partial_{xx}z(t,x) \\ z(t,0) = 0 \\ \partial_x z(t,1) = -\sigma(\partial_t z(t,1)), \\ z(0,x) = z_0(x), z_t(0,x) = z_1(x). \end{cases}$$

Some references

[Conrad, Leblond, Marmorat ; 1989], [Zuazua ; 1990], [Lasiecka, Tataru ; 1993], [Martinez ; 1999], [Pierre, Vanconstenoble ; 2000], [Haraux ; 2009], [Alabau-Boussouira ; 2012], [Cheng-Zhong Xu, Gen Qi Xu ; 2019], etc...

Most of these results hold in $H^1(0,1) \times L^2(0,1)$.

Our results

Our viewpoint

We transform the wave equation into a **discrete-time equation**.

which allows us to

Contributions

1. **Well-posedness** and **asymptotic stability** in $\mathsf{X}_p := W^{1,p}_*(0,1) \times L^p(0,1)$, with $W^{1,p}_*(0,1) := \{u \in L^p(0,1) \mid u' \in L^p(0,1) \text{ and } u(0) = 0\}$ for all $p \in [1,\infty]$.
2. **Necessary and sufficient conditions** for such results.
3. **Optimal** decay rates.
4. Input-to-state stability (**ISS**) results.

D'Alembert decomposition

D'Alembert decomposition

$$z(t,x) = \frac{1}{\sqrt{2}} \int_0^{t+x} f(s)ds + \frac{1}{\sqrt{2}} \int_0^{t-x} g(s)ds,$$

where f and g are **Riemann invariants**.

From this decomposition, one has :

$$\begin{pmatrix} g(t-x) \\ -f(t+x) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{=R} \begin{pmatrix} \partial_t z(t,x) \\ -\partial_x z(t,x) \end{pmatrix}.$$

R is a **rotation matrix** of angle $\frac{\Pi}{4}$.

Equivalent definition

Boundary conditions

$$
\begin{aligned}
z(t,0) = 0 &\iff g(t) = -f(t) \\
(\partial_t z(t,1), -\partial_x z(t,1)) \in \Sigma &\iff (g(t-1), -f(t+1)) \in R\Sigma
\end{aligned}
$$

The system can be expressed only with g. Denoting $S : \mathbb{R} \rightrightarrows \mathbb{R}$ the set-valued map whose graph is $R\Sigma$:

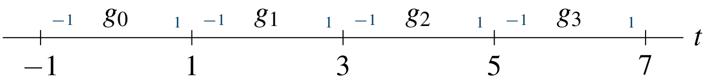
$$g(t) \in S(g(t-2)) \qquad \forall t \geq 1$$

The solution is **completely characterized by** g !

Discrete-time system

$g : [-1, +\infty) \to \mathbb{R}$ can be equivalently described by a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n : [-1, 1) \to \mathbb{R}$ and

$$g_n(t) = g(t + 2n)$$



$$g(t) \in S(g(t-2)) \quad \Longleftrightarrow \quad g_{n+1}(t) \in S(g_n(t))$$

The solution of the wave equation is completely characterized by the sequence $(g_n)_{n\in\mathbb{N}}$!

Iterations of the discrete-time multi-valued dynamical system $x_{n+1} \in S(x_n)$

Existence

$$\begin{cases} \partial_{tt} z = \partial_{xx} z \\ z(t,0) = 0 \\ (z_t(t,1), -z_x(t,1)) \in \Sigma. \end{cases} \Leftrightarrow \begin{cases} g_{n+1}(t) \in S(g_n(t)), \\ S = \mathrm{graph}(R\Sigma) \end{cases}$$

Existence [Chitour, Marx, Mazanti,2020]

Assume that $R\Sigma$ contains the graph of a **universally measurable** function with **linear growth**. Then, for every initial condition in X_p, there exists a solution to the wave equation.

- φ is universally measurable $\Leftrightarrow$ $\varphi \circ g$ is Lebesgue meas., $\forall g$ Lebesgue meas.
- φ with linear growth : $g \in L^p \implies \varphi \circ g \in L^p$
- "Conversely", if $\exists$ a solution for every initial condition in X_p, then $R\Sigma$ contains the graph of a universally measurable function and the graph of a function with linear growth

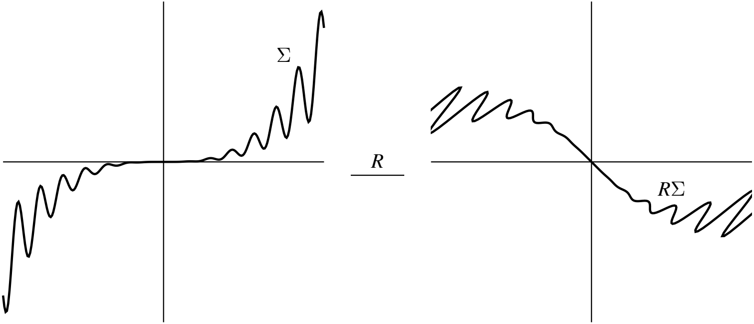
Uniqueness

$$\begin{cases} \partial_{tt} z(t,x) = \partial_{xx} z(t,x) \\ z(t,0) = 0 \\ (z_t(t,1), -z_x(t,1)) \in \Sigma. \end{cases} \Leftrightarrow \begin{cases} g_{n+1}(t) \in S(g_n(t)), \\ S = \text{graph}(R\Sigma) \end{cases}$$

Uniqueness [Chitour, Marx, Mazanti,2020]

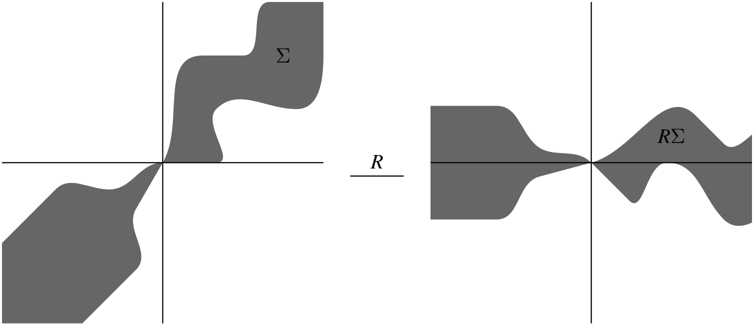
For every initial condition in X_p, there exists a unique solution to the wave equation if and only if $R\Sigma$ is equal to the graph of a universally measurable function with linear growth.

- Necessary and sufficient condition in terms of $R\Sigma = \text{graph}(S)$
- Both statements only for $p < +\infty$; also hold for $p = +\infty$ replacing linear growth by a weaker assumption

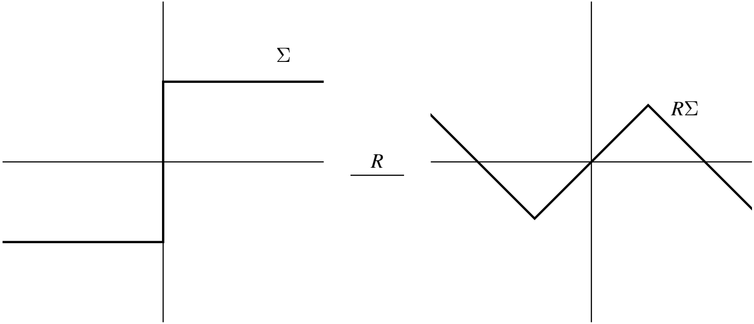


Σ is the graph of a function σ

Existence, not uniqueness



Σ and $R\Sigma$ are not graphs of functions
Existence, not uniqueness



Σ is the sign set-valued map, $R\Sigma$ is the graph of a function
Existence and uniqueness. No need of **Filippov solutions**.

Lyapunov stability

From now on : Σ is such that we have existence of solutions (not necessarily uniqueness)

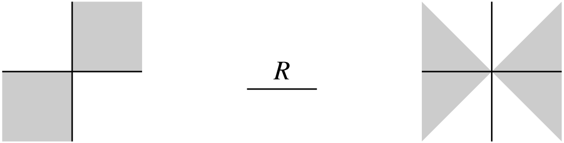
Lyapunov stability [Chitour, Marx, Mazanti,2020]

For every solution, $t \mapsto \|z(t)\|_{\mathsf{X}_p}$ is nonincreasing

$\iff \forall (x,y) \in \Sigma, xy \geq 0$

$\iff \forall (x,y) \in R\Sigma, y \leq x$

Generalization of the condition $s\sigma(s) \geq 0$ (known as a **damping condition**).



Asymptotic Stability notions

- Strong stability : For every solution z,

$$\lim_{t\to+\infty}\|z(t)\|_{X_p}=0$$

- Uniform global asymptotic stability (UGAS) : $\exists\beta\in\mathcal{KL}$ such that, for every solution z, and for all $t\geq 0$

$$\|z(t)\|_{X_p}\leq\beta(\|z(0)\|_{X_p},t)$$

- Global exponential stability (GES) : UGAS with $\beta(x,t)=Cxe^{-\gamma t}$

$\mathcal{KL}$ function $(x,t)\mapsto\beta(x,t)$: continuous, increasing in x, decreasing in t, $\beta(0,t)=0$, $\beta(x,t)\to 0$ as $t\to+\infty$.

Real iterated sequences

Technique : study real iterated sequences $(x_n)_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}$

$$x_{n+1} \in S(x_n) \qquad n \in \mathbb{N}$$

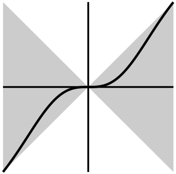
- $\forall t$, $(g_n(t))_n$ is a real iterated sequence
- Provide particular solutions to the wave equation in which $g_n(t) = x_n$ is constant $\forall n$
- Simple convergence to 0 : Every real iterated sequence $(x_n)_n$ converges to 0 as $n \to \infty$
- Uniform convergence to 0 (on compact sets) : $\forall r > 0$, $\forall \varepsilon > 0$, $\exists N = N(r, \varepsilon)$ s.t. $\forall$ real iterated seq. $(x_n)_n$ with $x_0 \leq r$, one has $x_n \leq \varepsilon$ for every $n \geq N$

Equivalent characterizations

Theorem [Chitour, Marx, Mazanti, 2020]

1. If $p < +\infty$:
   Strong stability $\iff$ real iterated sequences converge simply to 0
2. If $p = +\infty$:
   Strong stability $\iff$ UGAS $\iff$ real iterated sequences converge uniformly to 0

Strong stability is not equivalent to UGAS if $p < +\infty$



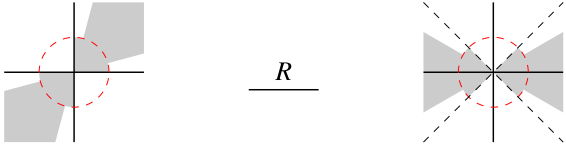
Asymptotic stability

Theorem [Chitour, Marx, Mazanti,2020]

Let $p \in [1, +\infty)$. If the wave equation is UGAS in X_∞ and Σ satisfies a sector condition at infinity, then the wave equation is UGAS in X_p

Sector condition at infinity :

- In terms of Σ : $\exists 0 < a < b < +\infty$ and $M > 0$ such that $ax \leq y \leq bx$ for every $(x, y) \in \Sigma$ with $(x, y) \geq M$
- In terms of S : $\exists \mu \in (0, 1)$ and $M > 0$ such that $y \leq \mu x$ for every $y \in S(x)$ with $(x, y) \geq M$



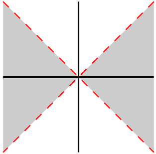
Refined condition

Theorem [Chitour, Marx, Mazanti, 2020]

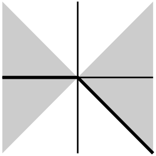
Let $p \in [1,+\infty]$ and assume that the graph of $S^{[2]} = S \circ S$ is closed. Then strong stability in $\mathsf{X}_p \iff \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ with $y \in S^{[2]}(x)$, one has $y < x \iff$ UGAS in X_∞

$$y \in S^{[2]}(x) \iff \exists z \text{ s.t. } y \in S(z) \text{ and } z \in S(x)$$

- $S^{[2]}$ is a “strict damping” :



- S is a strict damping $\implies$ strong stability, but the converse is false :



Conclusion

- We have transformed the wave equation into a discrete-time equation

$$\begin{cases} \partial_{tt} z(t,x) = \partial_{xx} z(t,x) \\ z(t,0) = 0 \\ (z_t(t,1), -z_x(t,1)) \in \Sigma. \end{cases} \iff \begin{cases} g_{n+1}(t) \in S(g_n(t)), \\ S = \mathrm{graph}(R\Sigma) \end{cases}$$

- Necessary and sufficient conditions for the well-posedness and the asymptotic stability.

Open questions

Is this framework useful for **hyperbolic systems** :

$$\begin{cases} \partial_t z + \Lambda \partial_x z = 0 \\ (z(t,1), -z(t,0)) \in \Sigma \subset \mathbb{R}^{2n} \end{cases}$$

?

To be continued...

Thank you for your attention

Any question ?