Waves and domain of determinacy for a timelike curve

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joint work with
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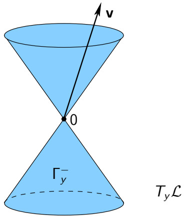
- In $\mathcal{L} = \mathbb{R}^{1+d}$ set $P = \partial_t^2 - \operatorname{div}(A(t,x)\,\mathrm{grad}\,)$ with symbol $p = -\tau^2 + a^{ij}(t,x)\xi_i\xi_j$
- More generally, $g(t,x)$ Lorentzian metric: signature $= -1, \underbrace{+1, \dots, +1}_{d \text{ times}}$ with $g(t,x)(\partial_t, \partial_t) < 0$
  $p(y,\eta) = g^{ij}\eta_i\eta_j$ with $g^* = (g^{ij}) = (g_{ij})^{-1}$, $y = (t,x)$ and $\eta_0 = \tau$, $\eta_k = \xi_k$, $k = 1, \dots, d$.
- Example – Minkowski space: $g = -|dt|^2 + |dx_1|^2 + \cdots |dx_d|^2$.
- time varying Riemannian metric $g(y) = (g_{ij})(y)$ on $\mathbb{R}^d$: $p(y,\eta) = -\tau^2 + g^{ij}(y)\xi_i\xi_j$.

At $y \in \mathcal{L}$ a vector $\mathbf{v}$ is

- timelike if $g(y)(\mathbf{v},\mathbf{v}) < 0$
- null if $g(y)(\mathbf{v},\mathbf{v}) = 0$
- spacelike if $g(y)(\mathbf{v},\mathbf{v}) > 0$

Set $\Gamma_y^- = \{\text{timelike } \mathbf{v} \text{ at } y\}$

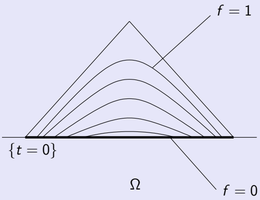


$s \mapsto y(s) \in \mathcal{L}$ is

- a timelike curve if $\frac{d}{ds}y(s) \in \Gamma_y^-$, for s a.e.

A hypersurface $\mathcal{S} = \{f(y) = 0\}$ is spacelike if $p(y, df(y)) < 0$ at all points $y \in \mathcal{S}$.

Theorem



$$\begin{cases} Pu = 0, & \textit{in } \mathcal{L}, \\ u_{|t=0} = 0 & \textit{in a neighborhood of } \Omega, \\ \partial_t u_{|t=0} = 0 & \textit{in a neighborhood of } \Omega, \end{cases}$$

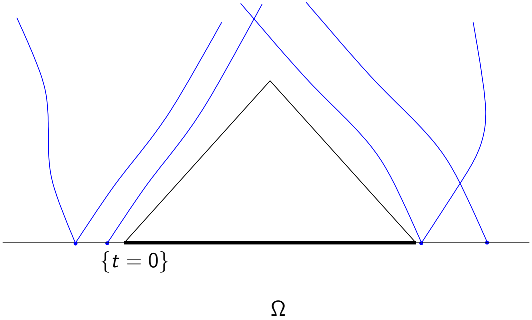
Assume $p(y, df(y)) < 0$ *if* $y \in f^{-1}([0,1])$. *Then* $u = 0$ *in* $f^{-1}([0,1])$.

Definition

A domain of determinacy of $\Omega \subset \{t = 0\}$ is an open set where the wave u vanishes if the Cauchy data are zero in a neighborhood of Ω.

The domain of determinacy of Ω is the largest such open sets.

A second characterization of the domain of determinacy is given as follows.



where blue curves are timelike and initiated in $\{t = 0\} \setminus \Omega$.

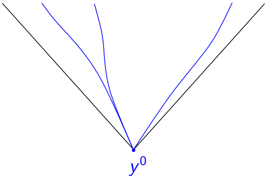
Joly-Métivier-Rauch (05) show that these two formulations are equivalent for general hyperbolic systems

Tool: Let $y^0 = (t^0, x^0)$
first arrival time $x \mapsto \zeta(x) = \zeta^F_{y^0}(x)$ solution to the Eikonal equation

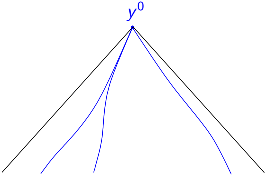
$$\begin{cases} p(\zeta(x), x, 1, -\nabla(\zeta(x)) = 0 \quad \text{a.e.} \\ \zeta(x^0) = t^0. \end{cases}$$

ζ is Lipschitz and smooth near $x = x^0$.

$$\zeta(x) = \inf\{t;\ \exists\ \gamma(s) \text{ timelike with } \gamma(0) = y^0 \ \text{ and } \ \gamma(1) = (t, x)\}$$

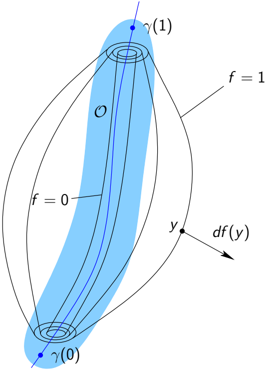


future of y^0
$\mathcal{F}(y^0) = \{(t, x);\ t > \zeta^F_{y^0}(x)\}$



past of y^0
$\mathcal{P}(y^0) = \{(t, x);\ t < \zeta^P_{y^0}(x)\}$

Assume coefficients are smooth with respect to x and analytic with respect to t.



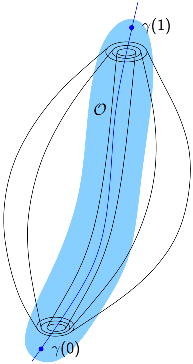
Theorem (Robbiano, 91; Hörmander, 92; Tataru, 95; Hörmander, 97; Robbiano-Zuily, 98; Tataru, 99)

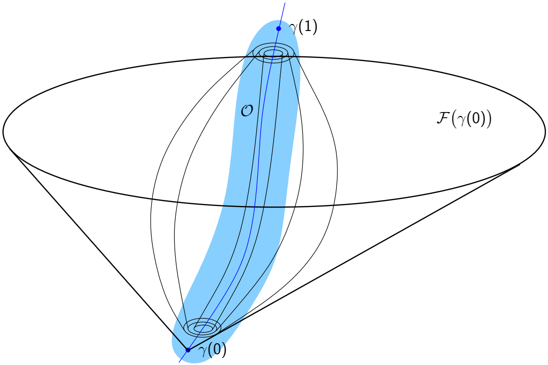
1. $\{f = 0\} \subset \mathcal{O}$ *where* $u = 0$,
2. $df(y)$ *spacelike if* $f^{-1}([0,1])$.

Then $u = 0$ *in* $f^{-1}([0,1])$.

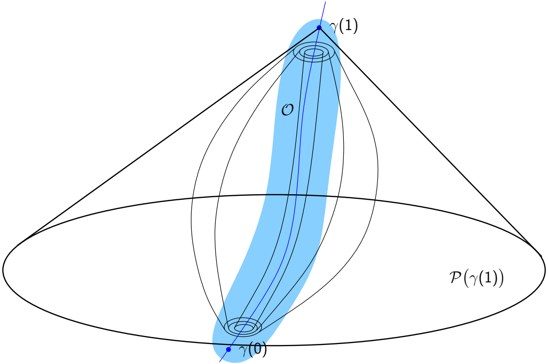
- See Laurent-Léautaud (19) for a quantitative version of this theorem.
- Motivation: approximate controllability, inverse problems (BC method of Belishev and Kurylev)

Solution vanishing in a neighborhood of a timelike curve

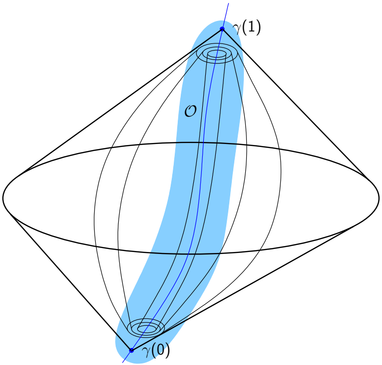




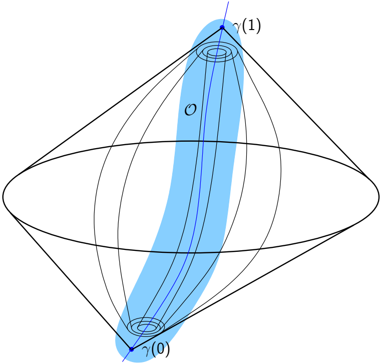
Future of $\gamma(0)$



Past of $\gamma(1)$

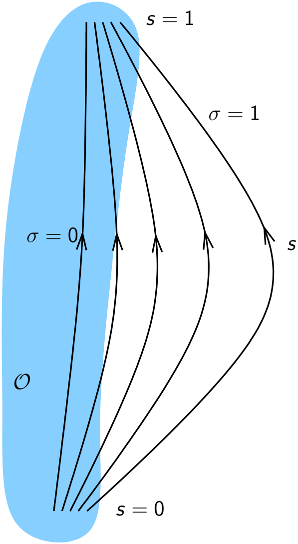


$\mathcal{F}\big(\gamma(0)\big) \cap \mathcal{P}\big(\gamma(1)\big) = \mathbb{D}\big(\gamma(0), \gamma(1)\big)$: double-cone



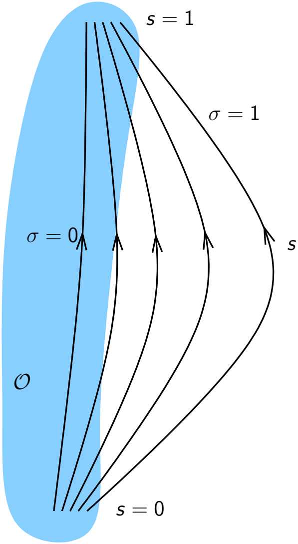
$\mathcal{F}\big(\gamma(0)\big) \cap \mathcal{P}\big(\gamma(1)\big) = \mathbb{D}\big(\gamma(0), \gamma(1)\big)$: double-cone

Is this notion of double-cone the answer to our question ?
It looks so here.



- $\mathbb{X} \in \mathscr{C}^0\big([0,1]; \mathscr{C}^1([0,1]; \mathbb{R}^{1+d})\big)$
- $[0,1] \ni s \mapsto \mathbb{X}(\sigma, s)$ timelike for all $\sigma \in [0,1]$
- $\sigma \in [0,1]$ is the homotopy parameter, and $s \in [0,1]$ is the parameter of each timelike curve.

If $\mathcal{O} \subset \mathbb{R}^{1+d}$ is an open set, $Z_{\mathcal{O}}$ denotes the domain of determinacy, that is, the largest open set where $u = 0$ if it holds in $\mathcal{O}$.



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Theorem (LR-Rauch)

Assume

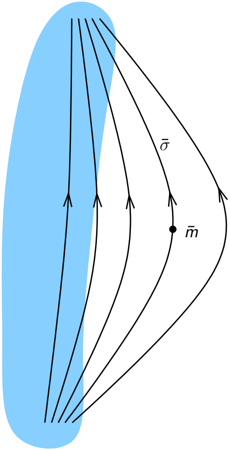
$$\mathbb{X}\big(\{0\} \times [0,1]\big) \cup \mathbb{X}\big([0,1] \times \{0\}\big) \cup \mathbb{X}\big([0,1] \times \{1\}\big) \subset \mathcal{O}.$$

Then $\mathbb{X}([0,1]^2) \subset Z_{\mathcal{O}}$.

If $\mathcal{O} \subset \mathbb{R}^{1+d}$ is an open set, $Z_{\mathcal{O}}$ denotes <u>the</u> domain of determinacy, that is, the largest open set where $u = 0$ if it holds in $\mathcal{O}$.

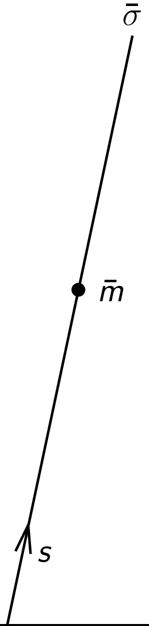
Set $J = \{\sigma \in [0,1];\ \mathbb{X}([0,\sigma] \times [0,1]) \subset Z_{\mathcal{O}}\}$.

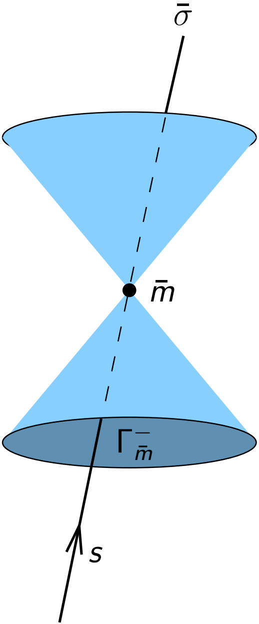
- J is a nonempty interval
- $\mathbb{X}$ continuous implies J open in $[0,1]$
- It remains to prove that J is closed.

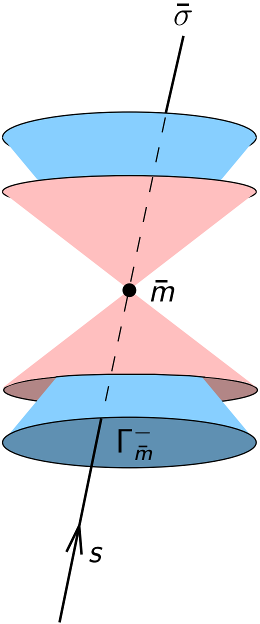


Consider $\bar{\sigma}$ such that $\sigma_n \to \bar{\sigma}$ with $\sigma_n \in J$ and $\bar{s} \in]0,1[$.

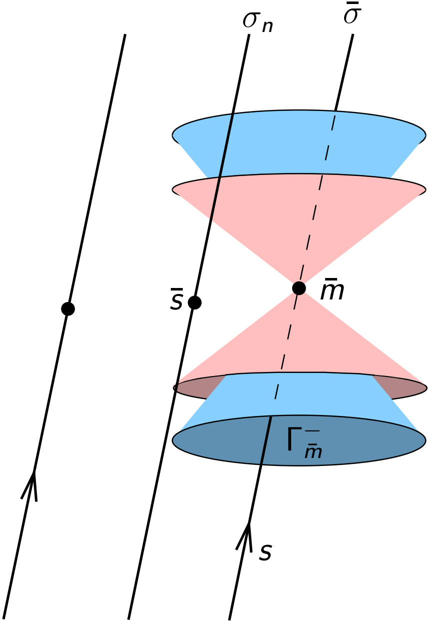
Set $\bar{m} = \mathbb{X}(\bar{\sigma}, \bar{s})$. We prove $\bar{m} \in Z_{\mathcal{O}}$.

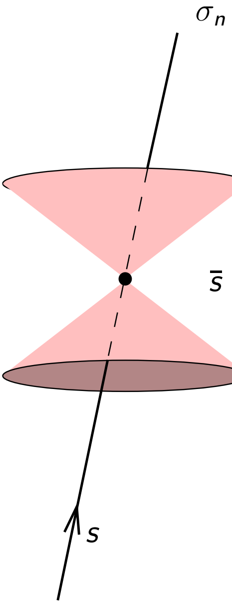

$\bar{\sigma}$
$\bar{m}$
s

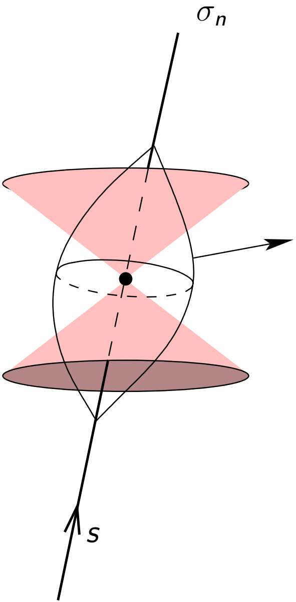

$\bar{\sigma}$
$\bar{m}$
$\Gamma^-_{\bar{m}}$
s

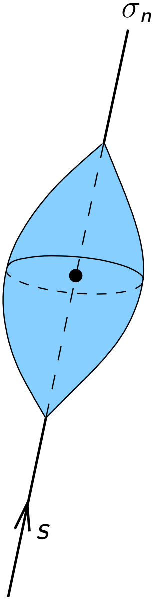

$\bar{\sigma}$
$\bar{m}$
$\Gamma^-_{\bar{m}}$
s

Looking very closely at $\bar{m}$

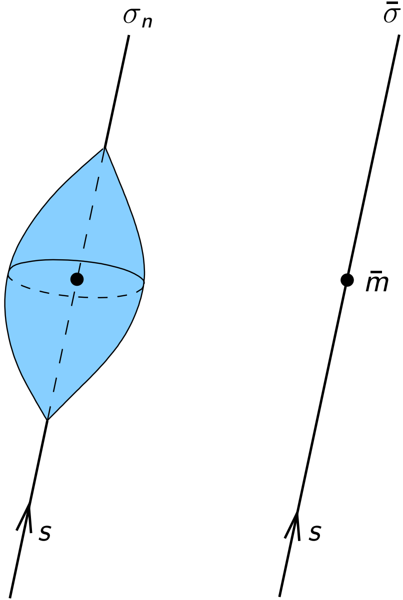


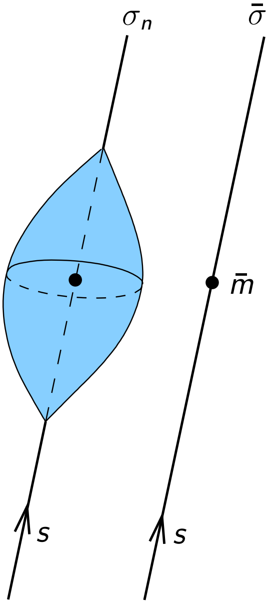

σ_n
$\bar{s}$
s


σ_n
S

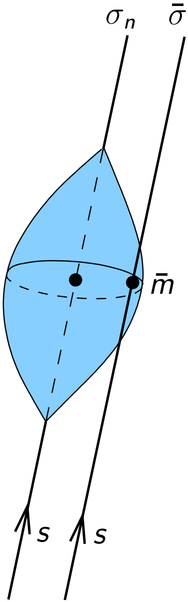

σ_n
S

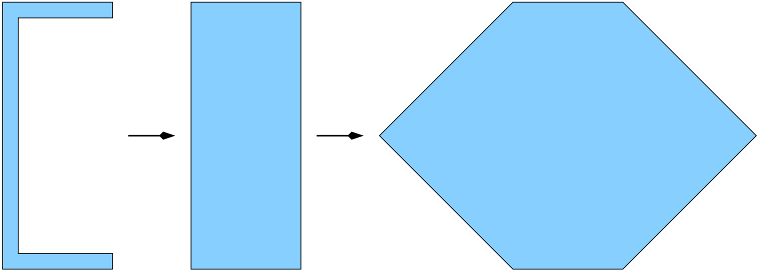
Looking very closely at $\bar{m}$

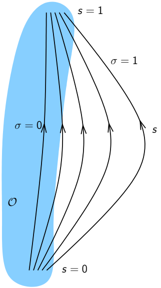


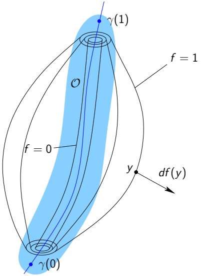

σ_n
$\bar{\sigma}$
$\bar{m}$
s
s

Looking very closely at $\bar{m}$









Proposition

If $\mathbb{X}([0,1]^2)$ is an embedded submanifold then one can construct a hypersurface deformation that reaches the same points.

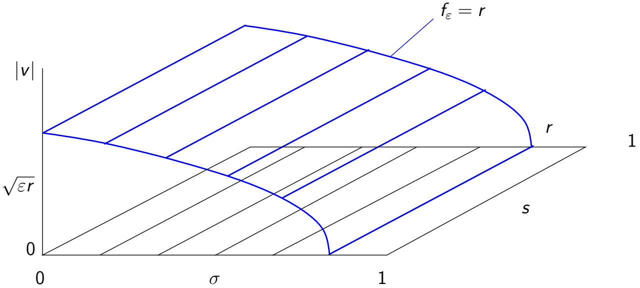
Proposition

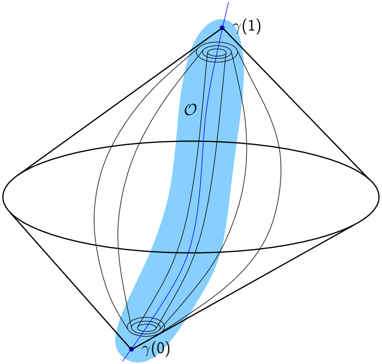
From a hypersurface deformation that reaches a point y^0 one can construct a homotopy of timelike curves that also reaches that point.

- Consider a tubular neighborhood of $\mathcal{M} = \mathbb{X}([0,1]^2)$:

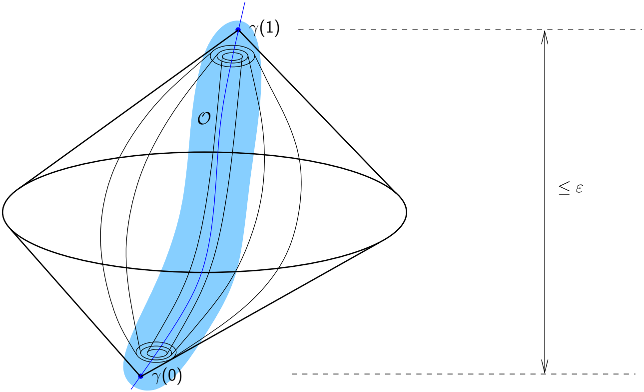
$$\begin{aligned} \theta : N^\delta \mathcal{M} &\to U \\ (y, v) &\mapsto y + v, \end{aligned}$$

  with $N^\delta \mathcal{M} = \{(y, v) \in N\mathcal{M};\ |v| < \delta\}$.
- Set $f_\varepsilon(z) = \sigma + \varepsilon^{-1}|v|^2$ with $z = \mathbb{X}(\sigma, s) + v$
- $\varepsilon > 0$ is chosen small so one remains in U
- the hypersurfaces $\{f_\varepsilon = r\}$ with $r \in [0,1]$ unfolds along with the homotopy parameter $r = \sigma$.


$f_\varepsilon = r$
$|v|$
$\sqrt{\varepsilon r}$
0
0
σ
1
r
1
s

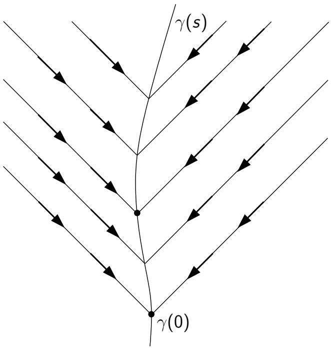


Are they domain of determinacy ? Can we use the homotopy theorem to prove it ?

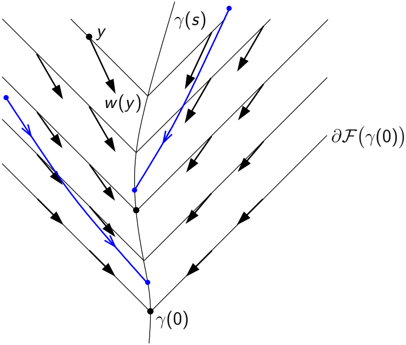


Lemma

Let K^{1+d} be a compact set of $\mathbb{R}^{1+d}$. There exists $\varepsilon > 0$ such that if $y^0, y^1 \in K^{1+d}$ with $y^0 = (t^0, x^0)$ and $y^1 = (t^1, x^1)$ and $t^0 \leq t^1 \leq t^0 + \varepsilon$ then $\overline{\mathbb{D}(y^0, y^1)} \subset \cap_{\mathcal{O}} Z_{\mathcal{O}}$ where the intersection is made for all open neighborhood $\mathcal{O}$ of any timelike curve connecting y^0 and y^1.



- $y \to \zeta^F_{\gamma(s)}(x)$ smooth near the tip
- $f_s(y) = t - \zeta^F_{\gamma(s)}(x)$ characteristic
- $v(y) = \nabla_{\mathcal{L}} f_s(y)$
  $$\in \big(T_y \partial\mathcal{F}(y^0)\big)^{\perp} \cap T_y \partial\mathcal{F}(y^0)$$
  for y such that $f_s(y) = 0$
- $v(x)$ null



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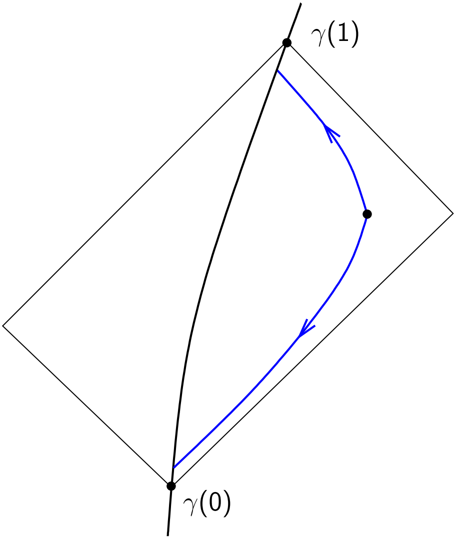
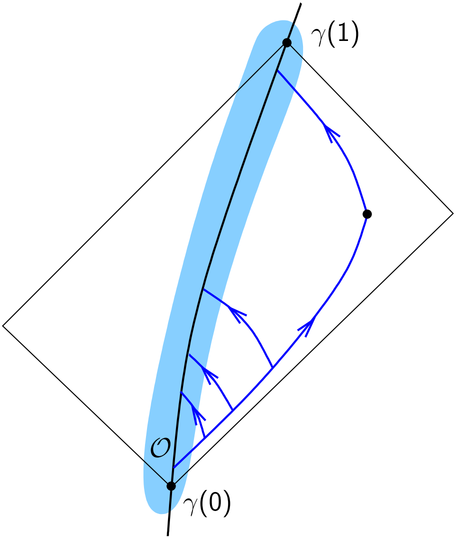
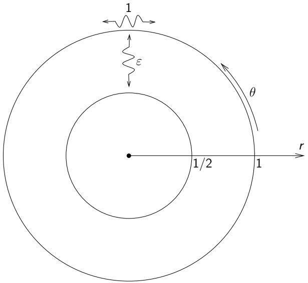
  $$\in \big(T_y\partial\mathcal{F}(y^0)\big)^\perp \cap T_y\partial\mathcal{F}(y^0)$$
  for y such that $f_s(y) = 0$
- $v(x)$ null
- $w(y) = v(y) - \delta s^2 \partial_t$

Small double-cones are domains of determinacy

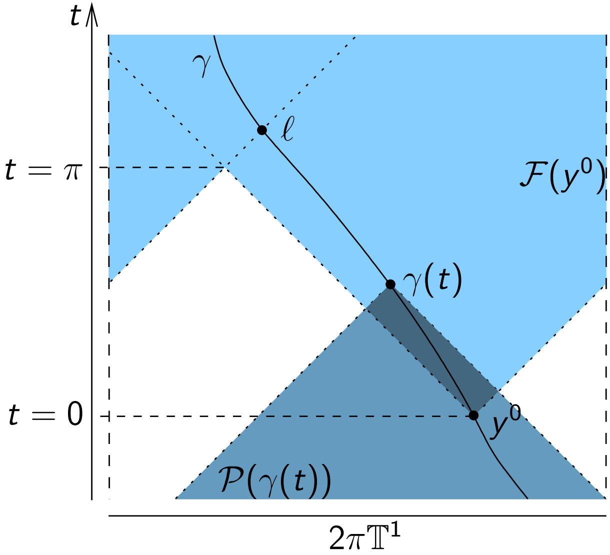


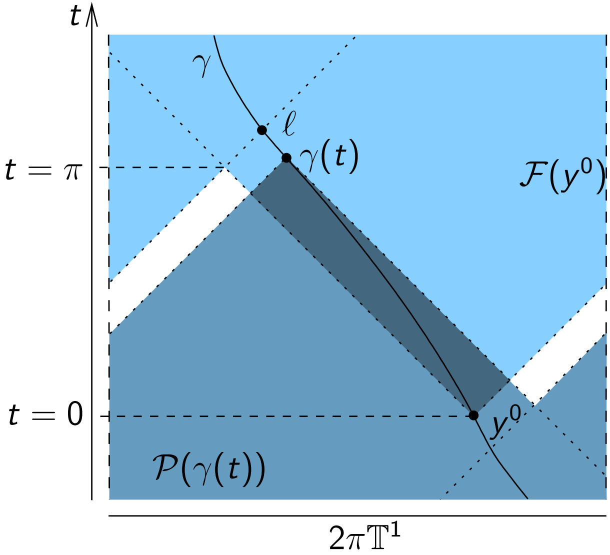

$\gamma(1)$
$\mathcal{O}$
$\gamma(0)$

A 'doldrums' model

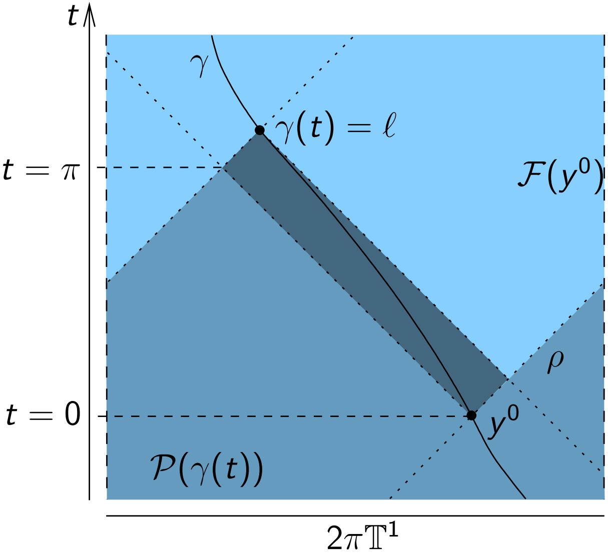


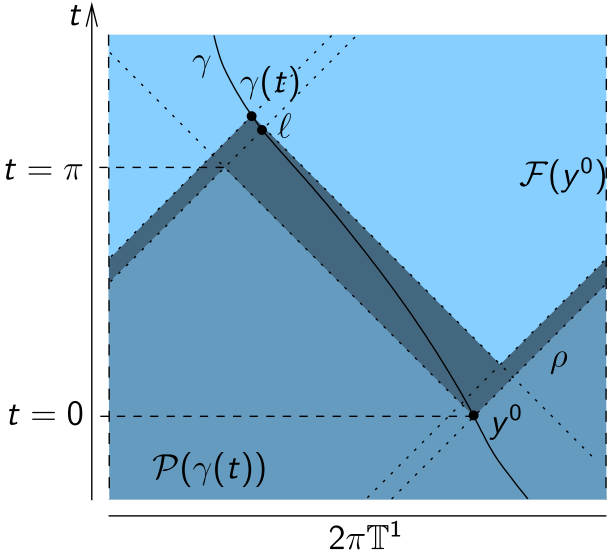
- $\mathcal{L} = \mathbb{R}^{1+2}$
- $g^{\mathcal{L}} = -|dt|^2 + g^R_{ij}(x) dx^i dx^j$
- $\Delta_g = \varepsilon^2 \partial_r^2 + \partial_\theta^2$ in polar coordinates for $r = |x| \geq 1/2$
- $0 < \varepsilon \ll 1$.

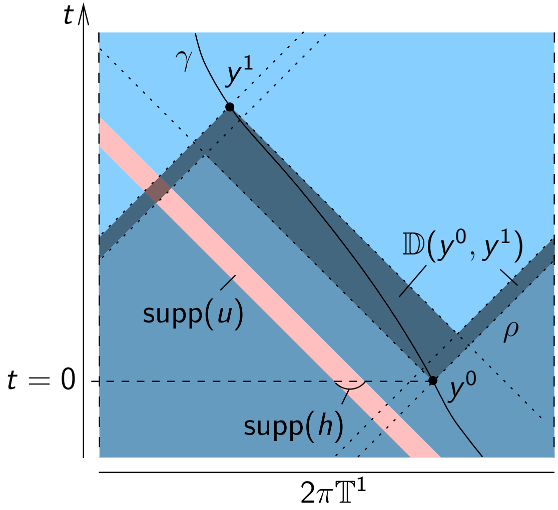

t
$t = \pi$
$t = 0$
γ
ℓ
$\mathcal{F}(y^0)$
$\gamma(t)$
y^0
$\mathcal{P}(\gamma(t))$
$2\pi\mathbb{T}^1$



t
γ
ℓ
$\gamma(t)$
$t = \pi$
$\mathcal{F}(y^0)$
$t = 0$
y^0
$\mathcal{P}(\gamma(t))$
$2\pi\mathbb{T}^1$


t
γ
$\gamma(t) = \ell$
$t = \pi$
$\mathcal{F}(y^0)$
ρ
$t = 0$
y^0
$\mathcal{P}(\gamma(t))$
$2\pi\mathbb{T}^1$


t
$t = \pi$
$t = 0$
γ
$\gamma(t)$
ℓ
$\mathcal{F}(y^0)$
ρ
y^0
$\mathcal{P}(\gamma(t))$
$2\pi\mathbb{T}^1$



Theorem (LR-Rauch)

Double-cones are not domains of determinacy in general