Control in Times of Crisis

Exponential bounds for gradient of solutions to linear elliptic and parabolic equations

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Plan

1. The Landis conjecture
2. Sharp observability estimates for elliptic and parabolic equations
3. Main results: gradient estimates for solutions of elliptic and parabolic equations
4. Proofs of the main results
5. Applications, Extensions

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The Landis conjecture on exponential decay

Conjecture (Landis, 1960's)

$$V \in L^\infty(\mathbb{R}^N), \begin{cases} -\Delta u + V(x)u = 0 \textit{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{cases} \implies u \equiv 0.$$

Example: $u(x) = \exp(-|x|)$ in $\{|x| > 1\}$, extended to a smooth positive function in $\mathbb{R}^N$.

Meshkov's counterexample and optimality

Theorem (Meshkov, 1991)

There exist $V \in L^\infty(\mathbb{R}^2; \mathbb{C})$ *and* $u \in L^\infty(\mathbb{R}^2; \mathbb{C})$ *such that*

$$\begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^2, \\ |u(x)| \leq \exp(-|x|^{4/3}), \\ u \neq 0. \end{cases}$$

Theorem (Meshkov, 1991)

$$V \in L^\infty(\mathbb{R}^N), \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{4/3+\varepsilon}), \ \varepsilon > 0, \end{cases} \Longrightarrow u \equiv 0.$$

Proof: Carleman estimates. □

Landis conjecture for real-valued potentials

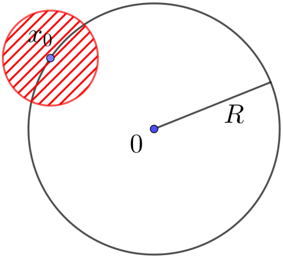
Open questions (Kenig, Bourgain, 2005):

- Is the (**qualitative**) Landis conjecture true for real-valued potentials?

$$V \in L^\infty(\mathbb{R}^N; \mathbb{R}), \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{cases} \implies u \equiv 0.$$

- **Quantitative** Landis conjecture: for $|V| \leq 1$ real-valued and $|u| \leq 1$ such that $-\Delta u + Vu = 0$, $|u(0)| = 1$, do we have: $\forall R >> 1$, $\forall |x_0| = R$,

$$\sup_{|x-x_0|<1} |u(x)| \geq \exp(-R\log^\alpha(R))?$$



Recent results on the Landis conjecture

- Is the (**qualitative**) Landis conjecture true for real-valued potentials?

$$V \in L^\infty(\mathbb{R}^N;\mathbb{R}), \begin{cases} -\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, \\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}), \ \varepsilon > 0, \end{cases} \implies u \equiv 0.$$

  For elliptic operators $\mathcal{L}$:
  - Rossi (2020): $N = 1$, or $u = u(|x|)$, or $\mathcal{L} = \mathcal{L}(|x|)$, or $u \geq 0$, or $\lambda_1(\mathcal{L}) \geq 0$. See also Arapostathis, Biswas, Ganguly (2018).
  - Sirakov, Souplet (2020), MP for $\mathcal{L}$ and unbounded coefficients.
- **Quantitative** Landis conjecture: for $|V| \leq 1$ real-valued and $|u| \leq 1$ such that $-\Delta u + Vu = 0$, $|u(0)| = 1$, do we have: $\forall R >> 1$, $\forall |x_0| = R$,

$$\sup_{|x-x_0|<1} |u(x)| \geq \exp(-R\log^\alpha(R))?$$

  - Kenig, Silvestre, Wang (2015), $V \geq 0$ in the plane $\mathbb{R}^2$. See also Davey, Zhu.
  - Logunov, Malinnikova, Nadirashvili, Nazarov (2020) in the plane $\mathbb{R}^2$.

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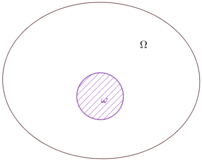
Sharp observability estimates for elliptic equations

$V \in L^\infty(\Omega)$, $\begin{cases} -\Delta\varphi + V(x)\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$



Theorem

$$\|\varphi\|^2_{L^2(\Omega)} \leq \exp\left(C(\Omega,\omega)\|V\|^{2/3}_{L^\infty}\right)\|\varphi\|^2_{L^2(\omega)}.$$

- Sharp for $V \in L^\infty(\Omega;\mathbb{C})$, $\Omega \subset \mathbb{R}^2$. Indeed, Meshkov's counterexample:

$$\|\varphi_R\|_{L^2(\Omega)} \approx (1/R^2),\ \|V_R\|_{L^\infty(\Omega)} = R^2,\ \|\varphi_R\|_{L^2(\omega)} \leq \exp(-R^{4/3}).$$

- **Open question:** $\|V\|^{2/3}_{L^\infty} \leftarrow \|V\|^{1/2}_\infty$ for $V \in L^\infty(\Omega;\mathbb{R})$?
  - $N = 1$,
  - V constant (Donnelly, Fefferman, 1990)
  - 2-D manifolds without boundary (Logunov and al, 2020): $\|V\|^{1/2}_\infty \log(\|V\|^{1/2}_\infty)$.

Sharp observability estimates for parabolic equations

Let $V \in L^\infty(\Omega)$, $\begin{cases} -\partial_t \varphi - \Delta\varphi + V(x)\varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases}$

Theorem (Fernández-Cara, Zuazua, 2000)

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} \leq C \left(\int_0^T \|\varphi(t,\cdot)\|^2_{L^2(\omega)} \, dt \right), \qquad \forall \varphi_T \in L^2(\Omega),$$

$$C = C(\Omega,\omega,T,V) = \exp\left(C(\Omega,\omega)\left(1 + \frac{1}{T} + T\|V\|_{L^\infty} + \|V\|_{L^\infty}^{2/3}\right)\right).$$

- Duyckaerts, Zhang, Zuazua (2008): $\|V\|_\infty^{2/3}$ sharp for $\varphi_T \in L^2(\Omega;\mathbb{C})$, $V \in L^\infty(\Omega;\mathbb{C})$ by using Meshkov's counterexample.
- **Open question:** $\|V\|_{L^\infty}^{2/3} \leftarrow \|V\|_\infty^{1/2}$ for $\varphi_T \in L^2(\Omega;\mathbb{R})$, $V \in L^\infty(\Omega;\mathbb{R})$?
  - $N = 1$ by transmutation's method,
  - $\varphi_T \in L^2(\Omega;\mathbb{R}^+)$, $V \in L^\infty(\Omega;\mathbb{R})$ (Le Balc'h, 2020).

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Proof of the Landis conjecture in 1-D (M. Pierre)

$$-u'' + V(x)u = 0 \text{ in } \mathbb{R},\ |u(x)| \leq \exp(-|x|^{1+\varepsilon}). \qquad \text{Goal: } u \equiv 0.$$

By integrating, we easily get

$$|u'(x)| \leq C \exp(-|x|^{1+\varepsilon}).$$

Let us define ϕ the solution to the Cauchy problem

$$-\phi'' + V\phi = \mathrm{sign}(u),\ \phi(0) = \phi'(0) = 0.$$

$$\text{Gronwall} \Rightarrow |\phi(x)| + |\phi'(x)| \leq C \exp(C|x|).$$

Then, by integrating by parts,

$$\int_{-R}^{R} |u| = \int_{-R}^{R} u \cdot \mathrm{sign}(u) = \int_{-R}^{R} u(-\phi'' + V\phi)$$
$$= -\phi'(R)u(R) + \phi'(-R)u(-R) + \phi(R)u'(R) - \phi(-R)u'(-R)$$
$$\leq \exp(R)\exp(-R^{1+\varepsilon}).$$

$$R \to +\infty \Rightarrow \int_{-\infty}^{+\infty} |u| = 0. \quad \square$$

Question on exponential bounds

Can we extend exponential bounds for solutions to elliptic equations and parabolic equations in the multi-dimensional case?

Exponential bounds for solutions of elliptic equations

Theorem (Le Balc'h, 2020)

Let $W \in L^\infty(\Omega;\mathbb{R}^N)$, $V \in L^\infty(\Omega;\mathbb{R}^+)$ *and* $F \in L^\infty(\Omega;\mathbb{R})$. *Consider*

$$\begin{cases} -\Delta\phi + W\cdot\nabla\phi + V\phi = F & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists an universal positive constant $C > 0$ *depending on* N *such that*

$$\forall x \in \Omega,\ |\nabla\phi(x)| \leq \exp\left(C\left(1 + \|W\|_{L^\infty} + \|V\|_{L^\infty}^{1/2}\right)\mathrm{diam}(\Omega)\right)\|F\|_{L^\infty}.$$

From maximal regularity in L^p, $p >> 1$, and Sobolev embeddings, $\exists C > 0$,

$$\left(|\phi|_{W^{2,p}(\Omega)} \leq C|F|_{L^\infty(\Omega)} \text{ and } |\phi|_{W^{1,\infty}(\Omega)} \leq C|\phi|_{W^{2,p}(\Omega)}\right) \Rightarrow |\phi|_{W^{1,\infty}(\Omega)} \leq C|F|_{L^\infty(\Omega)}.$$

Contribution: Give an estimate of the constant C in function of the parameters.

Exponential bounds for solutions of parabolic equations

Theorem (Le Balc'h, 2020)

Let $W \in L^\infty(\Omega;\mathbb{R}^N)$, $V \in L^\infty(\Omega;\mathbb{R})$, $F \in L^\infty(Q_T;\mathbb{R})$ *and* $\phi_0 \in W_0^{1,\infty}(\Omega)$. *Consider*

$$\begin{cases} \partial_t \phi - \Delta\phi + W\cdot\nabla\phi + V\phi = F & \text{in } (0,T)\times\Omega, \\ \phi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \phi(0,\cdot) = \phi_0 & \text{in } \Omega. \end{cases}$$

Then there exists an universal positive constant $C > 0$ *depending on* N *such that*

$$\|\nabla\phi\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \exp\left(C\left(T\|V\|_\infty + \left(1 + \|W\|_\infty + \|V\|_\infty^{1/2}\right)\mathrm{diam}(\Omega)\right)\right)\left(\|\nabla\phi_0\|_\infty + \|F\|_\infty\right)$$

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Proof for $V = 0$, $\Omega = B_R(0)$

$$\begin{cases} -\Delta\phi + W \cdot \nabla\phi = F & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

$$\text{Goal: } \forall x \in \Omega,\ |\nabla\phi(x)| \leq e^{C(|W|_\infty+1)R}|F|_\infty.$$

Andrews, Clutterbuck's argument:

$$\forall (x,y) \in \overline{\Omega} \times \overline{\Omega},\ Z(x,y) := \phi(y) - \phi(x) - 2\theta\left(\frac{|y-x|}{2}\right),$$

where θ is the solution to

$$\begin{cases} -\theta'' = (|W|_\infty + 1)\theta' + 2|F|_\infty & \text{in } (0,+\infty), \\ \theta(0) = 0,\ \theta'(0) = \lambda := e^{C(|W|_\infty+1)R}|F|_\infty. \end{cases} \quad (\Rightarrow \theta' > 0 \text{ in } (0,R)).$$

If we prove that

$$\forall (x,y) \in \overline{\Omega} \times \overline{\Omega},\ Z(x,y) \leq 0,$$

then

$$\forall x \in \Omega,\ |\nabla\phi(x)| \leq \theta'(0) \leq e^{C(|W|_\infty+1)R}|F|_\infty. \quad \square$$

$$\begin{cases} -\Delta\phi + W\cdot\nabla\phi = F & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad \begin{cases} -\theta'' = (|W|_\infty + 1)\theta' + 2|F|_\infty & \text{in } (0,+\infty), \\ \theta(0) = 0,\ \theta'(0) = \lambda. \end{cases}$$

$$Z(x,y) := \phi(y) - \phi(x) - 2\theta\left(\frac{|y-x|}{2}\right). \qquad \text{Goal: } Z(x,y) \leq 0.$$

$$Z(x_0,y_0) = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} Z(x,y).$$

If $x_0 = y_0$ then $Z(x,y) \leq Z(x_0,y_0) = 0$.
If $x_0 \neq y_0$ and $x_0 \in \partial\Omega$ or $y_0 \in \partial\Omega$ then ... Contradiction.
Assume that $x_0 \neq y_0$, and $x_0, y_0 \in \Omega$. Choose an orthonormal basis $(e_i)_{1\leq i\leq N}$ with $e_N = \frac{y_0-x_0}{|y_0-x_0|}$ and write optimality conditions

$$\nabla Z(x_0,y_0) = 0 \Leftrightarrow (\nabla\phi(y_0) = \theta' e_N,\ \nabla\phi(x_0) = \theta' e_N),$$

$$\frac{d^2}{ds^2} Z(x_0 + se_N, y_0 - se_N)_{|s=0} \leq 0 \Leftrightarrow \partial_N\partial_N\phi(y_0) - \partial_N\partial_N\phi(x_0) - 2\theta'' \leq 0,$$

$$\frac{d^2}{ds^2} Z(x_0 + se_i, y_0 + se_i)_{|s=0} \leq 0 \Leftrightarrow \partial_i\partial_i\phi(y_0) - \partial_i\partial_i\phi(x_0) \leq 0,\ (i = 1,\dots,n-1).$$

We sum: $0 \geq \Delta\phi(y_0) - \Delta\phi(x_0) - 2\theta''$
$\geq -2|W|_\infty\theta' - 2|F|_\infty + 2((|W|_\infty + 1)\theta' + 2|F|_\infty) > 0$. Contradiction.

Reduction of the case $V \geq 0$ to the case $V = 0$

Lemma

Let $W \in L^\infty(B_R(0); \mathbb{R}^N)$ *and* $V \in L^\infty(B_R(0); \mathbb{R}^+)$. *Then there exists* ψ *satisfying*

$$-\Delta\psi + W \cdot \nabla\psi + V\psi = 0 \text{ in } B_R(0),$$
$$\exp(-CR(|W|_\infty + |V|_\infty^{1/2})) \leq \psi \leq \exp(CR(|W|_\infty + |V|_\infty^{1/2})) \text{ in } B_R(0),$$
$$\|\nabla \log(\psi)\|_{L^\infty(B_R(0))} \leq C(1 + |W|_\infty + |V|_\infty^{1/2}).$$

$$-\Delta\phi + W \cdot \nabla\phi + V\phi = F,$$

$$\widehat{\phi} := \frac{\phi}{\psi} \Rightarrow \begin{cases} -\Delta\widehat{\phi} + \widehat{W} \cdot \nabla\widehat{\phi} = \widehat{F} & \text{in } \Omega, \\ \widehat{\phi} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{with } \widehat{W} := W - 2\nabla\log(\psi),\ \widehat{F} := \frac{F}{\psi}.$$

$$\forall x \in \Omega,\ |\nabla\widehat{\phi}(x)| \leq \exp\left(C\left(1 + \left\|\widehat{W}\right\|_{L^\infty}\right)R\right)\left\|\widehat{F}\right\|_{L^\infty} \qquad (V = 0)$$
$$|\nabla\phi(x)| \leq \exp\left(C\left(1 + \|W\|_{L^\infty} + \|V\|_{L^\infty}^{1/2}\right)R\right)\|F\|_{L^\infty}. \quad \square$$

Existence of a positive multiplier for $\Omega = B_R(0)$

Find $\psi > 0$ such that

$$-\Delta\psi + W \cdot \nabla\psi + V\psi = 0 \text{ in } B_{2R}(0). \qquad (*)$$

First, find ψ_1 subsolution to (*) and ψ_2 supersolution to (*), with

$$\psi_1(x) := \exp(\|W\|_\infty + \|V\|_\infty^{1/2})x_1) \leq \exp(2R(\|W\|_\infty + \|V\|_\infty^{1/2})) =: \psi_2(x),$$

So, $\exists\psi$ solution to (*) and

$$\psi_1(x) \leq \psi(x) \leq \psi_2(x),$$

$$\Rightarrow \exp(-CR(|W|_\infty + |V|_\infty^{1/2})) \leq \psi \leq \exp(CR(|W|_\infty + |V|_\infty^{1/2})).$$

Interior regularity estimates and Harnack's inequalities give

$$\|\nabla \log(\psi)\|_{L^\infty(B_R(0))} \leq C(1 + |W|_\infty + |V|_\infty^{1/2}). \quad \square$$

Plan

Application of gradient estimates

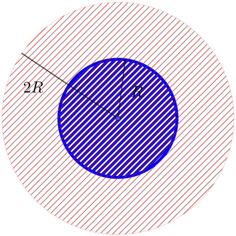
Proposition (Le Balc'h, 2020)

Let $W \in L^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $V \in L^\infty(\mathbb{R}^N;\mathbb{R}^+)$ *and* u *be such that*

$$\mathcal{L}u := -\Delta u - \nabla \cdot (Wu) + Vu = 0 \quad \text{in } \mathbb{R}^N.$$

Then there exists a universal positive constant $C > 0$ *such that*

$$\forall R \geq 1, \int_{|x|<R} |u(x)| dx \leq \exp\left(C\left(1 + \|W\|_{L^\infty} + \|V\|_{L^\infty}^{1/2}\right) R\right) \int_{R<|x|<2R} |u(x)| dx.$$



Proof:

$$\mathcal{L}u := -\Delta u - \nabla\cdot(Wu) + Vu = 0 \quad \text{in } \mathbb{R}^N.$$

Take

$$\begin{cases} \mathcal{L}^*\phi_R = -\Delta\phi_R + W\cdot\nabla\phi_R + V\phi_R = \mathrm{sign}(u) & \text{in } B_{2R}(0),\\ \phi = 0 & \text{on } |x| = 2R. \end{cases}$$

We have

$$\forall |x| \leq 2R,\ |\nabla\phi_R(x)| \leq \exp\left(C\left(1 + \|W\|_{L^\infty} + \|V\|_{L^\infty}^{1/2}\right)R\right).$$

Let χ be such that $\chi \equiv 1$ in $B_R(0)$ and $\chi \equiv 0$ for $|x| > (3/2)R$. We have

$$\int_{|x|<R} \chi|u(x)|dx \leq C(\|W\|_\infty + \|V\|_\infty)\,\|\phi_R\|_{W_0^{1,\infty}(B_{2R}(0))} \int_{R<|x|<2R} |u|$$

$$\int_{|x|<R} |u(x)|dx \leq \exp\left(C\left(1 + \|W\|_{L^\infty} + \|V\|_{L^\infty}^{1/2}\right)R\right) \int_{R<|x|<2R} |u(x)|dx. \quad \square$$

Another bound

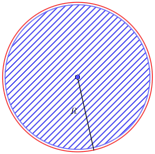
Proposition (Le Balc'h, 2020)

Let $W \in L^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $V \in L^\infty(\mathbb{R}^N;\mathbb{R}^+)$ *and* u *be such that*

$$\mathcal{L}u := -\Delta u - \nabla \cdot (Wu) + Vu = 0 \quad \text{in } \mathbb{R}^N.$$

Then there exists a universal positive constant $C > 0$ *such that*

$$\forall R \geq 1,\ \int_{|x|<R} |u(x)| dx \leq \exp\left(C\Big(1 + \|W\|_{L^\infty}\Big) R\right) \int_{|\sigma|=R} |u(\sigma)| d\sigma.$$



Conclusion and perspectives

What we have seen?

- Landis conjecture for $V \in L^\infty(\mathbb{R}^N;\mathbb{R})$,
  $\Big(-\Delta u + V(x)u = 0 \text{ in } \mathbb{R}^N, |u(x)| \leq \exp(-|x|^{1+\varepsilon})\Big) \Longrightarrow u \equiv 0.$
- Sharp observability estimates: $\|\varphi\|^2_{L^2(\Omega)} \leq \exp\left(C(\Omega,\omega)\|V\|^{1/2}_{L^\infty}\right)\|\varphi\|^2_{L^2(\omega)}$.
- Proof of Landis conjecture in 1-D, based on duality and exponential bounds,
- $|\nabla\phi(x)| \leq \exp\left(C\left(1+\|W\|_{L^\infty}+\|V\|^{1/2}_{L^\infty}\right)\mathrm{diam}(\Omega)\right)\|F\|_{L^\infty}$,
- $\int_{|x|<R}|u(x)|dx \leq \exp\left(C\left(1+\|W\|_{L^\infty}+\|V\|^{1/2}_{L^\infty}\right)R\right)\int_{R<|x|<2R}|u(x)|dx$.

Perspectives for gradient estimates:

- Other boundary conditions,
- $W(t,x)$, $V(t,x)$ for parabolic equations,
- optimality of the constants in function of V,
- more general $V \in L^\infty(\mathbb{R}^N;\mathbb{R})$ for elliptic equations?