

# Boundary Sliding Mode Control of hyperbolic systems

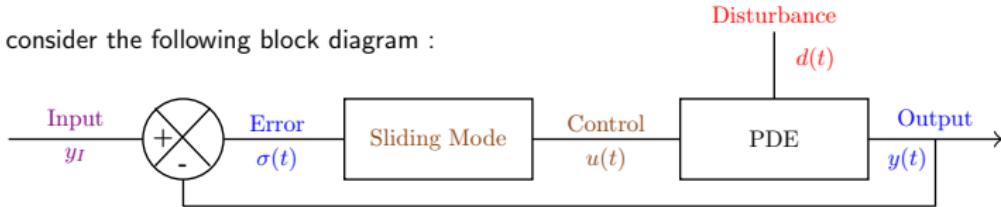
Thibault LIARD

*Post-doctoral researcher at LS2N*

*Joint work with Ismaïla Balogoun, Swann Marx, Vincent Perrollaz  
and Franck Plestan*

# Objective

We consider the following block diagram :



**Objective** : construct a boundary feedback law  $u(\cdot)$  which globally stabilizes **the input  $y_I$**

## Sliding Mode Strategy

The sliding controller  $u$  is designed to attain a **sliding surface** in finite time on which

- the boundary **disturbance  $d(\cdot)$**  is rejected
- the global asymptotic stability is ensured

**Question** : how is **the sliding surface** constructed ?

- The **sliding surface** is derived from the **gradient of a Lyapunov function**

# Outline of the presentation

- I. Sliding Mode Control to ODEs
- II. Sliding Mode Control to linear PDEs
- III. Simulations
- IV. Sliding Mode Control to scalar conservation laws

# I. Sliding Mode Control to ODEs

# Sliding Mode Control to ODEs

We consider the spring-mass system with unknown **disturbance**  $d(\cdot)$  :

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}(u(t) + d(t)) \\ x_1(0) = x_1^0, x_2(0) = x_2^0 \end{cases} \quad (\text{ODEs})$$

**The undisturbed case ( $d = 0$ )** : the feedback control law  $u(t) = -\nu x_2(t)$  with  $\nu > 0$

- The Lyapunov function  $V(x_1, x_2) = \frac{1}{2}(kx_1^2 + mx_2^2) \rightarrow (0, 0)$  is stable
- LaSalle's invariance principle  $\rightarrow (0, 0)$  is globally asymptotically stable

**The disturbed case ( $d \neq 0$ )** : a sliding mode strategy is used

- The sliding variable  $\sigma : (x_1, x_2) \in \mathbb{R}^2 \rightarrow (1, 1).\nabla V(x_1, x_2) = kx_1 + mx_2$
- The sliding surface  $\Sigma = \{ X = (x_1, x_2) \in \mathbb{R}^2 / \sigma(X) = 0 \}$
- The sliding controller  $u(t) = -(k + m)x_2(t) - K\text{sign}(\sigma(x_1(t), x_2(t)))$

where  $\text{sign}(z) = \begin{cases} -1 & \text{if } z(t) < 0 \\ [-1, 1] & \text{if } z(t) = 0 \\ 1 & \text{if } z(t) > 0 \end{cases}$   $\rightarrow$  Assuming that  $K > \|d\|_\infty$  then  $(0, 0)$  is globally asymptotically stable

# Sliding Mode Control to ODEs

**Proposition.** The solution  $(x_1(\cdot), x_2(\cdot))$  attains the **sliding surface**  $\Sigma$  in finite time and remains on it

*Proof.* The **sliding variable**  $\sigma : t \rightarrow \sigma(x_1(t), x_2(t))$  is a Filippov solution of

$$\begin{cases} \dot{\sigma}(t) \in -\sigma(t) - K \text{sign}(\sigma(t)) + d(t), \\ \sigma(0) = x_1^0 + x_2^0. \end{cases} \quad (\text{ODE})$$

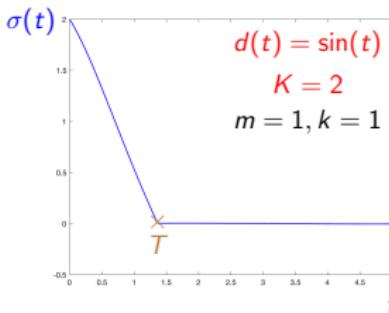
## Definition

A Filippov solution of (ODE) is an absolutely continuous map that satisfies (ODE) for almost all  $t$ .

Since  $d \in L^\infty(\mathbb{R}_+)$  and  $K > \|d\|_{L^\infty(\mathbb{R})}$ , we have

- $\sigma \in W^{1,\infty}(\mathbb{R})$
- $\exists T / \forall t > T, \sigma(t) = \dot{\sigma}(t) = 0$ .

$\implies (x_1(t), x_2(t)) \in \Sigma$  for any  $t > T$



# Sliding Mode Control to ODEs

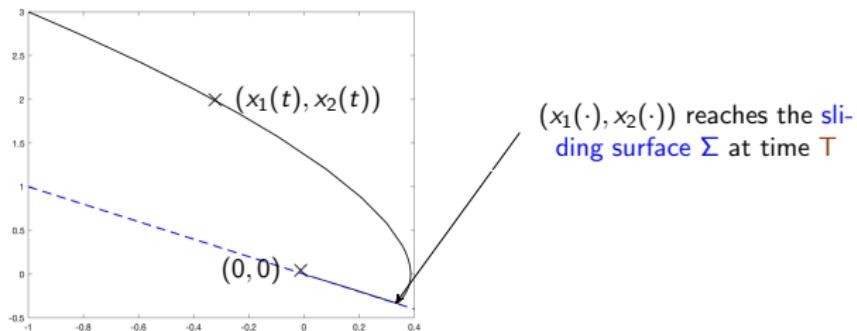
**Proposition.** The solution  $(x_1(\cdot), x_2(\cdot))$  is globally asymptotically stable **on** the *sliding surface*  $\Sigma$

*Proof.* On the *sliding surface*  $\Sigma$  the spring-mass system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{k}{m}x_1(t) + \frac{1}{m}(\textcolor{brown}{u}(t) + \textcolor{red}{d}(t)) \\ x_1(0) = x_1^0, x_2(0) = x_2^0 \end{cases} \quad \text{becomes}$$

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{k}{m}x_1(t) - \frac{k+m}{m}x_2(t) \end{cases}$$

which is globally asymptotically stable.



Plotting of  $(x_1(\cdot), x_2(\cdot))$  and the *sliding surface*  $\Sigma$  in the plane phase  $(x_1, x_2)$ .  $K = 2$  and  $d(t) = \sin(t)$

## II. Sliding Mode Control to PDEs

# References

In recent years, some authors have applied a **Sliding Mode Control** to PDEs

[Barbu, Colli, Cheng, Guo, Jin, Krstic, Liu, Levaggi, Orlov, Pisano, Utkin, Shu, Xu]

Considering **linear hyperbolic systems** with **boundary input disturbances**

- Construction of a **sliding surface  $\Sigma$**  and a **sliding controller  $u$** 
  - Backstepping method [Guo, Jin, Tang, Krstic]
  - Gradient of a Lyapunov function [T.L, Balogoun, Marx, Plestan]
- Existence and uniqueness of solutions
  - Semigroup theory  $C^0([0, T])$  with  $T$  the time when the solution of PDEs reaches the **sliding surface  $\Sigma$**  [Guo, Jin, Tang, Krstic]
  - Coupled PDEs-ODE  $C^0([0, +\infty))$  [T.L, Balogoun, Marx, Plestan]

# Linear $2 \times 2$ hyperbolic system

Let  $R_1^0, R_2^0 \in L^p(\mathbb{R})$  with  $p \in [1, +\infty]$ ,  $\lambda_1 > 0$ ,  $\lambda_2 > 0$  and  $k_2 \in \mathbb{R}$ . We consider the linear hyperbolic system

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0 \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0 \\ R_1(t, 0) = u(t) + d(t) \\ R_2(t, L) = k_2 R_1(t, L) \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x) \end{cases} \quad (\text{PDE})$$

**Objective :** find a **Sliding Surface  $\Sigma$**  on which (PDE) becomes

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0 \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0 \\ R_1(t, 0) = k_1 R_2(t, 0) \\ R_2(t, L) = k_2 R_1(t, L) \end{cases} \quad (\text{PDE-SS})$$

in finite time  $T > 0$  with  $k_1$  chosen such that  $|k_1 k_2| < 1$ . From [Bastin, Coron (2016)]  $(0, 0)$  is exponentially stable for (PDE-SS)

# Sliding Mode Control (SMC)

From [Bastin, Coron (2016)], a candidate **Lyapunov function** for (PDE-SS) is

$$V : (f, g) \in L^2(\mathbb{R})^2 \rightarrow \int_0^L \frac{p_1}{\lambda_1} f^2(x) e^{\frac{-\nu x}{\lambda_1}} + \frac{p_2}{\lambda_2} g^2(x) e^{\frac{\nu x}{\lambda_2}} dx$$

where  $p_1$  and  $p_2$  satisfies  $p_2 - p_1 k_1^2 > 0$  and  $p_1 > p_2 k_2^2 \exp(\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2}))$

## Sliding Mode Strategy

The sliding variable  $\sigma : (f, g) \in L^p(\mathbb{R})^2 \rightarrow \int_0^L \frac{1}{\lambda_1} f(x) e^{\frac{-\nu x}{\lambda_1}} + \frac{k_1}{\lambda_2} g(x) e^{\frac{\nu x}{\lambda_2}} dx$

The sliding surface  $\Sigma = \{(f, g) \in L^p(\mathbb{R})^2 / \sigma(f, g) = 0\}$

The sliding controller  $u : t \in \mathbb{R}_+ \rightarrow k_1 R_2(t, 0) - K \text{sign}(\sigma(R_1(t, \cdot), R_2(t, \cdot)))$

with  $k_1 = \frac{e^{-\nu(\frac{L}{\lambda_1} + \frac{L}{\lambda_2})}}{k_2}$ . Note that, for any  $\nu > 0$ ,  $|k_1 k_2| < 1$ .

# Existence and stabilization

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0) - K \text{sign}(\sigma(R_1(t, \cdot), R_2(t, \cdot))) + d(t) \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x), \end{cases} \quad (\text{PDE})$$

## Theorem (Existence)

Let  $p \in [1, \infty]$  and  $R_1^0, R_2^0 \in L^p(0, L)$ . If  $p \neq +\infty$ , the system (PDE) admits a weak solution

$$(R_1, R_2) \in C^0([0, \infty); L^p((0, L); \mathbb{R}^2)).$$

If  $p = \infty$ ,  $(R_1, R_2) \in C^0([0, \infty); L^1((0, L); \mathbb{R}^2))$ .

## Theorem (Stabilization)

For any  $K > \|d\|_\infty$ ,  $\exists C, \mu$  such that, for any  $R_1^0, R_2^0 \in L^p(0, L)$  with  $p \in [1, \infty]$ , for any  $t \in [0, \infty)$ ,

$$\|(R_1(t, \cdot), R_2(t, \cdot))^T\|_{L^p((0, L); \mathbb{R}^2)} \leq Ce^{-\mu t} \|(R_1^0(\cdot), R_2^0(\cdot))^T\|_{L^p((0, L); \mathbb{R}^2)},$$

for any  $(R_1(t, \cdot), R_2(t, \cdot))$  weak solution of (PDE).

# Existence of weak solutions : a coupled PDE-ODE

We consider the **weak** coupled **PDE-ODE**

$$\begin{cases} \partial_t S_1(t, x) + \lambda_1 \partial_x S_1(t, x) = 0 \\ \partial_t S_2(t, x) - \lambda_2 \partial_x S_2(t, x) = 0 \\ S_1(t, 0) = k_1 S_2(t, 0) + \dot{\gamma}(t) + \nu \gamma(t) \\ S_2(t, L) = k_2 S_1(t, L) \\ \dot{\gamma}(t) = -\nu \gamma(t) - K \text{sign}(\gamma(t)) + d(t) \\ \gamma(0) = \sigma(R_1^0, R_2^0) \\ S_1(0, x) = R_1^0(x), S_2(0, x) = R_2^0(x) \end{cases}$$

For any  $K > \|d\|_{L^\infty(\mathbb{R}_+)}$ , for any  $\gamma$  Filippov solution of ODE,

$\dot{\gamma} + \nu \gamma \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \implies$  existence of weak solutions for PDE-ODE

The function  $\sigma : t \rightarrow \sigma(S_1(t, \cdot), S_2(t, \cdot))$  satisfies

$$\begin{cases} \dot{\sigma}(t) = -\nu \sigma(t) + \dot{\gamma}(t) + \nu \gamma(t) \\ \sigma(0) = \sigma(R_1^0, R_2^0) \end{cases} \implies \begin{cases} \sigma(t) = \gamma(t) \\ \dot{\gamma}(t) + \nu \gamma(t) = -K \text{sign}(\sigma(t)) + d(t) \end{cases}$$

$\implies$  existence of weak solutions for PDE

# Stabilization

Let  $(R_1, R_2)$  a weak solution of (PDE). The **sliding variable**  $\sigma : t \rightarrow \sigma(R_1(t, \cdot), R_2(t, \cdot))$  is a Filippov solution of

$$\begin{cases} \dot{\sigma}(t) \in -\nu\sigma(t) - K\text{sign}(\sigma(t)) + d(t), \\ \sigma(0) = \sigma(R_1^0, R_2^0) \end{cases}$$

Since  $d \in L^\infty(\mathbb{R}_+)$  and  $K > \|d\|_{L^\infty(\mathbb{R})}$ ,

$$\exists T / \forall t > T, \sigma(t) = \dot{\sigma}(t) = 0.$$

The (PDE) becomes for any  $t \geq T$

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0) \\ R_2(t, L) = k_2 R_1(t, L), \end{cases}$$

with  $|k_1 k_2| < 1$ . From [Bastin, Coron (2016)]  $(0, 0)$  is exponentially stable

# Non uniforme 2\*2 hyperbolic system

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1(x) \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2(x) \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = k_1 R_2(t, 0) - K \text{sign}(\sigma(R_1(t, \cdot), R_2(t, \cdot))) + d(t) \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x), \end{cases} \quad (\text{PDE})$$

The sliding variable  $\sigma : (f, g) \in L^p(\mathbb{R})^2 \rightarrow \int_0^L \frac{1}{\lambda_1(s)} e^{- \int_0^x \frac{\nu}{\lambda_1(s)} ds} f(x) + \frac{k_1}{\lambda_2(x)} e^{- \int_0^x \frac{\nu}{\lambda_2(s)} ds} g(x) dx.$

The sliding controller  $u : t \in \mathbb{R}_+ \rightarrow k_1 R_2(t, 0) - K \text{sign}(\sigma(R_1(t, \cdot), R_2(t, \cdot)))$

with  $k_1 k_2 = \exp(-\nu \int_0^L \frac{1}{\lambda_1(s)} + \frac{1}{\lambda_2(s)} ds)$

## Theorem (Stabilization)

For any  $K > \|d\|_\infty$ ,  $\exists C, \mu$  such that, for any  $R_1^0, R_2^0 \in L^p(0, L)$  with  $p \in [1, \infty]$ , for any  $t \in [0, \infty)$ ,

$$\|(R_1(t, \cdot), R_2(t, \cdot))^T\|_{L^p((0, L); R^2)} \leq C e^{-\mu t} \|(R_1^0(\cdot), R_2^0(\cdot))^T\|_{L^p((0, L); R^2)},$$

with  $(R_1(t, \cdot), R_2(t, \cdot))$  solution of (PDE).

## IV. Simulations

# Simulation without controller

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = d(t) \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x), \end{cases}$$

The parameters are

- $L = 3$
- $\lambda_1 = 2, \lambda_2 = 1$
- $k_2 = -1$
- $d(t) = \sin(t)$

$R_1(T, \cdot)$

$R_2(T, \cdot)$

Plotting of an approximate solution ( $R_1(T, \cdot)$ ,  $R_2(T, \cdot)$ ) using an **upwind scheme**.

# Simulation with Sliding Mode Controller (SMC)

$$\begin{cases} \partial_t R_1(t, x) + \lambda_1 \partial_x R_1(t, x) = 0, \\ \partial_t R_2(t, x) - \lambda_2 \partial_x R_2(t, x) = 0, \\ R_1(t, 0) = \text{SMC} + d(t) \\ R_2(t, L) = k_2 R_1(t, L), \\ R_1(0, x) = R_1^0(x), R_2(0, x) = R_2^0(x), \end{cases}$$

The parameters are

- $L = 3$
- $\lambda_1 = 2, \lambda_2 = 1$
- $k_1 = -0.6376, k_2 = -1$
- $d(t) = \sin(t)$

$R_1(T, \cdot)$

$R_2(T, \cdot)$

Plotting of an approximate solution ( $R_1(T, \cdot), R_2(T, \cdot)$ ) using an **upwind scheme**.

## V. Sliding Mode Control to scalar conservation laws

# Sliding Mode Control to conservation laws

We consider the one-dimensional Burgers equation :

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = u_L(t) + d(t), \\ u(t,L) = u_R(t), \\ u(0,x) = u_0(x), \end{cases} \quad \text{with the flux } f : u \rightarrow \frac{u^2}{2} \quad (\text{PDE})$$

**Objective** : construct a boundary feedback law  $(u_L(\cdot), u_R(\cdot))$  which globally stabilizes the stationary entropy solution

$$y_I = \begin{cases} k & \text{if } 0 \leq x < p \\ -k & \text{if } p < x \leq L \end{cases} \quad k \geq 0, p \in (0, L)$$

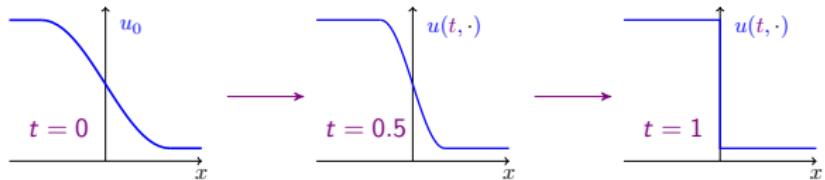
The sliding variable  $\sigma : f \in L^1(0, L) \rightarrow \int_0^L (f(x) - y_I(x)) dx$

The left controller  $u_L : t \in \mathbb{R}_+ \rightarrow y_I(0) - K \text{sign}(\sigma(u(t, \cdot)))$

The right controller  $u_R : t \in \mathbb{R}_+ \rightarrow y_I(L)$

# Conservation laws : weak solutions

## Barrier 1 : non-regularity of solutions



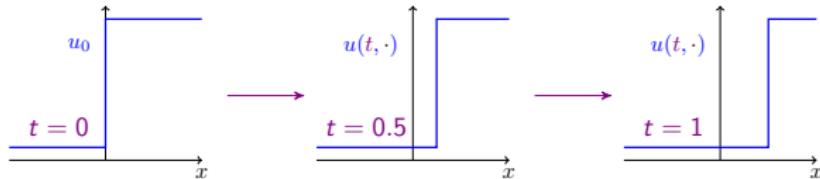
⇒ creation of a discontinuity (*called shock*) in finite time

The function  $u$  is a **weak solution** to (PDE), for  $(t, x) \in (0, +\infty) \times \mathbb{R}$ , i.e for all  $\varphi \in C_c^1(\mathbb{R}^2, \mathbb{R})$ ,

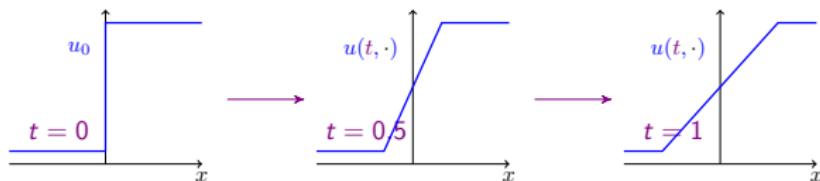
$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx = 0.$$

# Conservation laws : weak-entropy solution

**Barrier 2 : infinity of weak solutions**



**Solution 1 : moving of a discontinuity**



**Solution 2 : dissipation of a discontinuity**

The function  $u$  is a **weak-entropy solution** to (PDE) if  $\forall k \in \mathbb{R}, \forall \varphi \in C_c^1(\mathbb{R}^2, \mathbb{R}_+)$ ,

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} (|u - k| \partial_t \varphi + \operatorname{sgn}(u - k)(f(u) - f(k)) \partial_x \varphi) dx dt + \int_{\mathbb{R}} |u_0 - k| \varphi(0, x) dx \geq 0.$$

[S.N. Kruzkov, P.D. Lax, T.P. Liu, O.A. Oleinik]

# Conservation laws : boundary conditions

## Barrier 3 : entropy boundary conditions

For  $u \in \mathbb{R}$  we define  $Adm_l(u)$  and  $Adm_r(u)$  as follows

$$Adm_l(u) := \left\{ \begin{array}{ll} \{z \in \mathbb{R} / f'(z) \leq 0\} & \text{if } f'(u) \leq 0, \\ \{z \in \mathbb{R} / f'(z) < 0 \text{ and } f(z) \geq f(u)\} \cap \{u\} & \text{if } f'(u) > 0. \end{array} \right\}$$

$$Adm_r(u) := \left\{ \begin{array}{ll} \{z \in \mathbb{R} / f'(z) \geq 0\} & \text{if } f'(u) \geq 0, \\ \{z \in \mathbb{R} / f'(z) > 0 \text{ and } f(z) \geq f(u)\} \cap \{u\} & \text{if } f'(u) < 0. \end{array} \right\}$$

For almost all time  $t > 0$

$$u(t, 0+) = u_L(t) + d(t) \quad \text{becomes} \quad u(t, 0+) \in Adm_l(u_L(t) + d(t))$$

$$u(t, L-) = u_R(t) \quad \text{becomes} \quad u(t, L-) \in Adm_r(u_R(t))$$

# Sliding Mode Control to scalar conservation laws

We consider the one-dimensional Burgers equation :

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \\ u(t,0) = u_L(t) + d(t), \\ u(t,L) = u_R(t), \\ u(0,x) = u_0(x), \end{cases} \quad u(T,\cdot)$$

**Example :**  $L = 3$ ,  $u_0(\cdot) = -1$ ,  $d(t) = 0$ ,  
 $u_L(t) = 0.5$ ,  $u_R(t) = -1$

We have  $u_L(t) \neq u(t,0+)$  for any  $t \geq 0$

- Switching phenomena (inactive control)
- Delay phenomena

Let the stationary entropy solution  $y_I$  defined by  $y_I = \begin{cases} k & \text{if } 0 \leq x < p \\ -k & \text{if } p < x \leq L \end{cases}$

and  $u_L(t) = y_I(0) - K \text{sign}(\sigma(u(t,\cdot)))$  and  $u_R(t) = y_I(L)$

**Conjecture :** For any  $u_0 \in BV(0,L)$ , there exists a finite time  $T$  such that for any  $t \geq T$ ,  $u(t,\cdot) = y_I(\cdot)$

**Step 1.** Using conservation of mass, we have

$$\dot{\sigma}(t) = f(u(t, 0+)) - f(u(t, L-))$$

**Step 2.** Assuming that  $\|d\|_\infty$  is small enough

$$\exists T_1 > 0 \text{ such that } u(t, 0+) = u_L(t) + d(t) \text{ and } u(t, L-) = u_R(t)$$

**Step 3.** For every  $t \geq T_1$   $\sigma$  is a Filippov solution of

$$\begin{cases} \dot{\sigma}(t) = f(y_I(L) - K\text{sign}(\sigma(t)) + d(t)) - f(y_I(L)) \\ \sigma(T_1) = \int_0^T u(T, x) dx \end{cases}$$

**Step 4.** For every  $t \geq T_1$ , the (PDE) is rewritten as a coupled (PDE-ODE)

$$\exists T_2 > T_1 \text{ such that } \begin{cases} \sigma(t) = \dot{\sigma}(t) = 0 \\ u_L(t) = y_I(0) \text{ and } u_R(t) = y_I(L) \end{cases} \quad \forall t \geq T_2$$

$$\implies u(t, \cdot) = y_I(\cdot) \quad \forall t \geq T_2$$

# Simulation

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \\ u(t, 0) = \textcolor{violet}{y_I}(0) + \sin(t), \\ u(t, L) = \textcolor{violet}{y_I}(L), \\ u(0, x) = u_0(x), \end{cases}$$

$$\begin{cases} \partial_t u(t,x) + \partial_x f(u(t,x)) = 0, \\ u(t, 0) = \textcolor{red}{SMC} + \sin(t), \\ u(t, L) = \textcolor{brown}{y_I}(L), \\ u(0, x) = u_0(x), \end{cases}$$

$u(T, \cdot)$

$u(T, \cdot)$

Plotting of an approximate solution  $u(T, \cdot)$  using an **Godunov scheme**.

# Conclusion

**Summary :** three PDEs with **boundary disturbances**

- Linear  $2 \times 2$  hyperbolic systems
- Non uniforme  $2 \times 2$  hyperbolic systems
- Scalar conservation laws (*work in progress*)

Construction of a **boundary feedback law** which globally stabilizes a **stationary solution**

The **sliding controller** is designed to attain a sliding surface in finite time on which

- the boundary **disturbance  $d(\cdot)$**  is rejected
- the global asymptotic stability is ensured

The **sliding surface** may be derived from the gradient of a Lyapunov function

**Further works :**

- Generalization to semi-group
- Systems of conservation laws

Thank you for your attention