Some control results for Stokes equations with memory

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Outline

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System with nonlocal in time (memory) term:

$$\begin{cases} u_t - \Delta u - b \displaystyle\int_0^t e^{-a(t-s)} \Delta u(\cdot, s)\, ds + \nabla p = 0, \\ \nabla \cdot u = 0 \end{cases}$$

Figure: Saliva

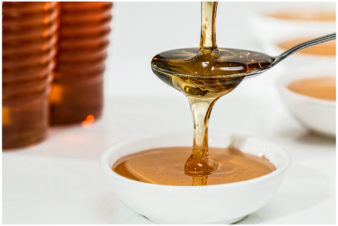

Figure: Honey

Control systems are those whose behavior can be influenced by the action of an additional variable (the control).

$$\begin{cases} u_t = Au + Bv, \\ u(0) = u_0 \quad + \quad B.C. \end{cases}$$

where $u : [0, T] \mapsto \mathcal{H}$ is the *state* and $v \in \mathcal{U}$ is the *control*.

- *Exact controllability (EC) at time T:*

  For any $u_0, u_T \in \mathcal{H}$, find $v \in \mathcal{U}$ such that $u(T) = u_T$.

- *Null controllability (NC) at time T:*

  For any $u_0 \in \mathcal{H}$, find $v \in \mathcal{U}$ such that $u(T) = 0$.

- *Approximate-finite dimensional controllability (A-FD) at time T:*

  For any $u_0, u_T \in \mathcal{H}$, any finite dimensional space $M \subset \mathcal{H}$ and for every $\varepsilon > 0$, find $v \in \mathcal{U}$ such that

  $$\|u(T) - u_T\|_{\mathcal{H}} \leq \varepsilon \text{ and } P_M(u(T)) = P_M(u_T).$$

Model nonlinear system:

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \nabla\cdot\tau + v1_\omega & \text{in} \quad Q, \\ \partial_t \tau + (u\cdot\nabla)\tau + g(\nabla u,\tau) + a\tau = 2bDu & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{on} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0, \quad \tau(0) = \tau_0 & \text{on} \quad \Omega. \end{cases}$$

Here, (u,τ) is the state, v is the control and

$$g(\nabla u,\tau) := \tau\, Wu - Wu\,\tau - k(Du\tau + \tau Du) \quad \text{and} \quad k\in[-1,1].$$

- it models a non-Newtonian homogeneous fluid with memory.
- More difficult to solve than NS ($\tau\equiv 0$).
- Much more complicate: $\exists$ global weak solution? (known for $k=0$)

[Guillopé-Saut, 1990; PL Lions-Masmoudi, 2000; EFC et al., 2002; $\cdots$]

Linearized viscoelastic Jeffreys fluids (Stokes + memory)

$$\begin{cases} u_t - \nu\Delta u + \nabla p = \nabla\cdot\tau + v1_\omega, \quad \nabla\cdot u = 0 & \text{in} \quad Q, \\ \partial\tau + a\tau = 2bDu & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0, \quad \tau(0) = \tau_0 & \text{in} \quad \Omega, \end{cases}$$

where $a, \; b > 0$. Rewritten as an integro-differential equation:

$$(S) \quad \begin{cases} u_t - \nu\Delta u - b\int_0^t e^{-a(t-s)}\Delta u(s)ds + \nabla p = e^{-at}\nabla\cdot\tau_0 + v1_\omega, & \text{in} \quad Q, \\ \nabla\cdot u = 0 & \text{on} \quad Q, \\ u = 0 & \text{on} \quad \Sigma, \\ u(0) = u_0 & \text{in} \quad \Omega. \end{cases}$$

Theorem (Doubova, Fernández-Cara, 2012)

$\forall\ \omega \subset \Omega$, (S) is Approximately-FD at any time $T > 0$.

- Same result for boundary controls $v1_\gamma$ on $\gamma \times (0, T) \subset \Sigma$.
- Equivalent to unique continuation property for adjoint problem.

When $\nu = 0$ (only elasticity) in (S) we have a *linear Maxwell fluids*.

Theorem (Boldrini, et al 2012)

Let $\nu = 0$. *For suitable large* ω, (S) *is exact controllable for large time.*

- When $b = 0$ (only viscosity) then (S) is the well-known *Stokes system*, that is NC.

Simplified case: Heat instead of Stokes

$$(H) \qquad \begin{cases} \partial_t y - \Delta y - b\int_0^t e^{-a(t-s)}\Delta y(s)ds = 0 & \text{in} \quad Q, \\ y = v1_\gamma & \text{on} \quad \Sigma, \\ y(0) = y_0, \quad \tau(0) = 0 & \text{on} \quad \Omega. \end{cases}$$

Theorem (Guerrero, Imanuvilov, 2013)

Let $T > 0$ *and* $\gamma \subset \partial\Omega$. $\exists u_0 \in L^2(\Omega)$ *such that* $u(T) \not\equiv 0$, $\forall v$.

- Same result for distributed controls $v1_\omega$, where $\overline{\omega} \subset \Omega$.

Question: Null controllability

Let us take $\tau_0 = 0$ in (S). Is there a control v such that $u(T) = 0$?

Outline

The null controllability of system (S) does not hold for $n = 2, 3$. Consider the similar boundary controlled problem

$$\begin{cases} u_t - \Delta u - b\int_0^t e^{-a(t-s)}\Delta u(s)ds + \nabla p = 0 & \text{in } Q, \\ \nabla \cdot u = 0 & \text{on } Q, \\ u = v & \text{on } \Sigma, \\ u(0) = u_0 & \text{on } \Omega. \end{cases}$$

Theorem (Fernández-Cara, Machado, Souza, 2020)

$\exists\ u_0 \in H$ such that $\forall\ v \in L^2(\Sigma)^n$ the solution u to above system satisfies

$$u(T) \neq 0.$$

Sketch of the proof: (Inspired by [Guerrero-Imanuvilov, 2013])

- Take Ω as a ball $\Omega = B(0; R)$. Not restrictive
- NC is equivalent to observability

$$\|\varphi(\,\cdot\,,0)\|^2 \leq C \iint_{\Sigma} \left| \left(-q\mathrm{Id} + \nabla\varphi + b\int_t^T e^{-a(s-t)} \nabla\varphi(\cdot, s)\, ds \right) \cdot \nu \right|^2 d\Gamma dt, \quad \forall \varphi_T \in H$$

where φ is solution to

$$\begin{cases} -\varphi_t - \Delta\varphi - b\displaystyle\int_t^T e^{-a(s-t)} \Delta\varphi(\cdot\,, s)\, ds + \nabla q = 0 & \text{in} \quad Q, \\ \nabla\cdot\varphi = 0 & \text{in} \quad Q, \\ \varphi = 0 & \text{on} \quad \Sigma, \\ \varphi(\cdot\,, T) = \varphi_T & \text{in} \quad \Omega. \end{cases}$$

- n=3;
- (φ_k, q_k): eigenfunctions; λ_k: eigenvalues of the Stokes operator with

$$\begin{cases} \varphi_k(x,y,z) = \dfrac{1}{\lambda_k^{1/2} r^2}\left(\cos(\lambda_k^{1/2} r) - \dfrac{1}{\lambda_k^{1/2} r}\sin(\lambda_k^{1/2} r)\right)(y-z, z-x, x-y), \\ q_k \equiv 0, \\ \lambda_k = \dfrac{\pi^2}{R^2}(k+1/2)^2 - \varepsilon_k, \quad \text{for some} \quad \varepsilon_k > 0 \quad \text{satisfying} \quad \varepsilon_k \to 0. \end{cases}$$

- Estimates for φ_k:

$$\|\varphi_k\|^2 \geq \frac{2\pi R}{\lambda_k},$$

$$\frac{\partial \varphi_k}{\partial \nu}\bigg|_{\partial\Omega} = -\frac{\sin(\lambda_k^{1/2} R)}{R^2}(y-z, z-x, x-y).$$

- Set $\varphi_T := \sum_{k\geq 1} \beta_k \varphi_k$

- Then: $\varphi(\cdot, t) = \sum_{k\geq 1} \alpha_k(t)\varphi_k, \quad q \equiv 0, \quad \forall t \in (0, T),$

$$\begin{cases} \alpha_k(t) \equiv 0 \quad \forall k < k_0, \\ \alpha_k(t) \equiv C_{1,k} e^{\mu_k^+(T-t)} + C_{2,k} e^{\mu_k^-(T-t)} \quad \forall k \geq k_0, \end{cases}$$

  where $\mu_k^+ \to -\infty$ and $\mu_k^- \to -(a+b)$.

- $\beta_k = 0$, for $k \neq 8M + \ell$ with $1 \leq \ell \leq 8$

GOAL: to construct, for M large, a sequence of solutions $\{\varphi^M\}$ such that:

$$(\star) \qquad \frac{C_0}{M^6} \leq \|\varphi^M(0)\|^2 \quad \text{and} \quad \iint_\Sigma \left|\frac{\partial \varphi^M}{\partial \nu}\right|^2 d\sigma dt \leq \frac{C_1}{M^{10}},$$

for some $C_0, C_1 > 0$ independent of M.

- Large $M \Longrightarrow$ Observability cannot hold.

The estimates from above

$$\iint_\Sigma e^{2(a+b)(T-t)} \left| \frac{\partial \varphi^M}{\partial \nu} \right|^2 d\Gamma\, dt \le A_1 + A_2$$

Lemma

There exists $C > 0$ such that, for M large enough, one has

$$A_1 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_k C_{1,k} e^{(a+b+\mu_k^+)(T-t)} \right)^2 dt \le \frac{C}{M^{10}},$$

Integrating by parts ten times

$$A_1 = \sum_{j=0}^{9} E_{1,j,M} g_M^{(j)}(0) - E_{2,j,M} g_M^{(j)}(T) + \int_0^T E_M(t)\, g_M^{(10)}(t)\, dt.$$

$$g_M^{(0)}(T) = g_M^{(1)}(T) = \cdots = g_M^{(8)}(T) = g_M^{(9)}(T) = 0.$$

Homogeneous system, with 5 linear equations and 8 unknowns $C_{1,8M+\ell}$ (or β_k).

Lemma

There exists $C > 0$ such that, for M large enough, one has

$$A_2 := 96\pi R^2 \int_0^T \left(\sum_M \gamma_k C_{2,k} e^{(a+b+\mu_k^-)(T-t)} \right)^2 dt \leq \frac{C}{M^{12}},$$

Expanding

$$\gamma_k C_{2,k} e^{(a+b+\mu_k^-)(T-t)} = \beta_k \left[E_{1,k} + E_{2,k}(T-t) + \mathcal{O}(\lambda_k^{-3}) \right]$$

$$\sum_M \beta_k E_{1,k} = 0 \quad \text{and} \quad \sum_M \beta_k E_{2,k} = 0$$

Homogeneous system, with 7 linear equations and 8 unknowns β_k.

It is possible to choose a nontrivial solution bounded independently of M.

The estimates from below

Lemma

There exists $C > 0$ such that, for M large enough, one has

$$\|\varphi^M(0)\|^2 \geq \frac{C}{M^6}$$

$$\|\varphi(\cdot,0)\|^2 = \sum_{k\geq k_0}(C_{1,k}e^{\mu_k^+ T} + C_{2,k}e^{\mu_k^- T})^2\|\varphi_k\|^2 \geq C\sum_{M}\frac{1}{\lambda_k}\left(\frac{3}{4}C_{2,k}^2 e^{2\mu_k^- T} - 3C_{1,k}^2 e^{2\mu_k^+ T}\right).$$

- $C_{2,k} \approx \mathcal{O}(\lambda_k^{-1})$
- $\mu_k^- \to -(a+b)$
- $C_{1,8M+\ell}^2 e^{2\mu_{8M+\ell}^+ T} \leq Ce^{-CM^2 T} \quad \forall \ell = 1,\ldots,8$

Construction of the initial condition

We define

$$u_0 = \sum_{j\geq 1} \frac{1}{j^{3/4}} \frac{\varphi_{8j+\ell_0}}{\|\varphi_{8j+\ell_0}\|}.$$

By contradiction, let $v \in L^2(\Sigma)$ be such that $u(T,\cdot) = 0$.

$$\underbrace{-\int_\Omega u_0 \varphi^M(0)}_{X_1} + \underbrace{\iint_\Sigma v \frac{\partial \varphi^M}{\partial \nu} + b \int_0^T \int_0^t e^{-a(t-s)} \left(\int_{\partial\Omega} v(s) \frac{\partial \varphi^M}{\partial \nu}(t) \right) dsdt}_{X_2} = 0$$

From $(\star)$

$$|X_1| \geq \frac{C_0}{M^{15/4}} \quad \text{and} \quad |X_2| \leq \frac{C}{M^5}.$$

Taking M large enough we have a contradiction.

Corollary

Same result for distributed controls $v1_\omega$, *where* $\omega \subset \Omega$ *is a non-empty open set with* $\Omega \setminus \overline{\omega} \neq \emptyset$*:*

NC is true $\forall\ \omega$ and $\forall\ u_0 \implies$ Contradiction

Remark

Theorem and Corollary still hold if we replace the integral (memory) term by

$$\int_0^t e^{-a(t-s)} u(\cdot, s)\, ds$$

Outline

- For two-dimensional ($n = 2$) case the proof is similar.

$$\varphi_k(x,y) = \frac{J_1(\lambda_k^{1/2} r)}{\lambda_k^{1/2} r}(-y,x) \qquad q_k \equiv 0 \qquad \lambda_k^{1/2} R = j_{1,k},$$

where J_1 is the first order Bessel function of the first kind and $j_{1,k}$ its k-th positive root.

- Using $n-1$ controls, AC in u is true.
  $n = 3$, using only one control? In general, NO! $\Omega = B(0;R) \times (0,L)$.

- Oseen equation with memory?

$$\begin{cases} u_t + (z \cdot \nabla) u - \Delta u - b \int_0^t e^{-a(t-s)} \Delta u(s) ds + \nabla p = 0 \\ \nabla \cdot u = 0 \\ etc. \end{cases}$$

- Is it possible to use moving controls for Stokes with memory?
  In [Chaves-Silva, Rosier, Zuazua, 2014] NC of a simplified similar system, with $\omega = \omega(t)$.

References

- A. Doubova, E. Fernández-Cara, *On the control of viscoelastic Jeffreys fluids*, Systems & Control Letters **61** (2012), no. 4, 573–579.
- S. Guerrero, O.Y. Imanuvilov, *Remarks on non controllability of the heat equation with memory*, ESAIM **19** (2013), no. 1, 288–300.
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Thank you very much!