Insensitizing controls for the heat equations with respect to boundary variations

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Control in Times of Crisis
Online
18 november 2021

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Summary

Summary

What is insensitizing control?

A way to treat uncertainty of measurements for PDE with partially unknown data (initial/boundary conditions, coefficients of the equation, domain...).

Global spirit

- Consider y_p^h solution of a controlled PDE, with control h, depending on an unknown parameter p.
- We are given a functional of observation $J_h(p) = \Phi(y_p^h)$, that is differentiable in p for any h.
- Goal: find a control h such that the functional is not too sensitive to small variations of p.
- More precisely: given a family of perturbations p_τ of a reference parameter p_0, ($\tau \geqslant 0$ small), find a control h such that

$$\partial_\tau J_h(p_\tau)_{|\tau=0} = 0.$$

(Partial) state of the art

Initiated by J.-L. Lions'89. Mainly studied for perturbations of the initial condition.

Approximate insensitizing for the semilinear heat equation (SHE): Bodart-Fabre'95 (JMAA).

Exact insensitizing result for SHE: de Teresa'00 (CPDE), de Teresa-Zuazua'09 (CPAA)...

Case of disjoint observation and control for SHE: de Teresa-Micu-Ortega'04 (AML).

Also study for models in fluid Mechanics by Carreno, Guerrero, Gueye... and for the wave equation by Alabau, Dager, Tebou...

The general problem under study

A quite natural question

Insensitization with respect to a perturbation of the domain?

Introduced by Lissy-Privat-Simporé'17 (COCV).

- $T > 0$ final time. ω, Θ open subsets of $\mathbb{R}^d$.
- Ω smooth bounded domain of $\mathbb{R}^d$.
- $\xi \in L^2((0,T)\times\Omega)$ source term, $h \in L^2(0,T;L^2(\omega))$ control.

$$\text{We introduce } J_h(\Omega) = \frac{1}{2}\int_0^T \int_\Theta y_{\Omega,h}(t,x)^2\,dxdt,$$

where $y_{\Omega,h}$, defined on $(0,T)\times\mathbb{R}^d$, is the extension by 0 of y verifying

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \xi + h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

Perturbations of a domain

- Ω_0 reference domain.
- $Q_0 = (0, T) \times \Omega_0$, $\Sigma_0 = (0, T) \times \partial\Omega_0$.
- $\mathbf{V}$ in $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\tau > 0$ small enough.
- $\mathbf{T}_{\mathbf{V},\tau} := \mathrm{Id} + \tau\mathbf{V}$ is a diffeo in $\mathbb{R}^d$
- $\Omega_{\mathbf{V},\tau} = (\mathrm{Id} + \tau\mathbf{V})(\Omega_0)$ is also a bounded smooth domain and is "small" perturbation of the domain.
- $Q_{\mathbf{V},\tau} = (0, T) \times \Omega_{\mathbf{V},\tau}$, $\Sigma_{\mathbf{V},\tau} = (0, T) \times \partial\Omega_{\mathbf{V},\tau}$.

$y_{\Omega_{\mathbf{V},\tau},h}$: extension by 0 of y verifying

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \xi + h\chi_\omega & \text{in } Q_{\mathbf{V},\tau}, \\ y = 0 & \text{on } \Sigma_{\mathbf{V},\tau}, \\ y(0,\cdot) = 0 & \text{in } \Omega_{\mathbf{V},\tau} \end{cases}$$

and we study $J_h(\Omega_{\mathbf{V},\tau}) = \dfrac{1}{2}\displaystyle\int_0^T \int_\Theta y^2_{\Omega_{\mathbf{V},\tau},h}.$

Three definitions of insensitizing control

Definition

The control h insensitizes J_h exactly if

$$\text{for all } \mathbf{V} \in \mathcal{W}^{3,\infty}, \quad \frac{d}{d\tau}\left(J_h\left(\Omega_{\mathbf{V},\tau}\right)\right)\Big|_{\tau=0} = 0. \qquad \text{(Insens-Ex)}$$

Let $\mathcal{E}$ be a linear subspace of $\mathcal{W}^{3,\infty}$. The control h insensitizes J_h for $\mathcal{E}$ if

$$\text{for all } \mathbf{V} \in \mathcal{E}, \quad \frac{d}{d\tau}\left(J_h\left(\Omega_{\mathbf{V},\tau}\right)\right)\Big|_{\tau=0} = 0. \qquad \text{(Insens-Ex-E)}$$

Given $\varepsilon > 0$, the control h ε-insensitizes J_h if

$$\text{for all } \mathbf{V} \in \mathcal{W}^{3,\infty}, \quad \left|\frac{d}{d\tau}\left(J_h\left(\Omega_{\mathbf{V},\tau}\right)\right)\Big|_{\tau=0}\right| \leq \varepsilon ||\mathbf{V}||_{\mathcal{W}^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)}. \qquad \text{(Insens-Appro)}$$

Clearly (Insens-Ex) implies (Insens-Ex-E) and (Insens-Appro) for any $\mathcal{E}$.

Reformulation in terms of a coupled problem (1)

Let $y_0 = y_{\Omega_0,h}$. Let us introduce

$$\begin{cases} \dfrac{\partial y_0}{\partial t} - \Delta y_0 = \xi + h\chi_\omega & \text{in } Q_0, \\ y_0 = 0 & \text{on } \Sigma_0, \\ y_0(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \qquad \text{(Coup-y)}$$

$$\begin{cases} -\dfrac{\partial q_0}{\partial t} - \Delta q_0 = y_0\chi_\Theta & \text{in } Q_0, \\ q_0 = 0 & \text{on } \Sigma_0, \\ q_0(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \qquad \text{(Coup-q)}$$

By superposition, $(y_0, q_0) = (y_\xi + y_h, q_\xi + q_h)$,

where (y_h, q_h) is the part depending on h (*i.e.* for $\xi = 0$), and (y_ξ, q_ξ) the part depending on ξ (*i.e.* for $h = 0$).

Reformulation in terms of a coupled problem (2)

(Insens-Ex) $\Leftrightarrow$ find $h \in L^2(Q_0)$ such that

$$\int_0^T \partial_n y_0 \partial_n q_0 \, dt d\sigma = 0, \quad \text{a.e. in } \partial\Omega_0.$$

(Insens-Ex-E) $\Leftrightarrow$ find $h \in L^2(Q_0)$ such that for all $\mathbf{V} \in \mathcal{E}$,

$$\int_{\partial\Omega_0} (\mathbf{V} \cdot \mathbf{n}) \left(\int_0^T \partial_n y_0 \partial_n q_0 \, dt \right) d\sigma = 0.$$

(Insens-Appro) $\Leftrightarrow$ find $h \in L^2(Q_0)$ such that for all $\mathbf{V} \in W^{3,\infty}$,

$$\left| \int_{\partial\Omega_0} (\mathbf{V} \cdot \mathbf{n}) \int_0^T \partial_n y_0 \partial_n q_0 \right| \leqslant \varepsilon ||\mathbf{V}||_{W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)}.$$

Remark

We are reduced the problem to some unusual controllability problems: even if (Coup-y)-(Coup-q) is linear, our conditions are bilinear in (y_0, q_0)!

Sketch of the proof

Let $\dot{y}_0$ be the derivative of y_0 in a direction $\mathbf{V}$.
Then, $\dot{y}_0$ verifies

$$\left\{\begin{array}{rcll} \dfrac{\partial \dot{y}_0}{\partial t} - \Delta \dot{y}_0 & = & 0 & \text{in } Q_0, \\ \dot{y}_0 & = & -\partial_n y_0 (\mathbf{V} \cdot \mathbf{n}) & \text{on } \Sigma_0, \\ \dot{y}_0\,(0,\cdot) & = & 0 & \text{in } \Omega_0. \end{array}\right.$$

Then, multiply the first equation of (Coup-q) by $\dot{y}_0$, integrate by parts to obtain

$$\frac{d}{d\tau}\left(J_h\left(\Omega_{\tau\mathbf{V}}\right)\right)\Big|_{\tau=0} = \int_{\partial\Omega_0} \mathbf{V}\cdot\mathbf{n}\left(\int_0^T \partial_n y_0 \partial_n q_0 \, dt\right) d\sigma,$$

The result easily follows.

A first negative result on exact insensitization

Theorem (ELP'20)

Assume that Ω_0 is smooth (of class $\mathscr{C}^\infty$), that $\Theta = \Omega_0$ and that $\omega \Subset \Omega_0$. Then, there exists $\xi \in L^2(Q_0)$ such that the exact insensitization problem cannot be solved.

Sketch of the proof.

- Assume $\int_0^T \partial_n y_0 \partial_n q_0 = 0$.
- Write $(y_0, q_0) = (y_\xi + y_h, q_\xi + q_h)$,
- Construct $\xi \in L^2(Q_0)$ such that

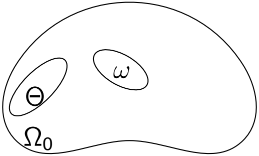
$$\int_0^T \partial_n y_0 \partial_n q_0 = -g(x)^2 + g(x) a_0(x) + a_1(x) = 0,$$

  for g of class $\mathscr{C}^4$ but nowhere $\mathscr{C}^5$, and for a_0, a_1 (depending linearly, quadratically on $\partial_n y_h$,$\partial_n q_h$) smooth.
- Contradiction since either $g(x) = \frac{1}{2}\Big(- a_0(x) + \pm\sqrt{a_0(x)^2 + 4a_1(x)}\Big)$ locally, or $g(x) = -a_0(x)/2$ everywhere.

Summary

A positive result in a particular geometrical setting

Assume $\omega \cap \Theta = \emptyset$, $\quad \Theta \Subset \Omega_0$, and $\Omega_0 \setminus \overline{\Theta}$ is connected.



- Difficult from the viewpoint of controllability (Ammar-Khodja-Benabdallah-González-Burgos-De Teresa'16 JMAA)
- And also insensitizing controls with respect to the initial condition (Kavian-De-Teresa'10 COCV, Micuo-Ortega-de Teresa'04 AML).

Theorem (ELP'20)

For all $\xi \in L^2(Q_0)$ and $\varepsilon > 0$, there exists $h \in L^2(0, T; L^2(\omega))$ that ε-approximately insensitizes J_h.

Sketch of the proof (1)

We have a stronger property.

Proposition

For any $(f_1, f_2) \in L^2(\Sigma_0)^2$, *any* $\varepsilon > 0$ *and any* $\xi \in L^2(Q_0)$, *there exists a control function* $h \in L^2(0,T; L^2(\omega))$ *such that the solution* (y,q) *of* (Coup-y)-(Coup-q) *satisfies*

$$\|\partial_n y - f_1\|_{L^2(\Sigma_0)} + \|\partial_n q - f_2\|_{L^2(\Sigma_0)} \leq \varepsilon. \qquad \text{(Approx-Boun)}$$

Choose $f_1 = f_2 = 0$, $\varepsilon \leftrightarrow \sqrt{\varepsilon}$ and CS in space:

$$\int_0^T \|\partial_n y_0 \partial_n q_0\|_{L^1(\partial\Omega_0)} \leq \varepsilon$$

and the desired result follows:

$$\left| \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \int_0^T \partial_n y_0 \partial_n q_0 \right| \leqslant \left\| \int_0^T \partial_n y_0 \partial_n q_0 \, dt \right\|_{L^1(\partial\Omega_0)} ||\mathbf{V}||_{L^\infty(\partial\Omega_0)}$$

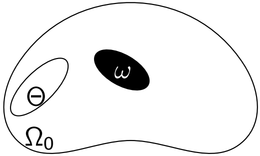
$$\leqslant \varepsilon ||\mathbf{V}||_{W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)}.$$

Sketch of the proof of the Proposition

Duality argument: if $(g_1, g_2) \in L^2(\Sigma_0)$, and (ψ, φ) solves

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_\Theta \varphi & \text{in } (0,T) \times \Omega_0, \\ \psi = g_1 & \text{on } (0,T) \times \partial\Omega_0, \\ \psi(T,\cdot) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T) \times \Omega_0, \\ \varphi = g_2 & \text{on } (0,T) \times \partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

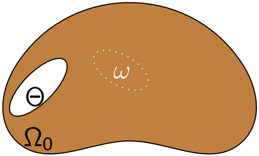
then $\psi = 0$ in $(0,T) \times \omega \Rightarrow g_1 = g_2 = 0$.



$\psi = 0$ in $(0,T) \times \omega$.

Sketch of the proof of the Proposition

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_{\Theta}\varphi & \text{in } (0,T)\times\Omega_0, \\ \psi = g_1 & \text{on } (0,T)\times\partial\Omega_0, \\ \psi(T,\cdot) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T)\times\Omega_0, \\ \varphi = g_2 & \text{on } (0,T)\times\partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

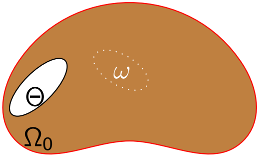


$$\psi = 0 \text{ in } (0,T)\times\omega,$$

$$-\frac{\partial \psi}{\partial t} - \Delta\psi = 0 \text{ on } (0,T)\times(\Omega_0 \setminus \overline{\Theta}).$$

Sketch of the proof of the Proposition

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_{\Theta} \varphi & \text{in } (0,T) \times \Omega_0, \\ \psi = g_1 & \text{on } (0,T) \times \partial \Omega_0, \\ \psi(T,\cdot) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T) \times \Omega_0, \\ \varphi = g_2 & \text{on } (0,T) \times \partial \Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

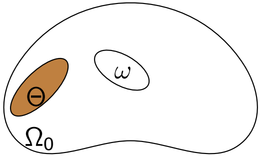


$$\Rightarrow \psi \equiv 0 \text{ in } (0,T) \times (\Omega_0 \setminus \overline{\Theta}).$$

$$\Rightarrow g_1 = 0.$$

Sketch of the proof of the Proposition

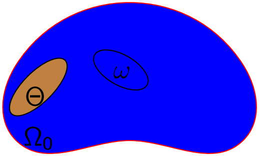
$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_{\Theta} \varphi & \text{in } (0,T) \times \Omega_0, \\ \psi = 0 & \text{in } (0,T) \times (\Omega_0 \setminus \overline{\Theta}), \\ \psi(T,\cdot) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T) \times \Omega_0, \\ \varphi = g_2 & \text{on } (0,T) \times \partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$



$$\int_0^T \int_\Theta |\varphi|^2 = \int_{Q_0} \varphi \left(-\frac{\partial \psi}{\partial t} - \Delta \psi \right) = 0 \text{ by IPP }.$$

Sketch of the proof of the Proposition

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_\Theta \varphi & \text{in } (0,T)\times\Omega_0, \\ \psi = 0 & \text{in } (0,T)\times(\Omega_0 \setminus \overline{\Theta}), \\ \psi(T,\cdot) = 0 & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T)\times\Omega_0, \\ \varphi = g_2 & \text{on } (0,T)\times\partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$



$\Rightarrow \varphi = 0$ in $(0,T)\times\Theta$.

$\Rightarrow \varphi = 0$ in $(0,T)\times\Omega_0$

$\Rightarrow g_2 = 0$.

A refinement of the previous result

In fact, we can do better.

Theorem (ELP'20)

Let $\mathcal{E}$ be a finite-dimensional linear subspace of $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Then, for all $\xi \in L^2(Q_0)$ and for all $\varepsilon > 0$, there exists a control $h \in L^2(0, T; L^2(\omega))$ that insensitizes J exactly for $\mathcal{E}$ and that ε-approximately insensitizes J.

Sketch of the proof.

- **First step:** Insensitization on $\mathcal{E}$. To simplify, assume that $\mathcal{E}$ is of dimension 1: $\mathcal{E} = \text{Span}(\mathbf{V})$. Goal: find h such that

$$\mathcal{U}(h) = \int_{\Sigma_0} (\mathbf{V}.n)\partial_n y_0 \partial_n q_0 = 0.$$

Sketch of the proof (1)

We write

$$\mathcal{U}(h) = Q(h) + L(h) + C, \text{ with}$$

$$\begin{aligned}
Q(h) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T \partial_n y_h \partial_n q_h \, dt \right) d\sigma, \\
L(h) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T (\partial_n y_\xi \partial_n q_h + \partial_n y_h \partial_n q_\xi) \, dt \right) d\sigma, \\
C &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T \partial_n y_\xi \partial_n q_\xi \, dt \right) d\sigma.
\end{aligned}$$

We will choose h_1 and h_2 in a two-dimensional space: $h = \lambda_1 h_1 + \lambda_2 h_2$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2$. We write

$$\lambda_1^2 Q_{11}(h_1) + \lambda_1\lambda_2 Q_{12}(h_1, h_2) + \lambda_2^2 Q_{22}(h_2) + \lambda_1 L_1(h_1) + \lambda_2 L_2(h_2) + C = 0.$$

Sketch of the proof (2)

Here,

$$
\begin{aligned}
Q_{11}(h_1) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T \partial_n y_{h_1} \partial_n q_{h_1} \, dt \right) d\sigma, \\
Q_{12}(h_1, h_2) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T (\partial_n y_{h_1} \partial_n q_{h_2} + \partial_n y_{h_2} \partial_n q_{h_1}) \, dt \right) d\sigma, \\
Q_{22}(h_2) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T \partial_n y_{h_2} \partial_n q_{h_2} \, dt \right) d\sigma, \\
L_1(h_1) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T (\partial_n y_{h_1} \partial_n q_{\xi} + \partial_n y_{\xi} \partial_n q_{h_1}) \, dt \right) d\sigma, \\
L_2(h_2) &= \int_{\partial\Omega_0} (\mathbf{V}\cdot\mathbf{n}) \left(\int_0^T (\partial_n y_{h_2} \partial_n q_{\xi} + \partial_n y_{\xi} \partial_n q_{h_2}) \, dt \right) d\sigma.
\end{aligned}
$$

Sketch of the proof (3)

If one can choose h_1, h_2 such that $Q_{11} = Q_{22} = 0$ and $Q_{12} = 1$, then choosing $\lambda_2 = \lambda_1$ leads to solve

$$\lambda_1|\lambda_1| + \lambda_1 L_1(h_1) + |\lambda_1| L_2(h_2) + C = 0.$$

This has a solution by the intermediate value theorem. Unfortunately, this program cannot be solved exactly but only approximately by (Approx-Boun), which turns out to be enough.
For a space of dimension M, we choose the control on a space of dimension $2M$ and we use a Brouwer fixed point argument. Our construction also leads to a continuity estimate

$$\|h\|_{L^2(0,T;L^2(\omega))} \leqslant C\|(\partial_n y_\xi, \partial_n q_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}. \qquad \text{(Cont-app)}$$

Sketch of the proof (4)

Second step: adding the approximate insensitization. Using (Approx-Boun), for some $\varepsilon_0 > 0$ small enough, choose h_{app} such that the corresponding solution (y, q) to (Coup-y)-(Coup-q) verifies

$$||(\partial_n y, \partial_n q)|| \leqslant \varepsilon_0.$$

Then, for the source term $\xi_1 = \xi + h_{app}$, choose $h_\mathcal{E}$ such that the corresponding solution (y_0, q_0) to (Coup-y)-(Coup-q) (with source ξ_1) verifies

$$\int_{\partial\Omega_0} (\mathbf{V} \cdot \mathbf{n}) \left(\int_0^T \partial_n y_0 \partial_n q_0 \, dt \right) \, d\sigma = 0.$$

$h = h_{app} + h_\mathcal{E}$ is suitable for exact insensitization on $\mathcal{E}$ of ξ. By (Cont-app) (for $h_\mathcal{E}$ and ξ_1) , if ε_0 is chosen small enough, we also have

$$||(\partial_n y_0, \partial_n q_0)|| \leqslant \sqrt{\varepsilon}$$

and so

$$\left| \int_{\partial\Omega_0} (\mathbf{V} \cdot \mathbf{n}) \int_0^T \partial_n y_0 \partial_n q_0 \right| \leqslant \varepsilon ||\mathbf{V}||_{W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)}.$$

Summary

Insensitiziation and controllability

Assume that

$$\omega \cap \Theta \neq \emptyset.$$

Theorem (ELP'20)

For every $\xi \in L^2(Q_0)$, $\varepsilon > 0$ and $y_T \in L^2(0, T)$, there exists a control h which ε-insensitizes J_h and which also ε-approximately controls the state y_0 of (Coup-y) at time T to y_T in the sense that

$$||y_0(T) - y_{T|}|_{L^2(\Omega_0)} \leq \varepsilon.$$

Besides, if the source term ξ is null-controllable in the sense that there exists $h_{nc} \in L^2(0, T; L^2(\omega))$ such that the solution y_{nc} of (Coup-y) satisfies

$$y_{nc}(T) = 0 \text{ in } \Omega_0,$$

then, there exists a control $h \in L^2(0, T; L^2(\omega))$ which ε-insensitizes J_h and which also steers the state y_0 of (Coup-y) exactly to 0 at time T.

Interpretation of the previous result

Remind that

$$\begin{cases} \dfrac{\partial y_0}{\partial t} - \Delta y_0 = \xi + h\chi_\omega & \text{in } Q_0, \\ y_0 = 0 & \text{on } \Sigma_0, \\ y_0(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

$$\begin{cases} -\dfrac{\partial q_0}{\partial t} - \Delta q_0 = y_0\chi_\Theta & \text{in } Q_0, \\ q_0 = 0 & \text{on } \Sigma_0, \\ q_0(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

What is the meaning of these theorems

In terms of controllability, we have a robustness result: We can have an approximate/null-control for a solution y_0 the heat equation with a source term (Coup-y), that ensures moreover that J_h is robust to small variations of the boundary.

sketch of the proof of the first result

For the approximate controllability, the proof is very similar to the previous case, prove a unique continuation property: if $(g_1, g_2, \psi_T) \in (L^2(0,T; L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0)$, and (ψ, φ) solves

$$\begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta\psi = \mathbb{1}_\Theta \varphi & \text{in } (0,T)\times\Omega_0, \\ \psi = g_1 & \text{on } (0,T)\times\partial\Omega_0, \\ \psi(T,\cdot) = \psi_T & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega_0, \\ \varphi = g_2 & \text{on } (0,T)\times\partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

then $\psi = 0$ in $(0,T)\times\omega \Rightarrow g_1 = g_2 = 0$ and $\psi_T = 0$.

Sketch of the proof of the second result (1)

- Introduce the functional K_ε defined for $(g_1, g_2, \psi_T) \in (L^2(0,T; L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0) =: X_0$ by

$$K_\varepsilon(g_1, g_2, \psi_T) = \frac{1}{2} \int_0^T \int_\omega |\psi(t,x)|^2 \, dxdt + \int_0^T \int_{\Omega_0} \mathbb{1}_\Theta y_0 \varphi + \varepsilon \|(g_1, g_2)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2},$$

- Endow X_0 with the norm

$$\|(g_1, g_2, \psi_T)\|_{obs}^2 = \int_0^T \int_\omega |\psi|^2 + \int_0^T \int_{\partial\Omega_0} \left(|g_1|^2 + |g_2|^2\right),$$

Then, define $X_{obs} = \overline{X_0}^{\|\cdot\|_{obs}}$.

- Extend K_ε continously on X_{obs}, by density argument and Carleman estimates.

Sketch of the proof of the second result (2)

- Prove by contradiction that

$$\liminf_{\|(g_1,g_2,\psi_T)\|_{obs}\to\infty} \frac{K_\varepsilon(g_1,g_2,\psi_T)}{\|(g_1,g_2,\psi_T)\|_{obs}} \geq \varepsilon.$$

- K_ε is continuous; strictly convex and coercive, so that it admits a unique minimizer (g_1^*, g_2^*, ψ_T^*). The Euler-Lagrange equation gives that if we set $h_1 = \psi^* \mathbb{1}_\omega$, then (y_1, q_1) solution of

$$\begin{cases} \dfrac{\partial y_1}{\partial t} - \Delta y_1 = h_1 \mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_1 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_1(0,\cdot) = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial q_1}{\partial t} - \Delta q_1 = \mathbb{1}_\Theta(y_{nc} + y_1) & \text{in } (0,T)\times\Omega_0, \\ q_1 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ q_1(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

satisfies $\|\partial_n y_1 + \partial_n y_{nc}\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n q_1\|_{L^2(0,T;L^2(\partial\Omega_0))} \leq \varepsilon$ and $y_1(T) = 0$. Setting $y = y_{nc} + y_1$, $q = q_1$ and $h = h_{nc} + h_1$ answers our question.

Summary

Other results obtained

Some other results are proved in Ervedoza-Lissy-Privat'20.

Exact insensitizing problem

There exists geometries where it can be solved positively:

- $\Theta \Subset \omega$.
- $\partial\Theta$ has only one connected component, $\Theta \Subset \Omega_0$, and $\partial\Theta \subset \omega$.

On the case $\omega \cap \Theta \neq \emptyset$, we can also obtain the following result.

Theorem (ELP'20)

*Let $\mathcal{E}$ be a finite-dimensional subspace of $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.
Then, for all $\xi \in L^2(Q_0)$ and $y_T \in L^2(\Omega_0)$, for all $\varepsilon > 0$, there exists a control $h \in L^2(0, T; L^2(\omega))$ that insensitizes J exactly for $\mathcal{E}$, ε-approximately insensitizes J, and that approximately controls y_0 at time T.
Besides, if the source term $\xi \in L^2(Q_0)$ is null-controllable, then, there exists a control $h \in L^2(0, T; L^2(\omega))$ that insensitizes J exactly for $\mathcal{E}$, ε-approximately insensitizes J, and that steers y_0 to 0 at time T.*

Open problems (1)

- Exact insensitization : understand better the influence on the geometric setting, notably when $\Theta \subsetneq \Omega_0$ and when $\Theta \cap \omega = \emptyset$. Characterization of the source terms ξ for which exact insentization is true.
- Obtain the same kind of results for other classes of equations (wave equation, semilinear heat equation, models coming from fluid mechanics, ...)
- Generalize our results for the case $\omega \cap \Theta \neq \emptyset$ to the case $\omega \cap \Theta = \emptyset$. Likely to be very difficult. A preliminary is to obtain unique continuation properties / null controllabiltiy for coupled systems of heat equations with disjoint coupling control region, which is far from being understood and out of reach in a general context.

Open problems (2)

- More general shape functionals, for instance involving also the gradient of the solution.
- More generally, for various control systems (finite or infinite-dimensional), study robustness issues for controllability, with respect to small perturbations.

Reference

Insensitizing control for the heat equation with respect to boundary variations, Sylvain Ervedoza, Pierre Lissy and Yannick Privat, submitted (2020),

`https://arxiv.org/abs/2012.14327`

`https://hal.archives-ouvertes.fr/hal-03083177v1`.

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Thank you for your attention.