Theoretical and numerical bi-objective optimal control: Nash equilibria

Irene Marín-Gayte[1]

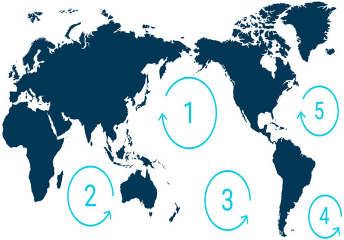
work in collaboration with
Enrique Fernández-Cara

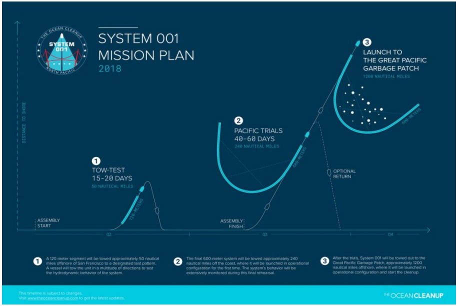
[1]Departamento EDAN, Universidad de Sevilla, Campus Reina Mercedes, 41012 Sevilla (SPAIN). E-mail: imgayte@us.es

Summary

Summary

Garbage Patch





Garbage Patch

Controls:
$f = f(x,t)$ in $\omega \times (0,T)$.
$u_\Gamma = u_\Gamma(t)$ in $(0,T)$.
State: (u,p,ψ)
Problem: Find (f, u_Γ) such that "minimize":

$$\int_\Omega |\psi(x,T) - \psi_d|^2$$

$$\iint_{\omega\times(0,T)} |f|^2 + \int_0^T |u_\Gamma|^2 + \iint_{\partial\Omega\times(0,T)} |\sigma(u,p)\cdot \overrightarrow{n}|^2$$

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{h}, \\ \nabla\cdot\mathbf{u} = 0, \\ \psi_t - \kappa\Delta\psi + \mathbf{u}\cdot\nabla\psi = -m(f,\psi), \\ u|_{\partial\Omega\times(0,T)} = u_\Gamma \quad + \dots \end{cases} \tag{1}$$

Summary

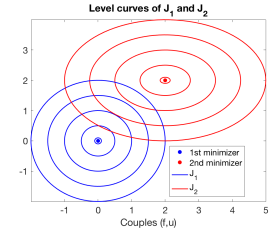
Introduction

- Multi-Objctive Optimal Control Problem

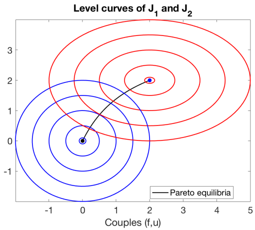
$$\min J_i(f) \qquad \begin{cases} a(u) = B(f) \\ + \text{ bounded conditions} \end{cases} \tag{2}$$

- Equilibria $\rightsquigarrow \begin{cases} \text{Nash} \\ \text{Pareto} \ldots \end{cases}$
- Bi-Objective Problem and Nash equilibria

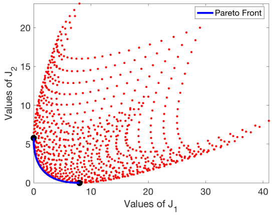
$$J_1(\widehat{f}_1, \widehat{f}_2) \leq J_1(f_1, \widehat{f}_2) \text{ and } J_2(\widehat{f}_1, \widehat{f}_2) \leq J_2(\widehat{f}_1, f_2) \quad \forall f_1, f_2 \tag{3}$$



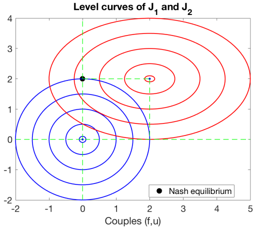
(c) Level curves



(d) Pareto equilibria



(e) Pareto front



(f) Nash equilibria

Introduction

Contents:

- Definitions and use control theory [Nash, Lions]
- Existence and characterization of Nash equilibria [Girsanov, Fursikov, Lions]
- Algorithms and numerical experiments [Fursikov-Pironneau, Glowinski]

Summary

Semilinear Case

$$\begin{cases} -\Delta u + \phi(u) = f_1 1_{\omega_1} + f_2 1_{\omega_2}, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases} \tag{4}$$

$$\begin{cases} \phi \in \mathcal{C}^1(\mathbb{R}), \quad \phi'(s) \geq 0 \quad \forall s \in \mathbb{R}, \\ |\phi(s)| \leq C + C|s| \quad \forall s \in \mathbb{R}. \end{cases} \tag{5}$$

$$J_1(\widehat{f_1}, \widehat{f_2}) \leq J_1(f_1, \widehat{f_2}) \text{ and } J_2(\widehat{f_1}, \widehat{f_2}) \leq J_2(\widehat{f_1}, f_2) \quad \forall f_1, f_2 \tag{6}$$

$$J_i(f_1, f_2, u) := \frac{a}{2} \int_{\mathcal{O}_i} |u - u_{id}|^2 + \frac{\mu}{2} \int_{\omega_i} |f_i|^2,$$

$\frac{a}{\mu}$ small (resp large) $\rightarrow$ easy (resp difficult).

Semilinear Case

Definition: Nash equilibria

Control pairs $(\widehat{f_1}, \widehat{f_2}) \in L^2(\omega_1) \times L^2(\omega_2)$ is a **Nash equilibria** if

$$\begin{cases} J_1(\widehat{f_1}, \widehat{f_2}) \leq J_1(f_1, \widehat{f_2}) & \forall f_1 \in L^2(\omega_1), \\ J_2(\widehat{f_1}, \widehat{f_2}) \leq J_2(\widehat{f_1}, f_2) & \forall f_2 \in L^2(\omega_2). \end{cases} \tag{7}$$

$(\widehat{f_1}, \widehat{f_2})$ Nash equilibria then

$$\frac{\partial J_1}{\partial f_1}(\widehat{f_1}, \widehat{f_2}) = 0, \quad \frac{\partial J_2}{\partial f_2}(\widehat{f_1}, \widehat{f_2}) = 0.$$

Semilinear Case

Definition:

$(\widehat{f}_1, \widehat{f}_2)$ is a Nash quasi-equilibria if $\dfrac{\partial J_1}{\partial f_1}(\widehat{f}_1, \widehat{f}_2) = 0, \quad \dfrac{\partial J_2}{\partial f_2}(\widehat{f}_1, \widehat{f}_2) = 0.$

Nash quasi-equilibria

$(\widehat{f}_1, \widehat{f}_2)$ is a **Nash quasi-equilibria** $\Rightarrow\ \exists\ \hat{\varphi}_1, \hat{\varphi}_2$ such that

$$\begin{cases} -\Delta \hat{u} + \phi(\hat{u}) = \widehat{f}_1 1_{\omega_1} + \widehat{f}_2 1_{\omega_2}, \quad x \in \Omega, \\ \hat{u} = 0, \quad x \in \partial\Omega, \\ -\Delta \hat{\varphi}_i + \phi'(\hat{u})\hat{\varphi}_i = (\hat{u} - u_{id}) 1_{\mathcal{O}_i}, \quad x \in \Omega, \\ \hat{\varphi}_i = 0, \quad x \in \partial\Omega, \\ \widehat{f}_i = -\dfrac{a}{\mu}\ \hat{\varphi}_i\big|_{\omega_i}. \end{cases} \tag{8}$$

Semilinear Case

Theorem

Assume: $N \leq 8$, ϕ is of class $\mathcal{C}^2$, $|\phi'| + |\phi''| \leq M$ and a/μ small. The following are equivalent:

(a) $(\widehat{f}_1, \widehat{f}_2)$ *is a Nash equilibrium.*

(b) $(\widehat{f}_1, \widehat{f}_2)$ *is a Nash quasi-equilibrium.*

Proof:

- **(a)** $\Rightarrow$ **(b)** Karush-Kuhn-Tucker Theorem.
- **(b)** $\Rightarrow$ **(a)** Define $\tilde{J}_1(f_1) := J_1(f_1, \widehat{f}_2)$ and $\tilde{J}_2(f_2) := J_2(\widehat{f}_1, f_2)$. Calculate $\tilde{J}_1''(\widehat{f}_1; g_1, g_1)$ and $\tilde{J}_2''(\widehat{f}_2; g_2, g_2)$ and prove $\tilde{J}_1''(\widehat{f}_1; g_1, g_1) > 0$ and $\tilde{J}_2''(\widehat{f}_2; g_2, g_2) > 0$ for all nonzero $g_i \in L^2(\omega_i)$. $\tilde{J}_1(\widehat{f}_1; g_1, g_1)$ and $\tilde{J}_2(\widehat{f}_2; g_2, g_2)$ are convex.

Semilinear Case

Theorem

(a) a/μ *small* $\Rightarrow \exists$ *Nash equilibria.*

(b) a/μ *very small* $\Rightarrow$ *Uniqueness.*

Proof:

Schauder's Fixed-Point Theorem, $\Psi : L^2(\omega_1) \times L^2(\omega_2) \mapsto L^2(\omega_1) \times L^2(\omega_2)$ with $(f_1, f_2) = \Psi(\tilde{f}_1, \tilde{f}_2)$ if and only if

$$f_i = -\frac{a}{\mu} \, \varphi_i|_{\omega_i}, \quad i = 1, 2.$$

Semilinear Case

- Find a Nash quasi-equilibria, i.e, solve the optimallity system

$$\begin{cases} -\Delta\widehat{u} + \phi(\widehat{u}) = \widehat{f}_1 1_{\omega_1} + \widehat{f}_2 1_{\omega_2}, \quad x \in \Omega, \\ \widehat{u} = 0, \quad x \in \partial\Omega, \\ \\ -\Delta\widehat{\varphi}_i + \phi'(\widehat{u})\widehat{\varphi}_i = (\widehat{u} - u_{id}) 1_{\mathcal{O}_i}, \quad x \in \Omega, \\ \widehat{\varphi}_i = 0, \quad x \in \partial\Omega, \\ \\ \widehat{f}_i = -\dfrac{a}{\mu}\, \widehat{\varphi}_i\big|_{\omega_i}. \end{cases} \tag{9}$$

Semilinear Case

Algorithms

$$\begin{cases} -\Delta u^n + \phi(u^n) = f_1^n 1_{\omega_1} + f_2^n 1_{\omega_2}, \quad x \in \Omega, \\ u^n = 0, \quad x \in \partial\Omega, \\ -\Delta \varphi_i^n + \phi'(u^n)\varphi_i^n = (u^n - u_{id}) 1_{\mathcal{O}_i}, \quad x \in \Omega, \\ \varphi_i^n = 0, \quad x \in \partial\Omega \end{cases} \tag{10}$$

- **ALG 1:** Fixed-Point

$$f_i^{n+1} = -\frac{a}{\mu}\, \varphi_i^n|_{\omega_i}\,. \tag{11}$$

Take

$$f_i^{n+1} = f_1^n - \rho_i J_i'(f_1^n, f_2^n)$$

with $J_i'(f_1^n, f_2^n) = a\varphi_i^n + \mu f_i^n$.

- **ALG 2:** of the Optimal Step Gradient kind

$$f_i^{n+1} = f_i^n - \rho_i^n g_i^n, \tag{12}$$

$$\begin{cases} g_i^n = a\ \varphi_i^n|_{\omega_i} + \mu f_i^n, \\ \rho_1^n = \underset{\rho\geq 0}{\text{argmin}}\ J_1(f_1^n - \rho g_1^n, f_2^n), \quad \rho_2^n = \underset{\rho\geq 0}{\text{argmin}}\ J_2(f_1^n, f_2^n - \rho g_2^n). \end{cases} \tag{13}$$

- **ALG 3:** of the Optimal Step Conjugate Gradient kind

$$f_i^{n+1} = f_i^n - \rho_i^n d_i^n, \tag{14}$$

$$\begin{cases} d_i^n = g_i^n + \gamma_i^n d_i^{n-1}, \quad \gamma_i^n = \dfrac{(g_i^n - g_i^{n-1}, g_i^n)_{L^2(\omega_i)\times L^2(\omega_i)}}{\|g_i^{n-1}\|^2_{L^2(\omega_i)}}, \\ g_i^n = a\ \varphi_i^n|_{\omega_i} + \mu f_i^n \end{cases} \tag{15}$$

$$\rho_1^n = \underset{\rho\geq 0}{\text{argmin}}\ J_1(f_1^n - \rho g_1^n, f_2^n), \qquad \rho_2^n = \underset{\rho\geq 0}{\text{argmin}}\ J_2(f_1^n, f_2^n - \rho g_2^n). \tag{16}$$

Summary

Navier-Stokes

Difficulties

- System

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_1 1_{\omega_1} + f_2 1_{\omega_2} \quad x\in\Omega, \\ \nabla\cdot u = 0 \quad x\in\Omega, \\ u=0, \quad x\in\partial\Omega. \end{cases} \tag{17}$$

- Non-linear
- Non-uniqueness

$$J_i(f,u) := \frac{a}{2}\int_{\mathcal{O}_i} |u-u_{id}|^2 + \frac{\mu}{2}\int_\omega |f|^2, \quad i=1,2, \tag{18}$$

- Existence of Nash equilibria ? OPEN
- Nash equilibria $\Rightarrow$ Nash quasi-equilibria
  $\rightsquigarrow$ Dubovitsky-Milyoutin formalism (technical proof)

Navier-Stokes

Nash quasi-equilibria: (f_1, f_2) such that

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_1 1_{\omega_1} + f_2 1_{\omega_2} \quad x\in\Omega, \\ \nabla\cdot u = 0 \quad x\in\Omega, \\ u = 0, \quad x\in\partial\Omega, \\ \\ -\nu\Delta\varphi_i + (u\cdot\nabla)\varphi_i + (\nabla u)^t\varphi_i + \nabla q_i = (u-u_{id})1_{\mathcal{O}_i} \quad x\in\Omega, \\ \nabla\cdot\varphi = 0 \quad x\in\Omega, \\ \varphi = 0, \quad x\in\partial\Omega, \\ \\ f_i = -\dfrac{a}{\mu}\varphi_i 1_{\omega_i}. \end{cases} \tag{19}$$

Navier-Stokes

Newton Method

- Fix ν sufficiently large.
- We try to find (f_1, f_2) and u such that $\dfrac{\partial J_i}{\partial f_i}(\widehat{f_1}, \widehat{f_2}) = 0$. Denote $F := \left(\dfrac{\partial J_1}{\partial f_1}, \dfrac{\partial J_2}{\partial f_2}\right)$ and compute till convergence

$$(f_1^{n+1}, f_2^{n+1}, u^{n+1}) = (f_1^n, f_2^n, u^n) - (F'(f_1^n, f_2^n, u^n))^{-1} F(f_1^n, f_2^n, u^n).$$

- Then modify ν and Newton iterates again.

Navier-Stokes

Newton Method

(a) Choose $(f_1^0, f_2^0) \in L^2(\omega_1)^N \times L^2(\omega_2)^N$ and $\nu^0 \in \mathbb{R}^+$ and compute the solution (u^0, p^0) to

$$\left\{ \begin{array}{l} -\nu^0 \Delta u^0 + \nabla p^0 = f_1^0 1_{\omega_1} + f_2^0 1_{\omega_2} \\ \nabla \cdot u^0 = 0, \quad u^0 = 0, \end{array} \right. \tag{20}$$

and the solution (φ_i^0, q_i^0) to

$$\left\{ \begin{array}{l} -\nu^0 \Delta \varphi_i^0 + \nabla q_i^0 = (u^0 - u_{id}) 1_{\mathcal{O}_i}, \\ \nabla \cdot \varphi_i^0 = 0, \quad \varphi^0 = 0, \end{array} \right. \tag{21}$$

and we take

$$f_i^0 = -\frac{a}{\mu}\, \varphi_i^0\Big|_{\omega_i} \quad \text{and} \quad \nu^1 = \max\{\tilde{\nu}, a\nu^0\}.$$

(b) For given $n \geq 0$, ν^n, $f_i^n \in L^2(\omega_i)^N$, (u^n, p^n) and (φ_i^n, q_i^n):

(b.1) Take $f_i^{n,0} = -\dfrac{a}{\mu}\varphi_i^n\,|_{\omega_i}$, $u^{n,0} = u^n$, $\varphi_i^{n,0} = \varphi_i^n$ and $\nu^{n+1} = \max(a\nu^n, \tilde{\nu})$.

(b.2) For given $k \geq 0$, $f_i^{n,k}, u^{n,k}, \varphi_i^{n,k}$, compute the solution $(v^k, h^k, \psi_i^k, \eta_i^k)$ to

$$\begin{cases} -\nu^{n+1}\Delta v^k + (u^{n,k}\cdot\nabla)v^k + (v^k\cdot\nabla)u^{n,k} + \nabla h^k = F^{n,k} & x \in \Omega, \\ -\nu^{n+1}\Delta\psi_i^k - (u^{n,k}\cdot\nabla)\psi_i^k - (v^k\cdot\nabla)\varphi_i^{n,k} + (\nabla u^{n,k})^t\psi_i^k & \\ \qquad\qquad + (\nabla v^k)^t\varphi_i^{n,k} + \nabla\eta_i^k = G_i^{n,k} & x \in \Omega, \\ \nabla\cdot v^k = 0, \quad \nabla\cdot\psi_i^k = 0 \quad x \in \Omega, & \\ v^k = 0, \quad \psi_i^k = 0, \quad x \in \partial\Omega & \end{cases} \tag{22}$$

$$F^{n,k} := -\nu^{n+1}\Delta u^{n,k} + (u^{n,k}\cdot\nabla)u^{n,k} - f_1^{n,k}1_{\omega_1} - f_2^{n,k}1_{\omega_2}$$

$$\begin{aligned} G_i^{n,k} &:= -\nu^{n+1}\Delta\varphi_i^{n,k} - (u^{n,k}\cdot\nabla)\varphi_i^{n,k} + (\nabla u^{n,k})^t\varphi_i^{n,k} \\ &\quad -(u^{n,k} - u_{id})1_{\mathcal{O}_i}, \end{aligned}$$

(b.3) Take

$$u^{n,k+1} = u^{n,k} - v^k, \quad \varphi_i^{n,k+1} = \varphi_i^{n,k} - \psi_i^k. \tag{23}$$

Navier-Stokes- Test 1

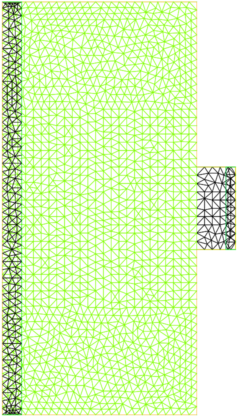


Figure: Test 1- The domain and the "rough" mesh; Ω is composed of the band ω, the large rectangle $\mathcal{O}_1$ and the small rectangle $\mathcal{O}_2$. Number of nodes: 1519 . Number of triangles: 2876.

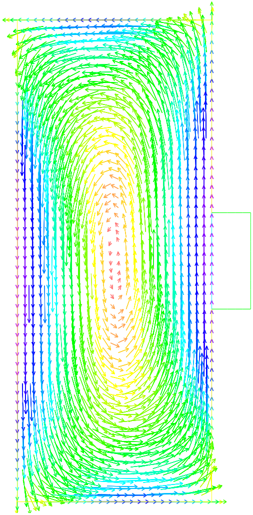
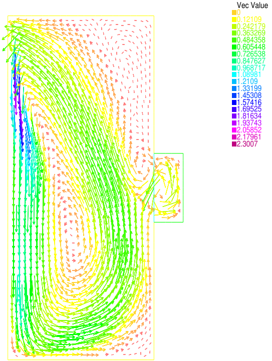


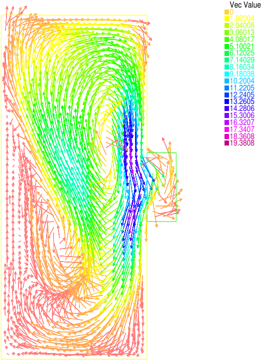
Figure: Test 1- The function $u_{1d} = \nabla \times \psi_d$; $u_{2d} \equiv 0$.



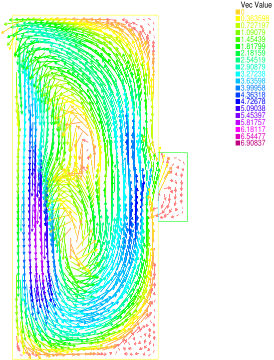
(a) Velocity

Figure: Test 1- The final velocity field and the adjoint states computed with Newton Method for $Re = 141.$. Now, $a = 1.5$ and $\mu = 2 - a = 0.5$.

Approximately $\nabla \times \psi_d$ in $\mathcal{O}_1$ and 0 in $\mathcal{O}_2$.



(a) First Adjoint State



(b) Second Adjoint State

Figure: Test 1- The final velocity field and the adjoint states computed with Newton Method for $Re = 141.$. Now, $a = 1.5$ and $\mu = 2 - a = 0.5$.

Navier-Stokes- Test 2

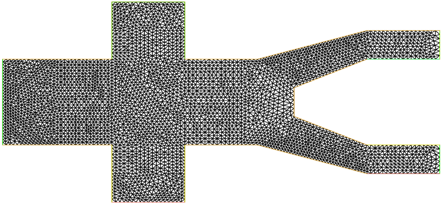


Figure: Test 2- The domain and the mesh; $\mathcal{O}_1$ and $\mathcal{O}_2$ are respectively the upper and lower small rectangles on the right. Number of nodes: 1993 . Number of triangles: 3685.

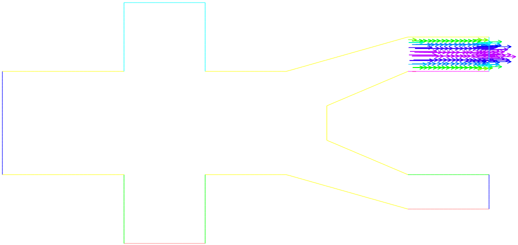
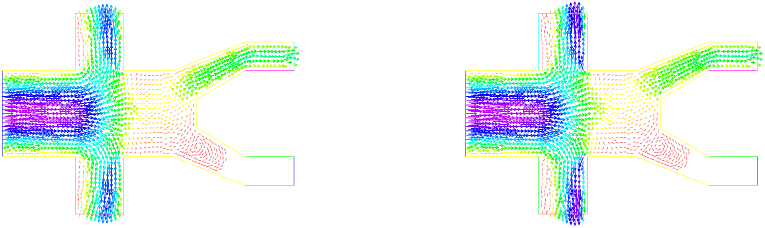
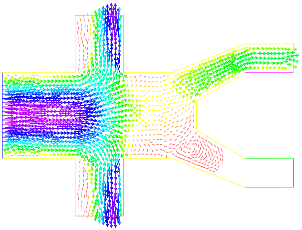


Figure: Test 2- The function u_{1d} (Poiseuille outflow in $\mathcal{O}_1$; $u_{2d} \equiv 0$ in $\mathcal{O}_2$.

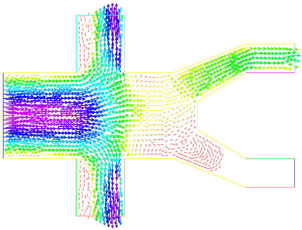


(a) $Re = 40.$ (b) $Re = 202.$

Figure: Test 2- The final velocity fields computed with Newton Algorithm for various Reynolds coefficients in the case $a = 1.95$, $\mu = 2 - a = 0.05$.



(a) $Re = 404.$



(b) $Re = 1011.$

Figure: Test 2- The final velocity fields computed with Newton Algorithm for various Reynolds coefficients in the case $a = 1.95$, $\mu = 2 - a = 0.05$.

Navier-Stokes- Test 3

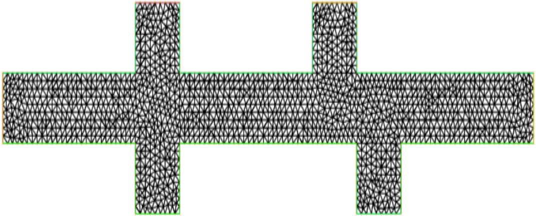


Figure: Test 3- The domain and the mesh; Ω is composed of the main pipe, the two first rectangles ω_1 and ω_2, the second upper rectangle $\mathcal{O}_1$ and the second lower rectangle $\mathcal{O}_2$. Number of nodes: 1541 . Number of triangles: 2774.

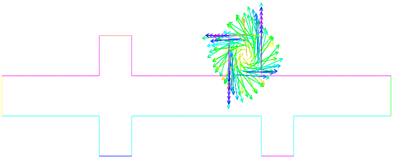
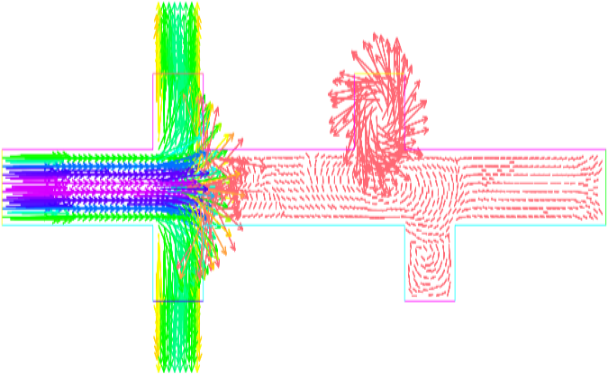
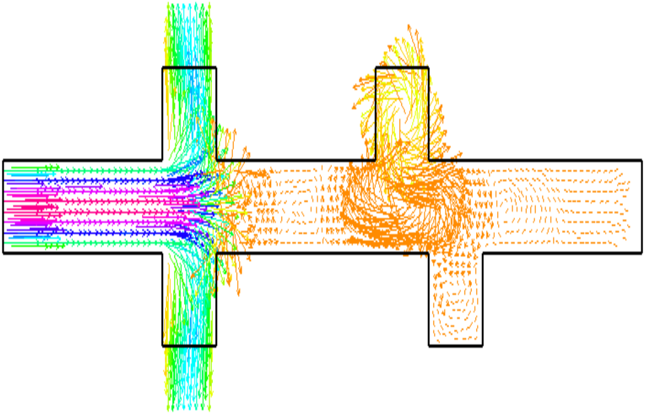


Figure: Test 3- The function $u_{1d} = \nabla \times \psi_d$ in $\mathcal{O}_1$; $u_{2d} \equiv 0$ in $\mathcal{O}_2$.



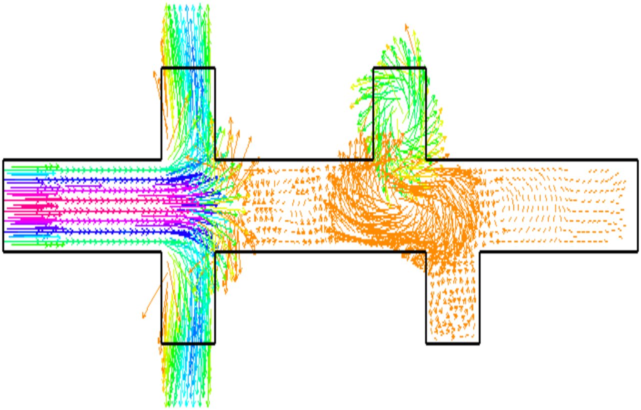
(a) $Re = 152.$

Figure: Test 3- The final velocity fields computed with Newton Algorithm for various Reynolds coefficients in the case $a = 1.99$, $\mu = 2 - a = 0.01$.



(a) $Re = 812.$

Figure: Test 3- The final velocity fields computed with Newton Algorithm for various Reynolds coefficients in the case $a = 1.99$, $\mu = 2 - a = 0.01$.



(a) $Re = 1625.$

Figure: Test 3- The final velocity fields computed with Newton Algorithm for various Reynolds coefficients in the case $a = 1.99$, $\mu = 2 - a = 0.01$.

Summary

Open Questions

- Navier-Stokes: Existence of Nash equilibria? Something can be said if ν is sufficiently large and a/μ is small.
- Navier-Stokes: When do we have Nash quasi-equilibria $\Rightarrow$ Nash equilibria?
- Evolution Navier-Stokes? In progress
- Stationary and evolution Boussinesq system? (back to plastic patches ...)

...

E. Fernndez-Cara, I. Marn-Gayte, "Theoretical and numerical results for some bi-objective optimal control problems",Communications on Pure and Applied Analysis, 19 **4** (2020), 2101-2126.

E. Fernndez-Cara, I. Marn-Gayte, "Theoretical and numerical bi-objective optimal control problems: Nash equilibria", preprints.

A. V. Fursikov, *"Optimal Control of Distributed Systems: Theory and Applications"*, American Mathematical Society Boston, MA, USA 2000.

A. V. Fursikov and O. Pironneau, *"Finite Element Methods for Navier-Stokes Equations"*, Annual Review of Fluid Mechanics, **24** (1992), 167-204.

I. V. Girsanov, *"Lectures on mathematical theory of extremum problems,"* Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, Berlin, **67** (1972).

- R. Glowinski, *"Finite element methods for incompressible viscous flow,"* Handbook of Numerical Analysis, **9** (2003).
- J. L. Lions, *"Optimal control of systems governed by partial differential equations"*, Springer-Verlag, New York, 1971.
- J. L. Lions, *"Some remarks on Stackelbergs optimization"*, Math. Models Methods Appl. Sci., **4** (1994), 477-487.
- J. F. Nash, *"Noncooperative games"*, Ann. Math., **54** (1951), 286-295.