Optimal Control for Management of Aquatic Population Models

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Outline

1. Introduction to some Optimal Control Ideas

2. River Model
with collaborator, Rebecca Pettit, U. S. Dept. of Defense

3. Black Sea Fishery Model
with collaborator, Mahir Demir, Michigan State U, postdoc in fishery group

4. Modeling the effects of habitat degradation
with collaborators: Michael Kelly, Transylvania University and Mike Neubert, Woods Hole Oceanographic Institute

Basic Idea behind Optimal Control

System of ODEs or PDEs
Decide on how to manage this system
—by choosing the terms to be controlled and bounds on the controls

Design an appropriate GOAL, objective functional
—balancing opposing factors in functional
—include (or not) terms at the final time

Derive necessary conditions for the optimal control
Compute the optimal control numerically

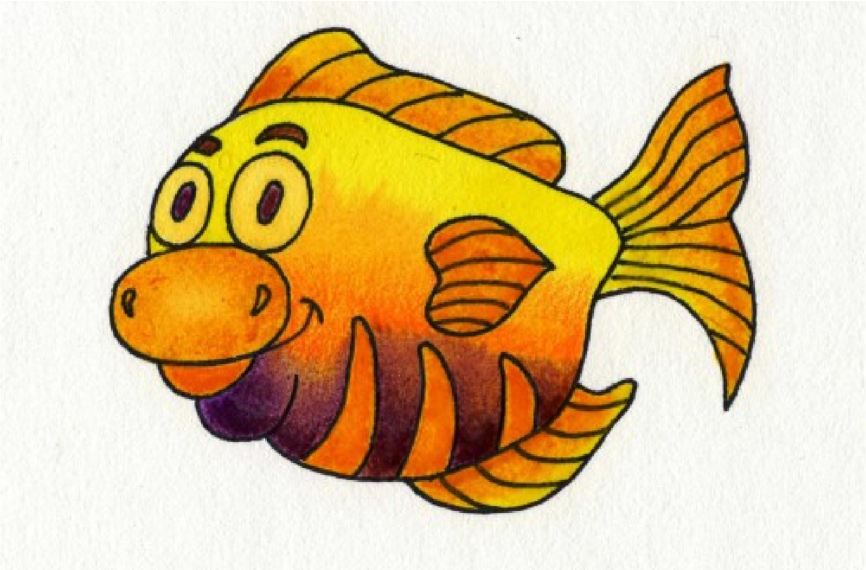
Optimal Control and Pontryagin's Maximum Principle

- Pontryagin and his collaborators developed optimal control theory for ODEs about 1950.
- Pontryagin's key idea was the introduction of the adjoint variables to attach the differential equations to the objective functional (like a Lagrange multiplier attaching a constraint to an optimization of a function).
- Instead of finding an optimal control to maximize the objective functional subject to dynamic equations, they maximize the Hamiltonian with respect to the control at each time.

Hamiltonian $H =$ (integrand of goal) $+\lambda$(RHS of state ODE).

Choosing Management Actions

Connect to Fish



Approach to this Optimal Control Problem

After setting up a PDE with a control in a specifed set and an objective functional, proving existence of an optimal control in an appropriate weak solution space is a first step.

To derive the necessary conditions , we need to differentiate the map

control $\rightarrow$ objective functional

Note that the state contributes to the objective functional, so we also must differentiate the map

control $\rightarrow$ state

The "sensitivity" is the derivative of the control-to-state map. The sensitivity solves a PDE, which is linearized version of the state PDE.

How to find and use the adjoint function

The formal **adjoint** of the operator in the sensitivity PDE is found.

Transversality Condition: final time condition $\lambda = 0$ at $t = T$

nonhomogeneous term

$$\frac{\partial(\text{ integrand of J })}{\partial \text{state}}$$

Differentiate the objective functional $J(control)$ with respect to the control.

Use the adjoint problem and the sensitivity problem to simplify and obtain the explicit **characterization** of an optimal control.

in a river...mixture of flows and depth

Pools are deep with slow water.

Riffles are shallow with fast, turbulent water running over rocks.

Runs are deep with fast water and little or no turbulence

First Example: Introduction to River Model

A conjecture from Lutscher et al. (2010) says that “a population can persist at any location in a homogeneous habitat if and only if it can invade upstream”.

In a pool-riffle river with fluctuating flows, if the time of low discharge is not enough for the population to invade from the riffle to the next upstream pool, then the population is washed back to its foothold in the downstream pool where it remains until the next low discharge time. (Yu Jin, Mark Lewis and collaborators, 2011, 2014)

Thus, the population stalls in the river but cannot spread further upstream, which indicates that the assumption of homogeneity in space or time in the conjecture is essential.

More on introducing river model

- Model adapted from - Jin, Hilker, Steffler, and Lewis (2014) SIAP
    - Seasonal invasion dynamics in a spatially heterogeneous river with fluctuation flows
    - PDE reaction-diffusion model
    - Incorporates both river and population dynamics
    - Use the water discharge flow to control the species
    - Motivated by invasive carp or zebra mussels

GOAL

Use flow control in our model to keep the invasive species downstream and prevent the population from moving upstream

Problem Formulation

STATE PDE: $N(x,t)$ population density in river at location x, time t

$$N_t = -A_t(x,t)\frac{N}{A(x,t)} + \frac{1}{A(x,t)}\left(D(x,t)A(x,t)N_x\right)_x - \frac{Q(t)}{A(x,t)}N_x + rN\left(1-\frac{N}{K}\right)$$

$$\begin{aligned} N(0,t) &= 0 && \text{on } (0,T), x=0, \ \text{(upstream)} \\ N_x(L,t) &= 0 && \text{on } (0,T), x=L, \ \text{(downstream)} \\ N(x,0) &= N_0(x) && \text{on } (0,L), t=0 \end{aligned}$$

in weak solution space $L^2((0,T); H^1_{\{0\}}(0,L))$ with time derivative in the dual space

$A(x,t)$ cross-sectional area of river
Free flow, downstream at $x = L$
$Q(t)$ water discharge rate, CONTROL

Our control set is

$$U = \{Q \in L^{\infty}(0,T) \,|\, m \leq Q(t) \leq M\}$$

with $0 \leq m < M$.

Our objective functional to minimize

$$J(Q) = \int_0^T \int_0^L W(x)N(x,t)dxdt + \int_0^T \epsilon Q^2(t)dt$$

where $\epsilon > 0$ is small.

The weight $W(x)$ is large near $x = 0$ to emphasize keeping the population low upstream.

Optimal Control Problem Formulation and Analysis

1. Differentiate the control-to-state map as a directional derivative, sensitivity PDE
2. Find our adjoint PDE from the sensitivity PDE
3. Characterize the Optimal Control by differentiate the map from control-to-J (goal)
4. Numerical simulation of state and adjoint system with optimal control

Mostly just showing a few results here

The Adjoint PDE and Optimal Control Characterization

$$
\begin{aligned}
& -\lambda_t - \left(DA\left(\frac{\lambda}{A}\right)_x\right)_x - Q^*\left(\frac{\lambda}{A}\right)_x + \frac{A_t\lambda}{A} - r\lambda + 2\frac{r\lambda}{K}N^* = W(x) \\
& \lambda(x,T) = 0 \qquad\qquad \text{on } (0,L), t = T, \\
& \lambda(0,t) = 0 \qquad\qquad \text{on } (0,T), x = 0, \\
& D(L,t)A(L,t)\left(\frac{\lambda(L,t)}{A(L,t)}\right)_x + Q^*(t)\frac{\lambda(L,t)}{A(L,t)} = 0 \text{ on } (0,T), x = L
\end{aligned}
$$

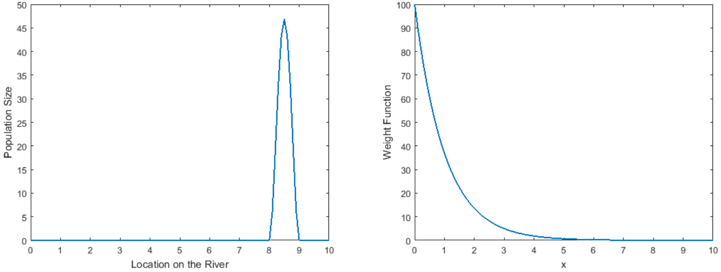
Optimal control characterization

$$
Q^*(t) = \min\left(M, \max\left(\frac{1}{2\epsilon}\int_0^L \frac{\lambda}{A}N_x^*(x,t)dx, m\right)\right)
$$

Forward-Backward Sweep method

- Initial guess for Q
- Solve the state PDE, N, forward in time starting with the initial condition
- Solve the adjoint PDE, λ, backward in time with the final time condition
- Update Q using N and λ in the optimal control characterization
- Check convergence
  - If the control values of the last iteration and this iteration are sufficiently close, we stop
  - If the control values are not close, we repeat

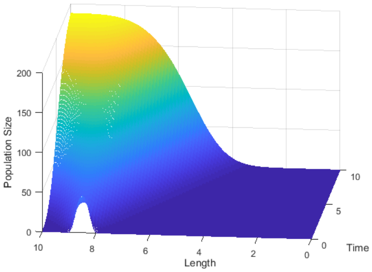
Initial Condition and Weight Function



(a) Initial Condition **(b)** Weight Function

Population starts downstream near $x = 9$ and weight function is high near $x = 0$ upstream.

Cross-sectional Area is Constant, Population, Control



(c) No Control **(d)** Optimal Control

Figure: Population plots for the no control population and the optimal control population. The parameter values are $T = 10$, $L = 10$, $r = 0.6$, $K = 200$, $D = 0.1$, $A = 20$, $\epsilon = 0.05$, and $0 \leq Q(t) \leq 10$.

Cross-sectional Area is Constant, Downstream

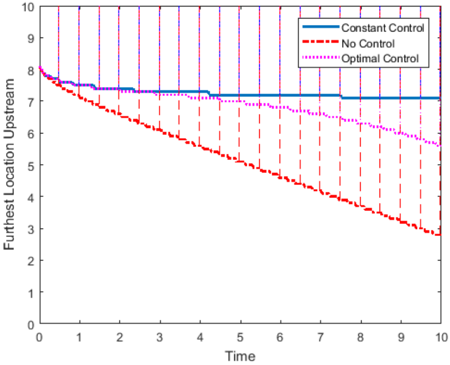
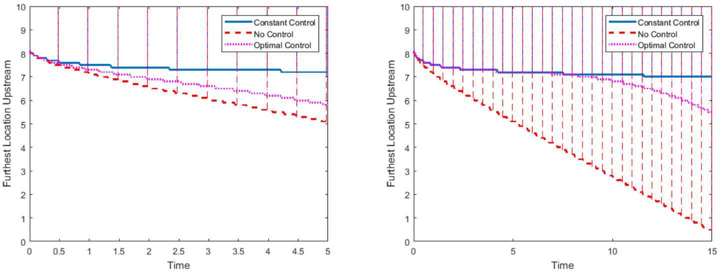


Figure: The upstream location of the constant control population with $Q = 10$ (solid blue line), no control population (red dashed line), and the optimal control population (magenta dotted line).

Detection level greater than 0.5



(a) T=5 **(b)** T=15

Figure: The upstream location of constant control population (solid blue line), no control population (red dashed line), and optimal control population (magenta dotted line).

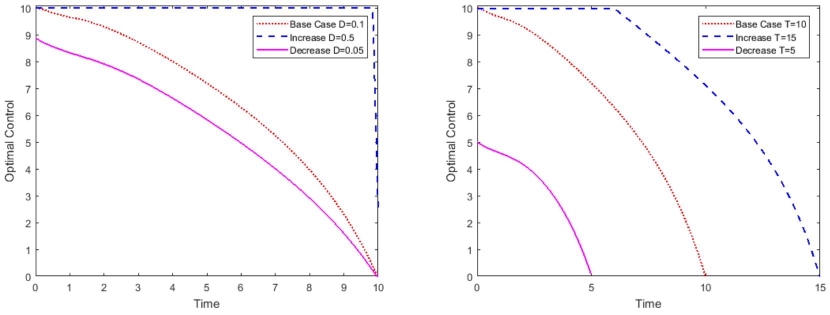
Cross-sectional Area is Constant

Table: The objective functional outputs for the cases tested where we changed parameter values for K and r (given below) with $T = 10$.

	Base Case	$K = 150$	$K = 250$	$r = 0.3$
No Control	239.52	204.28	269.14	55.18
Constant Control	56.63	56.16	56.97	52.44
Optimal Control	40.12	37.96	41.13	20.29
OC Improv. on CC	29%	32%	28%	61%

baseline $K = 200$ and $r = 0.6$

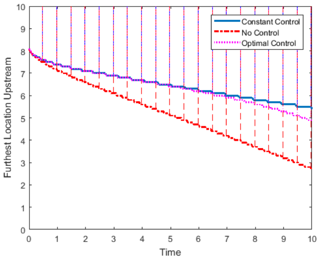
Cross-sectional Area is Constant, vary D and T



(a) Varying D **(b)** Varying T

Figure: The optimal control plots for varying of the parameters D and T. The base case (red dotted line), the increased value (dashed blue line), and the decreased value (solid magenta line).

Cross-sectional Area is Not Constant - $A(x,t) = (0.5x + 25) + (0.2t\,(10 - t))$



(a) $A(x,t)$ **(b)** $A = 20$

Figure: The upstream location of the constant control population (solid), no control population (red dashed), and the optimal control population (magenta) with $T = 10$.

J^* values are 66 and 40 resp.. Cross-sectional area bigger in (a).

Approximation of an Optimal Control

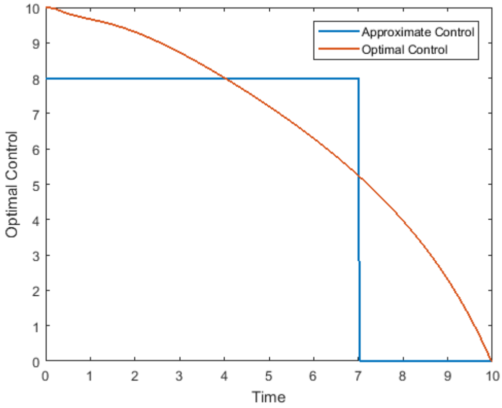


Figure: Comparing the optimal control and the approximate control case when $A = 20$

J(approximate control) is 6% higher than $J(Q^*)$

Conclusions and Future Work

Conclusions

- Successful in illustrating pushing an invasive species downstream compared to the no control case
- Various results with varying parameters, initial conditions, weight function, and the cross-sectional area

Future Work

- Want to use a more realistic $A(x, t)$
- Find data for an invasive species moving upstream
- Restrict flow to certain seasons in a year

publication: Pettit, R.; Lenhart, S. Mathematics 2019, 7, 975-993.

Connecting to Anchovy

Second Example: Background on Black Sea Anchovy

- The Anchovy family contributes to the global fisheries over 10% of landing.
- The European anchovy is the third most widely harvested species of the Anchovy family, and about 40% comes from the Black Sea.
- Fishery Season is open on the Turkish Coast of the Black Sea between September 1 and April 14, but for the commercial fishery of anchovy, the fishing season is about 3 months.
- Anchovy **plays a crucial role in the Black Sea pelagic food web** as a prey and predator of many species. It is also an **important consumer of zooplanktons** in the Black Sea.

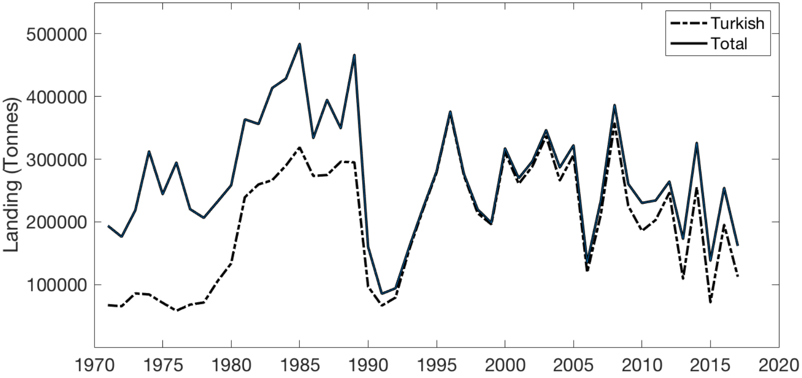


Figure: Landing of the Black Sea anchovy on Turkish coasts (dashed) and in the Black Sea.

Note the collapse of the fishery on north coast and decrease on Turkish coast due to M. leidyi jellyfish invasion about 1990.

Main goal of our study:

- The **main goal** is to investigate food chain-based optimal fishery management strategies for the anchovy fishing on the southern part of the Black Sea.

- **Tools**:
  - We built a **food chain model** with three trophic levels and with seasonal fishery **to track the effects of the fishery** on the Black Sea food web, and see the effect of predator-prey relations on the anchovy fishery, especially effect of the invasive Jellyfish.
  - Use OC tools to find the optimal harvesting strategy that maximizes the discounted net value of the anchovy population with seasonal harvesting.

Flow Diagram of Consumption in the System

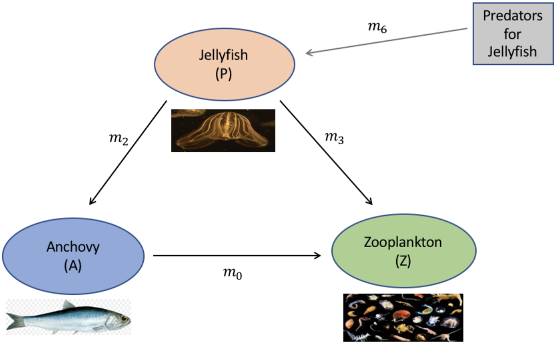


Figure 3: The flow diagram of consumption in our food chain model.

- $A(t)$: Anchovy biomass.
- $P(t)$: Predator biomass of anchovy (jellyfish).
- $Z(t)$: Zooplankton biomass.

Our Food Chain Model with Seasonal Harvesting

$$\frac{dA}{dt} = r_1 A(1 - \frac{A}{K_1}) + m_0 AZ - m_1 PA - hA$$
$$\frac{dP}{dt} = r_2 P(1 - \frac{P}{K_2}) + m_2 PA + m_3 PZ - m_6 P$$
$$\frac{dZ}{dt} = r_3 Z(1 - \frac{Z}{K_3}) - m_4 AZ - m_5 PZ$$

with the initial conditions:

$$A(0) = A_0, \quad P(0) = P_0, \quad Z(0) = Z_0$$

- $h(t)$: Harvest rate (effort), **OUR CONTROL**, $h = 0$ in the offseason.
- m_0, m_1, m_2, m_3, m_4, and m_5 are predation rates.
- m_6 denotes the predation rate on the jellyfish, P, from other predators

$$J(h) = \int_\Omega e^{-\alpha t}(phA - (\mu_1 + \mu_2 h)h)dt$$

- $\Omega = \cup_{i=1}^{T}[a_i, b_i]$ time intervals for seasonal harvesting.
- $[a_i, b_i]$ represents the fishery season, November-January
- $e^{-\alpha t}$ is the discount rate
- phA is the revenue from the yield of the fishery with price p
- $\mu_1 h + \mu_2 h^2$ denotes the cost of the harvest on Ω, and $h = 0$ on $[0, T] \setminus \Omega$

Find an optimal control, h^* in $\mathcal{A}$ such that

$$J(h^*) = \sup_{h \in \mathcal{A}} J(h)$$

$$\mathcal{A} = \{h : [0, T] \longrightarrow [0, M] \mid \text{h=0 on } [0, T] \setminus \Omega \text{ and h Leb. meas.}\}$$

Use Pontryagin's Maximum Principle and Hamiltonian

$$
\begin{aligned}
H = e^{-\alpha t}(hA - \mu_1 h - \mu_2 h^2) &+ \lambda_A \left[r_1 A - \frac{r_1}{K_1} A^2 + m_0 AZ - m_1 PA - hA \right] \\
&+ \lambda_P \left[r_2 P - \frac{r_2}{K_2} P^2 + m_2 PA + m_3 PZ - m_6 P \right] \\
&+ \lambda_Z \left[r_3 Z - \frac{r_3}{K_3} Z^2 - m_4 AZ - m_5 PZ \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_A}{dt} &= -\frac{\partial H}{\partial A} \\
\frac{d\lambda_2}{dt} &= -\frac{\partial H}{\partial P} \\
\frac{d\lambda_Z}{dt} &= -\frac{\partial H}{\partial Z}
\end{aligned}
$$

together with the transversality conditions, $\lambda_1(T) = \lambda_2(T) = \lambda_3(T) = 0$.

$$
h^*(t) = \min \left\{ M, \max \left\{ 0, \frac{A^*(1 - e^{\alpha t}\lambda_A^*) - \mu_1}{2\mu_2} \right\} \right\} \quad \text{on } [0, T]
$$

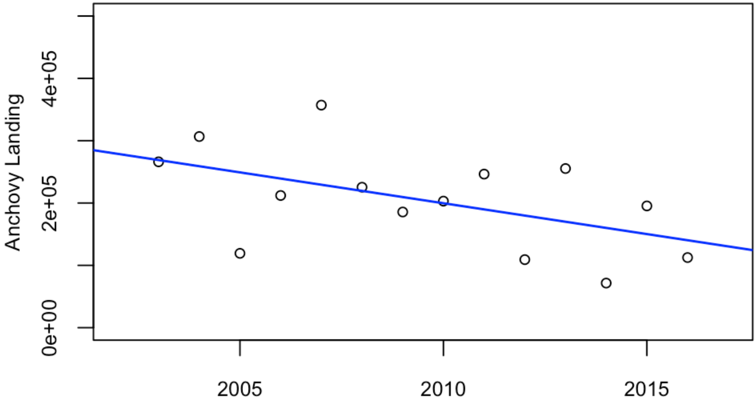


Figure: Annual landing data of the Black Sea anchovy and its linear regression.

The decreasing trend in landing indicates a need for better management.

- Using the annual **landing and fleets data** of the anchovy population on the southern part of Black Sea, 2003-2016, from
  obtained by the Scientific, Technical and Economic Committee for Fisheries (STECF),
  we estimated the parameters with constant h, and then did optimal control problem.
- **Estimated Parameters** are r_1, r_2, r_3 (intrinsic growth rates), m_i for $i = 0, 1, .., 6$ (interaction coefficients) with h constant.
- We did a **stability analysis** with constant harvest to see the threshold, above which the anchovy would decrease to 0. We used to this to set the upper bound on our control at 0.35.

Comparing Populations with Estimated Current Strategy and Optimal Control

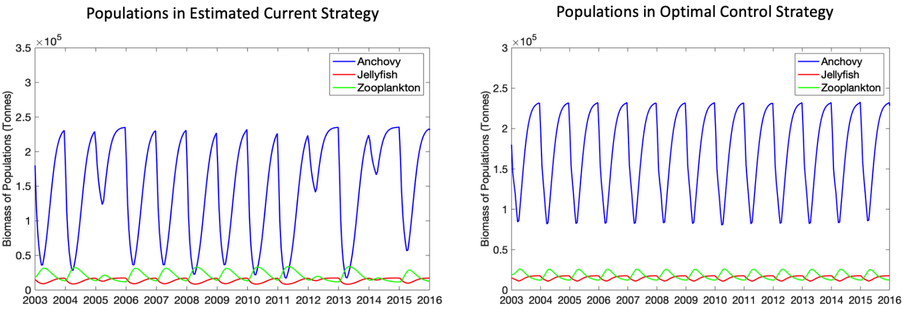


Figure: LHS: Populations with current strategy. See lower anchovy level on left.
RHS: Populations with Optimal control with $h_{max} = 0.35$.

Optimal Control Case

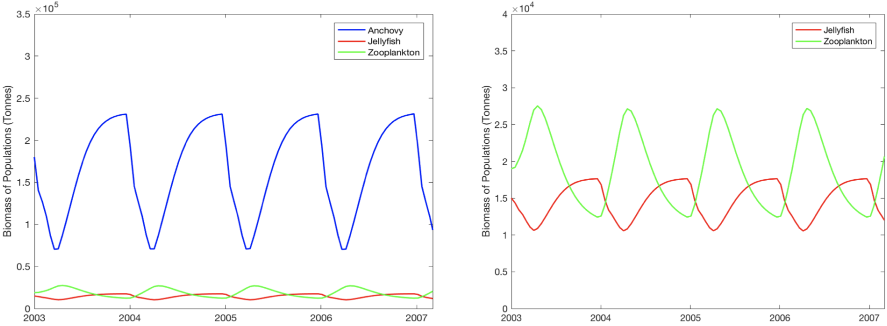


Figure: LHS: Landing of the Black Sea anchovy (blue) with OC case, $h_{max} = 0.35$. RHS: Biomass of Jellyfish (red), and Zooplankton (green) with OC case.

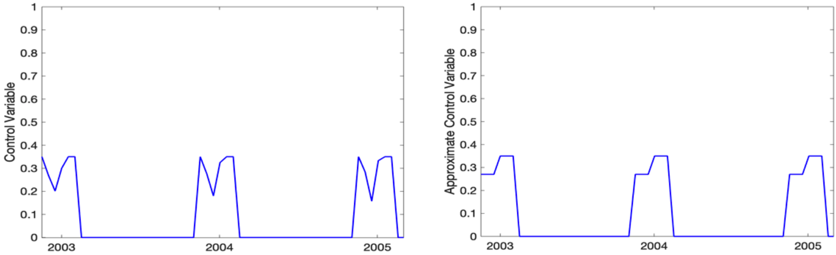


Figure: Left: The harvesting effort in OC case, $h_{max} = 0.35$. Right: The approximate harvesting effort with the first half of the fishing season, $h = 0.27$. At upper bound $h = 0,35$ on second half

J(approximate) is 2% less than $J(h^*)$

Estimation of fishing fleets (Effort $\approx h$)

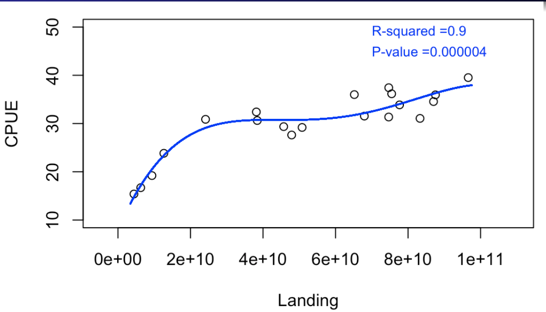


Figure: Non-linear regression between Catch per Unit Effort CPUE and landing of anchovy population depending on data.

The number of fishing fleets (Effort) is estimated as

$$Effort^* = \frac{Landing^*}{CPUE^*},$$

where, $Landing^*$ is our optimal landing, and $CPUE^*$ is our approximate $CPUE$ obtained from non-linear regression model.

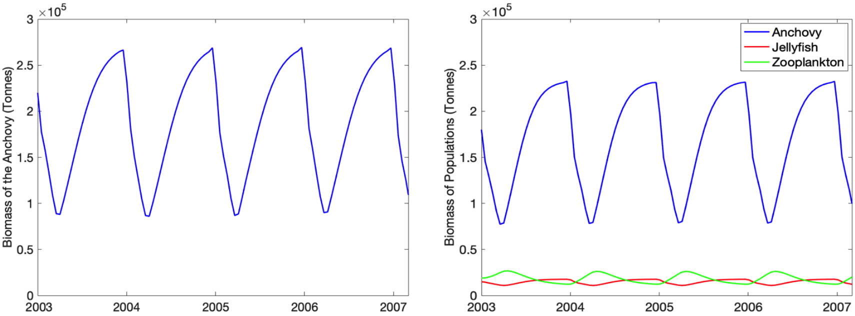
Compare with single DE case with only Anchovy

$$\frac{dA}{dt} = r_1 A(1 - \frac{A}{K_1}) - hA$$

We found the parameters to this model by fitting to the landing data and using the same objective functional.

We found optimal harvest for this model.

Populations with Single Species and Food Chain



Notice higher anchovy levels on the left plot (unrealistic).

Comparison of two models in optimal control case

Table: h_f and h_s denote the optimal harvesting strategy of food chain model and single species model, respectively. In the case of "Food Chain with h_s", we implement the optimal harvest strategy of the single species model in our food chain model.

Comparison of the Models with their Corresponding Optimal Harvesting Strategy			
Models	**Landing (Tonnes)**	**Net Cost (US Dollar)**	**Net Profit (US Dollar)**
Food Chain with h_f	3,111,500	360,710,000	607,890,000
Single Species with h_s	3,573,700	359,600,000	715,430,000
Food Chain with h_s	3,100,800	359,600,000	604,840,000

- We got more profit by using only the anchovy equation than by using our food chain model, but it is not realistic. The single species modeling framework overestimates the landing by 15% and the profit by 18%.

Conclusions for Anchovy part

- Instead of using the current strategy, if the optimal harvest strategy were used in anchovy fishery between 2003-2016, one could get 44% more profit and annually about 17,150 tonnes extra landing.
- Taking into account of the food web for the Black Sea anchovy **gives more reliable management information** than only using the anchovy equation.
- Optimal controls with too much variation may be difficult to implement and an approximation of an optimal control may be chosen and implemented effectively.

Publication: Mahir Demir and Lenhart, Natural Resource Modeling, December 2019 more theoretical

- How do both a dynamic habitat and habitat damage impact stock dynamics?
- What are optimal harvesting strategies that maximize discounted fishery revenue and/or habitat conservation?

Investigate resource management strategies for a dynamic system:

1. *fishery stock* on a spatial domain, and a
2. *habitat resource* on which the stock depends for reproduction.

Fishery-Habitat System Model

We use a system of parabolic, partial differential equations.

- dynamics changing in both space and time.

Model Considerations:

- dynamic stock *and* habitat
- movement and growth
- boundary conditions of spatial domain
- *dynamic* harvesting
- *effect of harvesting on the system*
  - harvesting lowers stock density
  - *habitat damage from fishing decreases the quantity or quality of spatial resources, and thus reducing stock carrying capacity.*

Fishery-Habitat System Model

The **fish stock density**, $u(x,t)$, is modeled by:

$$\begin{aligned} u_t = f(u,k) + (a_1(x,t)u_x)_x + (b_1(x,t)u)_x - h(x,t)u, &\quad Q = \Omega \times (0,T) \\ u(x,t) = 0 &\quad \partial\Omega \times (0,T) \end{aligned}$$

where
$$f(u,k) = r_1 u \left(1 - \frac{u}{M + k(x,t)}\right).$$

The **habitat density**, $k(x,t)$, is modeled by:

$$\begin{aligned} k_t = g(k) + (a_2(x,t)k_x)_x + (b_2(x,t)k)_x - \sigma k h(x,t), &\quad \Omega \times (0,T) \\ k(x,t) = 0 &\quad \partial\Omega \times (0,T) \end{aligned}$$

with appropriate initial conditions.

- carrying capacity in absence of habitat resource ($M << 1$)
- habitat sensitivity (σ)
- $g(k) = r_2 k(1-k)$

Objective Functional

Find the harvesting rate, $h(x,t)$, that maximizes the discounted fishery profit (revenue less cost) while conserving habitat.

Objective Functional

$$J(h) = \int_0^T \int_\Omega e^{-\delta t} \left[P_N h u - (W_0 + W_1 h) h + P_K k \right] dx dt$$

with discount term, $\delta \geq 0$, is maximized over the set of admissible controls:

$$\mathcal{H} = \{ h \in L^\infty(Q) \ : 0 \leq h(x,t) \leq h_{max} \}$$

- P_N, P_K weight coefficients and W_0, W_1 cost coefficients.
- $e^{-\delta t}$ discount factor puts more weight on money made earlier.
- Show only unexploited stock case.

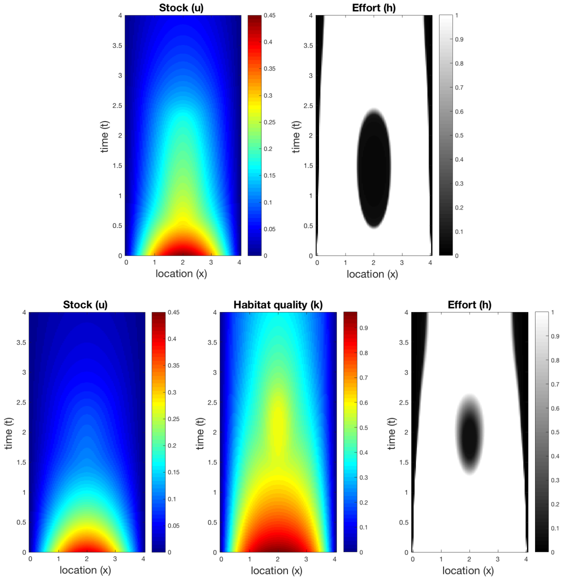


Figure: Stock, habitat and opt. harvesting: constant habitat ($k(x) \equiv 1$, $\sigma = 0$, top); dynamic vulnerable habitat ($\sigma = 0.5$, bottom). $P_K = 0$

After numerical runs, some conclusions

Publication: with Kelly, Neubert, in Theoretical Ecology 2018

1. Optimal spatial effort distribution and stock dynamics can change dramatically when including effects of habitat damage
2. Reserves can be part of optimal solutions, and the reserve area depends greatly on the habitat sensitivity.
3. Optimal reserves are prominent when the intrinsic value of the habitat is high.
4. Model can identify spatial management strategies that are beneficial to both conservation biologists and fishermen.

Some Discussion Points

- Other approaches: Viability modeling work of Luc Doyen and Pedro Gajardo, (state constraints)
- Adaptive management and learning, Paul Fackler, Jim Nichols, Michael Runge.
- NIMBioS Ecosystem Federalism working group
  studying two patches with external 'federal' control and local control,
  recent publication: Sanchirico, Blackwood, Fitzpatrick, Kling, Lenhart, Neubert, Shea, Sims, Springborn: Ecological Applications 2020.

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and Biological Synthesis

Adaptive management Tutorial at NIMBioS, October 26-29, 2020

LECTURERS:
Paul Fackler, Iadine Chayes, Mike Springborn, David Kling
VIRTUAL tutorial and SEE THE RECORDINGS

THANK YOU !