Controllability of subelliptic PDEs

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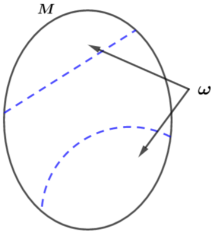
Controllability

We consider a linear PDE, of one of the following types:

$$i\partial_t u - \Delta u = f \qquad \text{(Schrödinger equation)}$$
$$\partial_t^2 u - \Delta u = f \qquad \text{(Wave equation)}$$



posed in a bounded subset $M \subset \mathbb{R}^n$
with $\partial M \neq \emptyset$ (and Dirichlet boundary conditions).

We fix $\omega \subset M$ an open subset and a time $T > 0$.

The question of controllability is: given any initial datum u_{init} and any final target u_{fin}, is it always possible to find f **supported in** ω such that the solution to the PDE with initial datum u_{init} is equal to u_{fin} at time T?

The answer depends on ω, T, and we have to say in which spaces $u_{\text{init}}, u_{\text{fin}}$ and f live.

Observability

Controllability is equivalent to observability.

Schrödinger: Given $T > 0$, $\omega \subset M$, we say that observability holds if $\exists C > 0$ such that for any solution u of the **free** equation

$$i\partial_t u - \Delta u = 0$$

there holds

$$\|u(0,\cdot)\|_{L^2(M)} \leqslant C \int_0^T \int_\omega |u(t,x)|^2 \, dxdt$$

Waves: Given $T > 0$, $\omega \subset M$, we say that observability holds if $\exists C > 0$ such that for any solution u of the **free** equation

$$\partial_t^2 u - \Delta u = 0$$

there holds

$$\|(u_0, u_1)\|^2_{\mathcal{H} \times L^2} \leqslant C \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt.$$

In the sequel we only consider observability, but there are dual controllability results.

Euclidean/Riemannian results

For the Euclidean Laplacian, and more generally for the Laplace-Beltrami operator (Laplacian) on compact Riemannian manifolds, observability results were proved in the 90es. Roughly:

- **Wave equation:** observability holds in time $T > 0$ in ω if and only if the so-called geometric control condition (GCC) holds;
- **Schrödinger equation:** if GCC holds in ω for some $T_0 > 0$, then observability in ω holds in any time $T > 0$. Converse false !

In this talk, we are going to investigate observability for PDEs with Δ a **sub-Laplacian** (= sub-Riemannian Laplacian), typically the Laplacian in the Heisenberg group.

I - Sub-Laplacians

II - Subelliptic wave equations are never observable

III - Subelliptic Schrödinger equations (joint with Chenmin Sun and Clotilde Fermanian Kammerer)

I - Sub-Laplacians

Sub-Laplacians

Let M be a smooth connected compact manifold of dimension n and μ be a smooth volume on M. Let $X_1, \ldots, X_m$ be smooth vector fields on M, and $\mathcal{D} = \mathrm{Span}(X_1, \ldots, X_m)$ (the "distribution"). We assume

$$\mathrm{Lie}(\mathcal{D}) = TM.$$

We define the **sub-Laplacian**

$$\Delta = -\sum_{i=1}^{m} X_i^* X_i = \sum_{i=1}^{m} X_i^2 + \mathrm{div}_\mu(X_i) X_i,$$

where

- Star = transpose in $L^2(M, \mu)$;
- $\mathrm{div}_\mu X$ is defined by $L_X \mu = (\mathrm{div}_\mu X)\mu$.

Sub-Laplacians are hypoelliptic: $\Delta u \in C^\infty \Rightarrow u \in C^\infty$. Examples:

- Heisenberg: $X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = \partial_y - x\partial_z$ in $\mathbb{R}^3$. $[X_1, X_2] = -\partial_z$. Then
- Grushin: $X_1^2 + X_2^2$ with $X_1 = \partial_x$ and $X_2 = x\partial_z$ in $\mathbb{R}^2$.

There can be complicated relations between brackets: $[X_1, X_3] = X_1$, etc ; sometimes, more brackets are required to generate the tangent space.

Sub-Riemannian distance

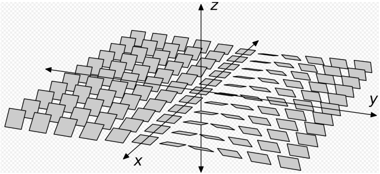
There is a metric g associated to the X_j, namely

$$g_q(v) = \inf \left\{ \sum_{j=1}^m u_j^2, \quad v = \sum_{j=1}^m u_j X_j(q) \right\},$$

and an associated distance

$$d_{\mathrm{sR}}(q,q') = \inf_{\substack{\gamma(0)=q,\gamma(1)=q' \\ \dot\gamma(t)\in\mathcal{D}, \text{ a.e. } t}} \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}dt.$$

According to Chow-Rashevsky, $d_{\mathrm{sR}}(q,q') < +\infty$ for any $q, q' \in M$.



II - Subelliptic wave equations are never observable

In a few words

We consider the **wave equation**

$$\partial_{tt}^2 u - \Delta u = 0, \quad (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1), \quad u = 0 \text{ on } (0,T) \times \partial M$$

in a manifold M equipped with a volume μ. Here, Δ is a sub-Laplacian. We fix $\omega \subset M$ (measurable). The natural **energy** of a solution is

$$E(u(t,\cdot)) = \frac{1}{2} \int_M \left(|\partial_t u(t,x)|^2 + |\nabla^{sR} u(t,x)|^2 \right) d\mu(x).$$

where

$$\nabla^{sR} \phi = \sum_{j=1}^{m} (X_j \phi) X_j.$$

Then $\frac{d}{dt} E(u(t,\cdot)) = 0$. **Initial data:**

$$\|(u_0, u_1)\|_{\mathcal{H} \times L^2}^2 = \|u_0\|_{\mathcal{H}}^2 + \|u_1\|_{L^2(M,\mu)}^2$$

with

$$\|v\|_{\mathcal{H}} = \left(\int_M |\nabla^{sR} v(x)|^2 d\mu(x) \right)^{\frac{1}{2}}.$$

Main result

Recall that (exact) observability holds in time T_0 if $\exists C > 0$,

$$C\|(u_0, u_1)\|^2_{\mathcal{H}\times L^2} \leqslant \int_0^{T_0} \int_\omega |\partial_t u(t,x)|^2 d\mu(x) dt.$$

We take a sub-Laplacian $-\sum X_i^* X_i$ such that $\mathcal{D} = \mathrm{Span}(X_1, \ldots, X_m)$ satisfies the following **unavoidable assumption**:

The set of $x \in M$ s.t. $\mathcal{D}_x \subsetneq T_x M$ is **dense** in M.

Theorem (C.L.-2020)

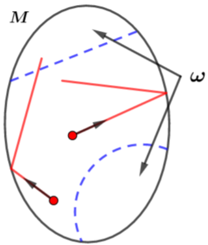
Let $T_0 > 0$ and let $\omega \subset M$ be a measurable subset such that $M \backslash \omega$ has nonempty interior. Then the subelliptic wave equation is **not exactly observable** *on ω in time T_0.*

NB: Also works for Grushin and the almost-Riemannian case as soon as $M \setminus \omega$ contains in its interior a point x such that $\mathcal{D}_x \subsetneq T_x M$.

How does one usually disprove an observability inequality?

If Δ is a **Riemannian** Laplacian,

$$\text{(Observability in time } T_0 \text{ in } \omega) \Leftrightarrow$$
$$(\omega \text{ satisfies GCC in time } T_0)$$



where (GCC):

any ray of geometrical optics (=geodesic)
travelled at speed 1 meets ω within time T_0.

(Bardos-Lebeau-Rauch, 1992).

Proof that if GCC does not hold, then observability fails: If GCC is not satisfied, we take a geodesic not entering ω in time T_0. We construct a sequence of solutions of the wave equation whose energy is concentrated along this geodesic. This sequence contradicts the observability inequality.
Many (equivalent) constructions: Gaussian beams, coherent states, WKB.

Reminder: A geodesic is a local minimizer of the sR distance d_{sR}.

Structure of the proof

Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M\backslash\omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.
  Therefore observability does not hold.
  More generally, along any (normal) sub-Riemannian geodesic, one may construct Gaussian beams.

Remark: Second point is not surprising (although not explicitly in the literature), first point is new.

Proof: Existence of spiraling geodesics

Forget about the wave equation ! This is pure **geometry**.

Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M \backslash \omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.

Proposition

For any $T_0 > 0$, any $q \in M$ and any open neighborhood V of q in M, there exists a geodesic $t \mapsto x(t)$ of $(M, \mathcal{D}, g)$ travelled at speed 1 and such that $x(t) \in V$ for any $t \in (0, T_0)$.

Remark: These geodesics lose quickly their optimality.
Main idea: Isolate a "Heisenberg structure".
Two steps: Nilpotent case and then general case.

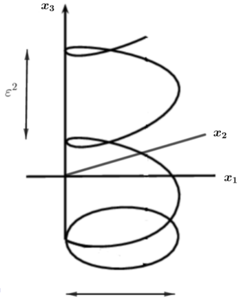
Example: if $X_1 = \partial_x$ and $X_2 = \partial_y - (x + x^2)\partial_z$ we compare the geodesics with those of Heisenberg, i.e., X_2 is replaced by $\partial_y - x\partial_z$.

Example of spiraling: the 3D Heisenberg case

Example: $M_H = (-1,1)_{x_1} \times \mathbb{T}_{x_2} \times \mathbb{T}_{x_3}$, with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.
Vector fields $X_1 = \partial_{x_1}$ and $X_2 = \partial_{x_2} - x_1\partial_{x_3}$.
Laplacian $\Delta = X_1^2 + X_2^2$. [Measure μ =Lebesgue.]
Distribution $\mathcal{D}_H = \text{Span}(X_1, X_2)$.
Metric g_H: (X_1, X_2) is a g_H-orthonormal frame of $\mathcal{D}_H$.
Then, $(M_H, \mathcal{D}_H, g_H)$= "Heisenberg manifold with boundary".
We note that $[X_1, X_2] = -\partial_{x_3}$ ($\Rightarrow \Delta$ subelliptic).

Spiraling geodesics:

$$
\begin{aligned}
x_1(t) &= \varepsilon \sin(t/\varepsilon) \\
x_2(t) &= \varepsilon \cos(t/\varepsilon) - \varepsilon \\
x_3(t) &= \varepsilon(t/2 - \varepsilon \sin(2t/\varepsilon)/4).
\end{aligned}
$$



They spiral around the x_3 axis $x_1 = x_2 = 0$.

Remark: This geodesic is travelled at speed 1.

Intuition in Heisenberg

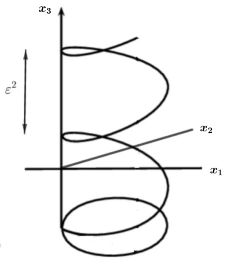
The **Hamiltonian** is $g^*(x,\xi) = \sigma_p(-\Delta)$.
Example: For Heisenberg, $g^* = \xi_1^2 + (\xi_2 - x_1\xi_3)^2$.
Geodesics are projections of null-bicharacteristics (travelled at speed 1), i.e., maximal integral curves of $\vec{p}_2$ lying in $\{p_2 = 0\}$ where $p_2 = -\tau^2 + g^*$. In Heisenberg, the bicharacteristic equations are

$$\begin{aligned} \dot{x}_1(t) &= 2\xi_1, & \dot{\xi}_1(t) &= 2\xi_3(\xi_2 - x_1\xi_3), \\ \dot{x}_2(t) &= 2(\xi_2 - x_1\xi_3), & \dot{\xi}_2(t) &= 0, \\ \dot{x}_3(t) &= -2x_1(\xi_2 - x_1\xi_3), & \dot{\xi}_3(t) &= 0. \end{aligned}$$

Take $\xi_3 = (2\varepsilon)^{-1}$. Since the geodesic is travelled at speed 1, i.e., $\xi_1^2 + (\xi_2 - x_1\xi_3)^2 = 1/4$, we take for example $\xi_1 = \cos(t/\varepsilon)/2$ and $\xi_2 = 0$. Then

$$x_1(t) = \varepsilon \sin(\frac{t}{\varepsilon}), \quad x_2(t) = \varepsilon\cos(\frac{t}{\varepsilon}) - \varepsilon$$

$$x_3(t) = \varepsilon(\frac{t}{2} - \frac{\varepsilon}{4}\sin(\frac{2t}{\varepsilon})).$$



Geodesics do not go far from their initial point !

Proof: Gaussian beams along normal sR geodesics

Two ingredients:

- Find a *sub-Riemannian* geodesic which does not enter ω within time T_0: in other words, (GCC) never holds because there exist *spiraling geodesics* which stay very long in $M \backslash \omega$.
- Construct a Gaussian beam along this geodesic: all the energy concentrates near this geodesic, hence outside ω.

Two steps for constructing Gaussian beams (GBs):
- **Approximate solutions:** $\partial_{tt}^2 v_k - \Delta v_k \sim 0$ with energy concentrated along the geodesic;
- **Exact solutions:** $\partial_{tt}^2 u_k - \Delta u_k = 0$ and concentrated energy.

Important: The construction is the same as for the Riemannian wave equation since normal geodesics stay in the elliptic part of the symbol.

III - Subelliptic Schrödinger equations

Joint with Chenmin Sun and Clotilde Fermanian Kammerer.

[For subelliptic heat equations, see [Beauchard-Cannarsa-Gugliemi 2014] and [Beauchard-Cannarsa 2017] among others.]

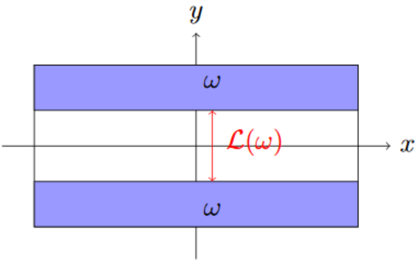
Ideas for subelliptic Schrödinger equation

Burq-Sun (2019) for the Grushin Schrödinger

$$i\partial_t u - (\partial_x^2 + x^2\partial_y^2)u = 0 \qquad \text{on } \mathbb{R}_t \times (-1,1)_x \times \mathbb{T}_y.$$

Observation set of the form $\omega = (-1,1)_x \times \omega_y$ (union of strips).

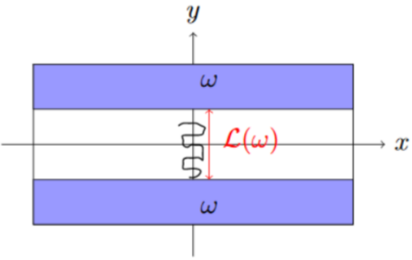
Result:
Existence of a minimal time of control $\mathcal{L}(\omega)$ related to the maximal height of the strips of $M_G \setminus \omega$.



Proof: Semiclassical analysis and construction of vertical Gaussian beams (along degenerated direction).

With spiraling geodesics: We find again their result heuristically.

Different frequencies travel at different speed (**dispersion**). If ξ_y is large, the spiraling geodesic is more "folded", makes very small meanders, but it is travelled more quickly. All in all, geodesics starting from 0 but with different ξ_y reach ω at the same time ($\neq$ waves).



With C. Fermanian: We proved a similar result in compact quotients of groups of Heisenberg type using non-commutative Fourier analysis (representation theory) and adapted semiclassical measures. We found finite speed propagation along degenerate directions. We also constructed "Gaussian beams" in these directions (quite robust technique, as soon as representations are known).

Subellipticity and speed of propagation

We consider the Baouendi-Grushin operator

$$\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2$$

in $M = (-1,1)_x \times \mathbb{T}_y$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

If $\gamma \in \mathbb{N}$, we need brackets of length $\gamma+1$ to recover TM along $x=0$. Step $= \gamma + 1$.

ω is a horizontal strip $(-1,1)_x \times I_y$ where $I \subsetneq \mathbb{T}$.

Fix $s \in \mathbb{N}$. We consider the Schrödinger-type equation $i\partial_t u - (-\Delta_\gamma)^s u = 0$ in M with Dirichlet boundary conditions. It is observable on ω in time $T > 0$ if $\exists C > 0$ such that for any solution u,

$$\|u(0,\cdot)\|^2_{L^2(M)} \leqslant C \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)}\, dt.$$

We denote by T_{ob} the infimum of all observable times $T > 0$.

Subellipticity and speed of propagation

Here is the main result concerning observability of

$$i\partial_t - (-\Delta_\gamma)^s = 0, \qquad \Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2.$$

Theorem (C.L., C. Sun (2020))

Assume that $\gamma \geqslant 1$. Then

- *If $s < \frac{\gamma+1}{2}$, then $T_{ob} = +\infty$.*
- *If $s = \frac{\gamma+1}{2}$, then $0 < T_{ob} < +\infty$.*
- *If $s > \frac{\gamma+1}{2}$, then $T_{ob} = 0$.*

Remark: we do not compute the exact value when $s = \frac{\gamma+1}{2}$.
Remark: it recalls the results of Beauchard-Cannarsa-Guglielmi for the heat equation. We recover part of it thanks to abstract results of Duyckaerts-Miller 2012.
Remark: if $s = \frac{1}{2}$, this is the wave result.

Ideas of proof

- Point 1 (and part of point 2) : to disprove observability, construction of analogues of Gaussian beams, directly in the degenerated directions.
- Point 3 (and part of point 1) : resolvent estimate

$$\|v\|_{L^2(M)} \leqslant C\|v\|_{L^2(\omega)} + Ch^{-(\gamma+1)}\|(h^2\Delta_\gamma + 1)v\|_{L^2(M)}$$

and then Burq-Zworski 2004. Also has consequences on damped wave equations.

Thank you very much for your attention !