Expansions for nonlinear systems, error estimates and convergence issues

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Program

Talk based on arXiv:2012.15653, will cover and compare:

- the Chen-Fliess expansion,
- the Magnus expansion,
- our variation around Magnus.

Each has its advantages and drawbacks for studying controllability; no free lunch principle!

This talk skips Sussmann's infinite product expansion, which is really interesting and covered in the (long) paper.

An old question

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, real-analytic, with $f_0(0) = 0$. Consider

$$\dot{x} = f_0(x) + u(t) f_1(x)$$

Notation $x(t; u, p)$ for init. data $p \in \mathbb{R}^n$ and control $u \in L^\infty(0, T)$.

Definition (Small-time local controllability)

For every $T, \eta > 0$, there exists $\delta > 0$ such that, for every $x^* \in \mathbb{R}^n$ with $|x^*| \leq \delta$, there exists $u \in L^\infty(0, T)$ such that $x(T; u, 0) = x^*$ and $\|u\|_\infty \leq \eta$.

Necessary and sufficient conditions on f_0, f_1 for STLC? Still open!

Definition is coordinate-invariant, so the answer must be too!

Goal of the talk: computing the final state

An essential point to study STLC is to compute $x(T; u, 0)$!
Thus, we must solve a time-varying ODE $\dot{x} = f(t, x)$.

Also important for stochastic ODEs and numerical splitting.

Key difficulty: lack of commutativity with time variation.
For example, in $\mathbb{R}^2$ the solution for $u(t) = t$ to

$$\dot{x} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + u(t) \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & t \\ 0 & 1 \end{pmatrix} x = A(t)x$$

is not given by

$$\left(\exp \int_0^t A(s) ds \right) x(0).$$

Here comes the commutator

For $f, g \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, the *Lie bracket* $[f, g] \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is

$$(f \cdot \nabla)g - (g \cdot \nabla)f = (Dg)f - (Df)g$$

For example, with $f = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 \\ x_2^2 \end{pmatrix}$,

$$[f, g] = \begin{pmatrix} 0 & 0 \\ 0 & 2x_2 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 + 2x_1x_2 \end{pmatrix}$$

Or, identifying f, g with first-order differential operators $f \cdot \nabla = x_1\partial_2$ and $g \cdot \nabla = \partial_1 + x_2^2\partial_2$, $[f, g]$ is identified with

$$\begin{aligned} [f, g] \cdot \nabla &= (x_1\partial_2)(\partial_1 + x_2^2\partial_2) - (\partial_1 + x_2^2\partial_2)(x_1\partial_2) \\ &= x_1\partial_{21} + 2x_1x_2\partial_2 + x_1x_2^2\partial_{22} - \partial_2 - x_1\partial_{12} + x_1x_2^2\partial_{22} \\ &= (-1 + 2x_1x_2)\partial_2 \end{aligned}$$

Sophus Lie (Norwegian)

Sophus Lie (1842-1899),
mistaken for a German spy in 1870,
released from prison by Darboux.

Lie brackets are the best (episode 1)

Consider

$$\dot{x} = u_1(t)f_1(x) + u_2(t)f_2(x)$$

For $\varepsilon > 0$, take controls (u_1, u_2) equal to

- $(1, 0)$ for $t \in [0, \varepsilon]$,
- $(0, 1)$ on $[\varepsilon, 2\varepsilon]$,
- $(-1, 0)$ on $[2\varepsilon, 3\varepsilon]$,
- $(0, -1)$ on $[3\varepsilon, 4\varepsilon]$.

Then

$$x(4\varepsilon; u, 0) = \varepsilon^2 [f_1, f_2](0) + \mathcal{O}(\varepsilon^3)$$

One can move in more complex directions, e.g. $[f_1, [f_2, [f_2, f_1]]]$.
Very important for control!

Lie brackets are the best (episode 2)

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$ and $x(0) = 0$.

Let ϕ be a smooth diffeomorphism $\mathbb{R}^n \to \mathbb{R}^n$ with $\phi(0) = 0$.
Then $y := \phi(x)$ solves $y(0) = 0$ and $\dot{y} = g_0(y) + u(t) g_1(y)$ where

$$g_i(y) := (\phi_* f_i)(y) := (D\phi)_{|\phi^{-1}(y)} f_i(\phi^{-1}(y))$$

Moreover, $[g_0, g_1] = \phi_*([f_0, f_1])$ and this holds for any Lie bracket!
For a formal bracket b, we write $g_b = \phi_* f_b$.
In particular, there exists a matrix $L \in GL_n(\mathbb{R})$, $L = D\phi_{|0}$ such that

$$\forall b, \quad g_b(0) = L f_b(0)$$

For example,

$$[[g_1, g_0], [[g_1, g_0], g_0]](0) = L[[f_1, f_0], [[f_1, f_0], f_0]](0)$$

Lie brackets are the best (episode 3)

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$ and $x(0) = 0$.
Consider $\dot{y} = g_0(y) + u(t) g_1(y)$ with $g_0(0) = 0$ and $y(0) = 0$.

Theorem (Krener, 1973)

Assume that there exists $L \in GL_n(\mathbb{R})$ such that

$$\forall b, \quad g_b(0) = L f_b(0).$$

Then there exists a (local) diffeomorphism ϕ from $\mathbb{R}^n \to \mathbb{R}^n$ with $\phi(0) = 0$ such that, for all $u \in L^\infty$ and t small enough,

$$y(t; u, 0) = \phi(x(t; u, 0)).$$

Important consequence: STLC can only depend on $\{f_b(0); b\}$.

Goal of the talk (revisited)

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$ and $x(0) = 0$.

Goal: Find a formula for $x(T; u, 0)$ for T small enough which depends only on the vectors $f_b(0) \in \mathbb{R}^n$, so on

- $f_1(0)$ ($f_0(0) = 0$ is useless),
- $[f_0, f_1](0)$,
- $[f_1, [f_0, f_1]](0)$,
- $\dots$

Classical approach from an analyst's point of view: find an asymptotic expansion with respect to a small parameter, e.g. $T \ll 1$ or $\|u\|_\infty \ll 1$.

A best-seller: the Chen-Fliess expansion

Very widely used in the control theory community to prove necessary conditions for STLC.

- Sussmann, 1983. First quadratic obstruction.
- Stefani, 1986. Strongest obstruction at each even order.
- Kawski, 1987. Second quadratic obstruction.
- Giraldi, Lissy, Moreau, Pomet, 2019. 2-controls obstruction.

A best-seller: the Chen-Fliess expansion

Could also be called *iterated Duhamel expansion* (in time).
Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $x(0) = 0$.

- *Order 1:* Since $x(t) \approx 0$, $\dot{x}(t) \approx f_0(0) + u(t) f_1(0)$ so

$$x(t) \approx x^1(t) := t f_0(0) + (\int_0^t u) f_1(0)$$

- *Order 2:* Hence $\dot{x}(t) \approx f_0(x^1) + u(t) f_1(x^1)$ so

$$\begin{aligned} x(t) \approx\ & t f_0(0) + (\int_0^t u) f_1(0) \\ & + \frac{t^2}{2} (f_0 \cdot \nabla) f_0(0) + (\int_0^t \int_0^s u)(f_0 \cdot \nabla) f_1(0) \\ & + (\int_0^t s u(s))(f_0 \cdot \nabla) f_1(0) + (\int_0^t u(s) \int_0^s u)(f_1 \cdot \nabla) f_1(0). \end{aligned}$$

Involves non-intrinsic quantities such as $(f_1 \cdot \nabla) f_1(0)$**!**

Why it is so widely used?

- It is easy to derive. Few algebraic prerequisites.
- It comes with nice explicit expressions (functionals of u) of the scalar coefficients in front of the involved vectors.
- It converges locally in time for analytic vector fields.

But ... it does not depend solely on the $\{f_b(0)\}$. So, to obtain control results, each author must work to reconstruct the Lie brackets and hide the undesired terms.

So we look for something else!

Lossless linearization?

Usually,

$$\text{nonlinear problem} \approx \text{linearized problem in the same space} \\ + \text{small remainder}$$

Here, we use another version

$$\text{nonlinear problem} = \text{linearized problem in a larger space} \\ + \text{nothing}$$

Linearization trick

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$, which is a nonlinear problem.

Define the zero-order operator

$$L(t) : \begin{cases} C^\infty(\mathbb{R}^n;\mathbb{R}) \to C^\infty(\mathbb{R}^n;\mathbb{R}) \\ \varphi \mapsto (p \mapsto \varphi(x(t;u,p))) \end{cases}$$

Then, $\forall \varphi \in C^\infty(\mathbb{R}^n;\mathbb{R})$, $\forall p \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(L(t)\varphi)(p) &= D\varphi(x(t;u,p))\,(f_0(x(t;u,p)) + u(t) f_1(x(t;u,p))) \\ &= (L(t)(f_0 \cdot \nabla + u(t) f_1 \cdot \nabla)\varphi)(p) \end{aligned}$$

So, in the weak-weak sense, $\dot{L}(t) = L(t)(f_0 \cdot \nabla + u(t) f_1 \cdot \nabla)$.
We are solving a linear differential equation in $\mathrm{Op}(C^\infty(\mathbb{R}^n;\mathbb{R}))$.

Welcome to algebra

Let $X := \{X_0, X_1\}$ be non-commutative indeterminates.

Let $\mathcal{A}(X)$ the free algebra over X, i.e. the vector space of non-commutative polynomials of X_0, X_1.
For example $1 + 7X_0 + (3X_0X_1 + 2X_1X_0) + 42X_1^3 \in \mathcal{A}(X)$.
$\mathcal{A}(X) = \oplus_{n\in\mathbb{N}}\mathcal{A}_n(X)$ (spanned by monomials of degree n).

Let $\widehat{\mathcal{A}}(X)$ the formal series generated by $\mathcal{A}(X)$, i.e. sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathcal{A}_n(X)$. Notation $a = \sum_{n\in\mathbb{N}} a_n$.
For example

$$\left(\sum_{n\in\mathbb{N}} n^n X_0^n + e^{n^3} X_1^n\right) \in \widehat{\mathcal{A}}(X)$$

No[1] convergence issue!

[1] Always read the small notes

Formal linear differential equations

Let $T > 0$ and $u \in L^\infty(0,T)$.
Consider the formal differential equation in $\widehat{\mathcal{A}}(X)$

$$\begin{cases} \dot{x}(t) = x(t)(X_0 + u(t)X_1), \\ x(0) = 1. \end{cases}$$

Definition

The solution to this formal equation is the formal-series valued function $x : [0,T] \to \widehat{\mathcal{A}}(X)$ whose homogeneous components $x_n : [0,T] \to \mathcal{A}_n(X)$ are given by $x_0(t) = 1$ and

$$x_{n+1}(t) = \int_0^t x_n(s)(X_0 + u(s)X_1)\mathrm{d}s.$$

If $u(t) = 7t$, then $x_0(t) = 1$, $x_1(t) = tX_0 + \frac{7t^2}{2}X_1$, ...
If $u(t) = 0$, then $x(t) = \sum_{n\in\mathbb{N}} \frac{t^n}{n!}X_0^n$

Formal Chen-Fliess expansion

Theorem (Classical)

$$x(t) = \sum_{n \in \mathbb{N}} \sum_{\sigma \in \{0,1\}^n} \left(\int_0^t u_\sigma \right) X_{\sigma_1} \cdots X_{\sigma_n},$$

where

$$\int_0^t u_\sigma := \int_{0<\tau_1<\cdots<\tau_n<t} u_{\sigma_1}(\tau_1) \cdots u_{\sigma_n}(\tau_n) \mathrm{d}\tau$$

with $u_0 := 1$ *and* $u_1 := u$.

Proof: Use the integral definition of x_{n+1}.

Chen-Fliess expansion for nonlinear ODEs

Theorem (Sussmann, 1983)

Let $f_0, f_1 \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ *real-analytic. Let* $u \in L^\infty(0, T)$*. For* $t > 0$ *small enough*

$$x(t; u, 0) = \sum_{n \in \mathbb{N}} \sum_{\sigma \in \{0,1\}^n} \left(\int_0^t u_\sigma \right) \left((f_{\sigma_1} \cdot \nabla) \cdots (f_{\sigma_n} \cdot \nabla) \mathrm{Id} \right)(0),$$

where the sum converges absolutely.

Proof: Use that $x(t; u, 0) = (L(t)\varphi)(0)$ for $\varphi := \mathrm{Id}$ and that L solves a linear differential equation + analytic estimates for products of first order differential operators.

Absolute convergence implies that one can obtain error estimates with respect to t and/or u by summing partial series.

Beyond Chen-Fliess?

Recall that, when solving the formal linear differential equation for $u(t) = 0$, then $x(t) = \sum_{n\in\mathbb{N}} \frac{t^n}{n!} X_0^n$.
Thus

$$x(t) = \exp(tX_0)$$

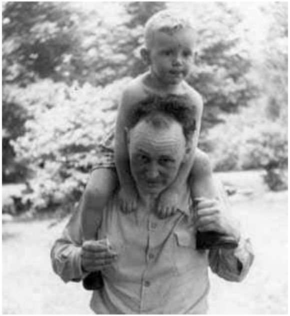
One can indeed define $\exp S$ for any $S \in \widehat{\mathcal{A}}(X)$ such that $S_0 = 0$ as

$$\exp S := \sum_{n\in\mathbb{N}} \frac{S^n}{n!}$$

Can we find a similar nice formula when $u \neq 0$?

Wilhelm Magnus (German)

On the exponential solution of differential equations for a linear operator, *Communications on pure and applied mathematics*, 1954



Wilhelm Magnus (1907-1990),
son of Alfred,
and his son Alfred.

He considers a formal linear equation $\dot{x} = A(t)x$ and finds $\Omega(t)$ such that $x(t) = \exp(\Omega(t))$.

Free Lie algebra

For $S_1, S_2 \in \mathcal{A}(X)$, we can define the Lie bracket

$$[S_1, S_2] := S_1 S_2 - S_2 S_1$$

Then,

- $\mathcal{L}(X) :=$ free Lie algebra over X, i.e. the smallest vector subspace of $\mathcal{A}(X)$ containing X and stable by Lie bracket
- $\mathcal{L}_n(X) := \mathcal{L}(X) \cap \mathcal{A}_n(X)$ its homogeneous components. For example $\mathcal{L}_3(\{X_0, X_1\})$ is spanned by $[[X_1, X_0], X_0]$ and $[X_1, [X_1, X_0]]$. So $\dim \mathcal{L}_3 = 2$ whereas $\dim \mathcal{A}_3 = 8$
- $\widehat{\mathcal{L}}(X)$ denotes the series $S \in \widehat{\mathcal{A}}(X)$ such that $S_n \in \mathcal{L}_n(X)$.

Formal Magnus expansion

Consider $\dot{x} = x(X_0 + u(t)X_1)$.

Theorem (Strichartz, 1987)

Then $x(t) = \exp Z(t,X,u)$ *where* $Z(t,X,u) \in \widehat{\mathcal{L}}(X)$ *is defined by*

$$Z(t,X,u) := \sum_{r=1}^{+\infty} \sum_{m=1}^{r} \sum_{\mathfrak{r} \in \mathbb{N}_r^m} \frac{(-1)^{m-1}}{mr} \int_{\mathcal{T}_{\mathfrak{r}(t)}}$$
$$[\cdots[X_0 + u(\tau_r)X_1, X_0 + u(\tau_{r-1})X_1], \ldots, X_0 + u(\tau_1)X_1] \mathrm{d}\tau$$

$\mathbb{N}_r^m := \{\mathfrak{r} \in (\mathbb{N}^*)^m \text{ of sum } r\}$ *and* $\mathcal{T}_{\mathfrak{r}}(t)$ *is a pyramidal domain.*

Proof:

- Using an abstract criterion, prove that $\log x(t) \in \widehat{\mathcal{L}}(X)$.
- Using C-F, prove the above formula without brackets and $1/r$.
- Apply Dynkin's theorem: $S \in \mathcal{L}_r(X) \Rightarrow S = \frac{1}{r} \times$ bracketing.

Magnus expansion for nonlinear ODEs

Consider $\dot{x} = f_0(x) + u f_1(x)$.

Theorem ($\approx$ Folklore)

Let $M \in \mathbb{N}$. If f_0, f_1 are smooth, then, for small t,

$$x(t; u, 0) = e^{Z_M(t,f,u)}(0) + \mathcal{O}(t^{M+1})$$

where $Z_M(t, f, u) \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is the sum $Z(t, X, u)$ truncated to $r \leq M$ (thus a finite sum) and where we replace X_i by f_i.

The formula involves a time-one flow: e^g,
i.e. $e^g(0) := y(1)$ where $y(0) = 0$ and $\dot{y} = g(y)$.

Fun facts

- If the Lie algebra generated by f_0 and f_1 is nilpotent of order M, then
$$x(t; u, 0) = e^{Z_M(t,f,u)}(0)$$
and this does not require analyticity of f_0, f_1 (not so easy).

- If $f_i(x) = A_i x$ where $A_i \in \mathcal{M}_n(\mathbb{R})$ and
$$\|A_0 + u(\cdot)A_1\|_{L^1(0,T)} < \frac{1}{4} \text{ or } \pi \text{ (harder)}$$
then $Z_M(t, A, u)$ converges as $M \to +\infty$.

Lack of convergence & control estimates

There exist f_0, f_1, u analytic and small such that, for every $t > 0$,

- first,

$$\lim_{M \to +\infty} |Z_M(t, f, u)(0)| = +\infty$$

  and

$$\lim_{M \to +\infty} |x(t; u, 0) - e^{Z_M(t,f,u)}(0)| = +\infty$$

  so even no weak convergence in the sense of the flow.

- and, the subseries within $Z(t, f, u)$ containing f_1 at most once does not converge, which prevents estimates

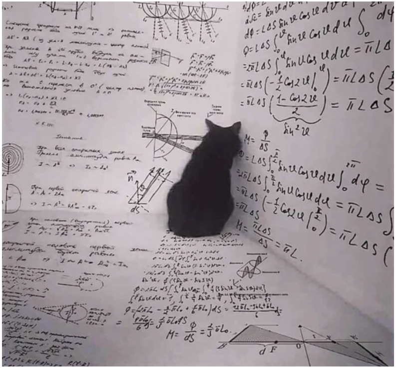
$$|x(t) - e^{\text{some convergent subseries}}| \leq C\|u\|^k$$

Really non convergent!

Interaction picture

A classical idea in quantum mechanics when one has a Hamiltonian $H(t) = H_0 + H_1(t)$ where H_0 is well understood / solvable.

- One introduces $|\psi_I(t)\rangle := e^{-iH_0 t}|\psi(t)\rangle$
- $H(t)$ becomes $e^{iH_0 t} H_1(t) e^{-iH_0 t}$



Cat planning his revenge

(© Open Culture)

Our formal Magnus in the interaction picture expansion

Consider $\dot{x} = x(X_0 + u(t)X_1)$.

Theorem

Then $x(t) = \exp(tX_0)\exp\mathcal{Z}(t,X,u)$ *where* $\mathcal{Z}(t,X,u)$ *is defined by*

$$\sum \frac{(-1)^{m-1}}{mr} \int_{\mathcal{T}_{\mathfrak{r}(t)}} \frac{(\tau_r - t)^{k_r}}{k_r!} \cdots \frac{(\tau_1 - t)^{k_1}}{k_1!} u(\tau_r)\cdots u(\tau_1)$$
$$[\cdots[M_{k_r}, M_{k_{r-1}}], \ldots, M_{k_1}]\mathrm{d}\tau$$

where $M_k = [X_0, [X_0, \ldots [X_0, X_1]\cdots]]$ *with* k *times* X_0 *and the sum is taken over* $r \geq 1$, $1 \leq m \leq r$, $\mathfrak{r} \in \mathbb{N}_r^m$ *and* $k_1, \ldots k_r \in \mathbb{N}$.

Proof: Introduce $y(s) := x(s)e^{(t-s)X_0}$. Then

$$\dot{y}(s) = y(s)\cdot e^{-(t-s)X_0}u(s)X_1e^{(t-s)X_0} = y(s)\cdot\sum_{k=0}^{+\infty}\frac{(s-t)^k}{k!}u(s)M_k$$

Back to bases

$\mathcal{Z}(t, X, u) \in \widehat{\mathcal{L}}(X)$ but the previous expansion is not directly written on a basis of $\widehat{\mathcal{L}}(X)$, e.g. because $[M_0, M_1] = -[M_1, M_0]$. Nevertheless,

Proposition

For any "monomial" basis $\mathcal{B}$ of $\mathcal{L}(X)$, there exists functionals $\eta_b : \mathbb{R}_+ \times L^\infty \to \mathbb{R}$ for $b \in \mathcal{B}$ such that

$$\mathcal{Z}(t, X, u) = \sum_{b \in \mathcal{B}} \eta_b(t, u) b$$

They are causal, and homogeneous.
For example, if $\mathcal{B}$ contains $\{X_0, X_1, [X_1, X_0]\}$,

- $\eta_{X_0}(t, u) = 0$,
- $\eta_{X_1}(t, u) = \int_0^t u$,
- $\eta_{[X_1, X_0]}(t, u) = \int_0^t \int_0^s u$, ...

Our expansion for nonlinear ODEs

Consider $\dot{x} = f_0(x) + u f_1(x)$.

Theorem

Let $M \in \mathbb{N}$. If f_0, f_1 are analytic, then, for small t and $u \in L^\infty$,

$$x(t; u, 0) = e^{\mathcal{Z}_M(t,f,u)}(e^{tf_0}(0)) + \mathcal{O}(\|u\|_{W^{-1,\infty}}^{M+1})$$

where $\mathcal{Z}_M(t, f, u)$ is the sum $\mathcal{Z}(t, X, u)$ truncated to $r \leq M$ and where we replace X_0 by f_0 and X_1 by f_1 (which converges in C^ω).

The formula involves the composition of two time-one flows.

Caution: As for the Magnus expansion, $\mathcal{Z}_M(t, f, u)$ may not converge as $M \to +\infty$ to some $\mathcal{Z}(t, f, u)$.

From an expansion to a representation formula

Consider $\dot{x} = f_0(x) + u(t) f_1(x)$ with $f_0(0) = 0$.

Theorem

If f_0, f_1 are analytic, then, for t and $u \in L^\infty$ small enough,

$$x(t; u, 0) = \mathcal{Z}_M(t, f, u)(0) + \mathcal{O}(\|u\|_{W^{-1,\infty}}^{M+1}) + o(|x(t; u, 0)|).$$

Proof: Use

- the previous formula and error estimates,
- $e^{tf_0}(0) = 0$,
- that $e^g(0) = g(0) + \mathcal{O}(\|g\||g(0)|)$ for any vector field.

This was the main goal of the talk!

Where to go from here... if you feel pessimist

$$x(T; u, 0) = \mathcal{Z}_M(T, f, u)(0) + \mathcal{O}(\|u\|^{M+1}) + o(|x(T; u, 0)|)$$

Assume that you have f_0, f_1, $M \in \mathbb{N}$ and $T > 0$ such that

$$\mathrm{Ran}\,(u \mapsto \mathcal{Z}_M(T, f, u)(0))$$

is included in a half-space $\mathbb{R}_+ \times \mathbb{R}^{n-1}$ of $\mathbb{R}^n$. Then not STLC?

Proposition

Yes. Under the condition that $\langle \mathcal{Z}_M(T, f, u)(0), e_1 \rangle \geq F(u)$ where F is such that the smallness assumption on the control implies $\|u\|^{M+1} \ll F(u)$.

This is the case for all known positive results.

Where to go from here... if you feel optimist

$$x(T;u,0) = \mathcal{Z}_M(T,f,u)(0) + \mathcal{O}(\|u\|^{M+1}) + o(|x(T;u,0)|)$$

Assume that you have f_0, f_1, $M \in \mathbb{N}$ and $T > 0$ such that

$$u \mapsto \mathcal{Z}_M(T,f,u)(0)$$

is locally onto from L^∞ to $\mathbb{R}^n$. Then STLC?

Proposition

Yes. Under the additional assumptions that $z \in \mathbb{R}^n \mapsto u_z \in L^\infty$ (for which $\mathcal{Z}_M(T,f,u_z)(0) = z$) is continuous, $u_0 = 0$ and $\|u_z\| \leq |z|^\alpha$ for $\alpha > \frac{1}{M+1}$.

This is the case for all known positive results.

We can work it out?

Next steps are to find bases $\mathcal{B}$ of the free Lie algebra over $\{X_0, X_1\}$ to overcome the following difficulties

- reflect the asymmetry between X_0 and X_1 ($\rightarrow$ not correctly embedded in known Philipp-Hall bases or Lyndon bases)
- ensure the absolute convergence of $\mathcal{Z}_M(T, f, u)$ ($\rightarrow$ linked with the growth of structure constants of the free Lie algebra)
- yield computable coefficients $\eta_b(T, u)$ for $b \in \mathcal{B}$ ($\rightarrow$ linked with the relation between pseudo-first kind coordinates and Sussmann's infinite product with coordinates of the 2nd kind)
- separate "good" and "bad" brackets ($\rightarrow$ nice question, open problem starting with $\mathcal{L}_{5,4}(X_0, X_1)$)

Thank you for your attention!

To study controllability:

- Chen-Fliess is attractive but involves undesired monomials,
- Magnus is intrinsic but does not converge and only yields error estimates in time and not in the size of the control,
- our variation
  - is intrinsic (only involves Lie brackets),
  - yields error estimates in the size of the control,
  - does not converge fully,
  - only provides an approximate representation of the state.

One can remember, for local results,

$$x(T; u, 0) \approx \mathcal{Z}_M(T, f, u)(0)$$