Control of a third order in time dynamics governing nonlinear acoustic waves - view from the boundary.

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In the Clouds
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THANKS:

- National Science Foundation -DMS,

Overview

- Motivation:Lithotripsy, High Frequency Focused Ultrasound.
- PDE models:Nonlinear Acoustics: Westervelt- Kuznetsov and Moore-Gibson-Thompson equations.

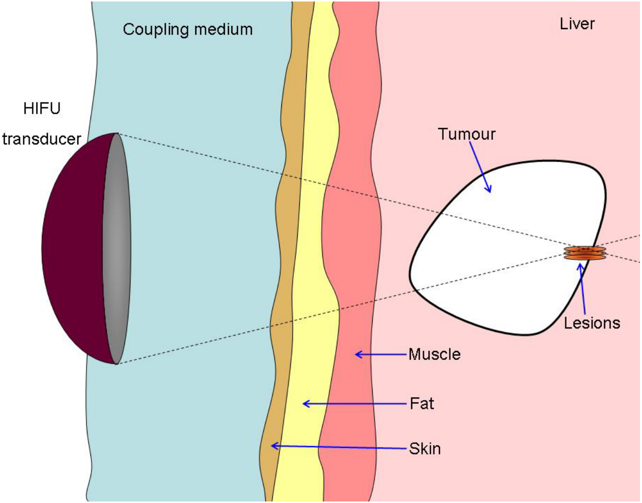
  - Westervelt/ Kuznetsov-2-nd order in time
    (infinite speed of propagation) -parabolic type
  - Moore-Gibson-Thompson equation -3-rd order in time
    (finite speed of propagation)-hyperbolic type

- Global well-posedness of these **quasilinear** PDE models
- Asymptotic analysis when **the relaxation** τ goes to zero.
- **Hidden regularity** from the boundary.
- Stabilization: **frictional, memory and boundary damping.**
- **Boundary control [singular]** [weak solutions] and feedback control.

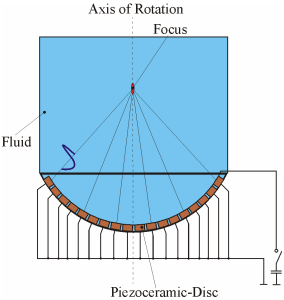
Challenges

- **Nonlinear** wave propagation
- **Singular perturbation** : Linear generator $\mathcal{A}_\tau$ becomes singular - from hyperbolic to parabolic.
- Weak solutions with **rough boundary data**.
- **Boundary Feedback Control Problems.** Singular -Nonstandard **Riccati Equations with Unbounded** coefficients.

1. PART I - review of MGT dynamics
2. PART II- MGT from the boundary
3. PART III - Feedback synthesis of optimal boundary control and Singular ARE Equations.


Coupling medium
Liver
HIFU
transducer
Tumour
Lesions
Muscle
Fat
Skin

Lithotripsy: Principle



⇝ shape optimization of the piezo mosaic/
optimal control of the excitation signal

Modeling

$$\begin{aligned}\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0\\ \rho(\mathbf{v_t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) &= \nabla \cdot \mathbf{T}\\ \rho\theta(\eta_t + (\mathbf{v} \cdot \nabla)\eta) &= -\nabla \cdot \mathbf{q} + \mathbf{T} : \mathbf{D} \end{aligned} \tag{1}$$

η -entropy, **q**-heat flux, **D** -deformation tensor, **T**-Cauchy Poisson stress tensor. $p_\sim$ ... pressure fluctuation

v... acoustic particle velocity

ψ ...acoustic velocity potential

ρ ... mass density

c ... speed of sound

b ... diffusivity of sound

B/A ... parameter of nonlinearity

$\mathbf{v} = -\nabla\psi\,, \qquad \rho D_t \mathbf{v} = -\nabla p_\sim$

Lesser&Seebass 1968, Kuznetsov 1971

Fourrier's Law

$$\mathbf{q} = -\mathbf{K}\nabla\theta$$

Cattaneo Law

$$\tau\mathbf{q_t} + \mathbf{q} = -\mathbf{K}\nabla\theta$$

$\tau > 0$ small relaxation time parameter

Equations of Nonlinear Acoustics

Westervelt -Kuznetsov equation

$$D_t^2 p_\sim - c^2\Delta p_\sim - bD_t\Delta p_\sim = \frac{1}{\rho c^2} D_t^2 \left((1 + \frac{B}{2A}) p_\sim^2 + |\rho c \mathbf{v}|^2 \right)$$

MGT-Moore -Gibson - Thompson equation

$$\tau D_t^3 p + D_t^2 p_\sim - c^2\Delta p_\sim - bD_t\Delta p_\sim = \frac{1}{\rho c^2} D_t^2 \left((1 + \frac{B}{2A}) p_\sim^2 + |\rho c \mathbf{v}|^2 \right)$$

Modeling: Jordan Pedro, Ivan Christov, Christo Christov, Brian Straughan.
Personal thanks also to Mauro Fabrizio.

Westervalt Equation - Fourier's Law ,"infinite" speed of propagation

Westervelt equation with Dirichlet boundary conditions

$$D_t^2 u - c^2 \Delta u - bD_t \Delta u = kD_t^2(u^2) \textbf{ in } (0, T) \times \Omega$$

Rewrite as a **degenerate-quasilinear**

$$(1 - 2ku)D_t^2 u - c^2 \Delta u - bD_t \Delta u = 2k(D_t u)^2$$

$$\alpha(t, x)D_t^2 u - c^2 \Delta u - bD_t \Delta u = f(D_t u)$$

- Degenerate : $\alpha(t, x) = (1 - 2ku)$ can vanish
- Nonlinear term $f(D_t u) = 2k[(D_t u)^2 + uD_t^2 u]$

Global Well-posedness

Let $\Omega \subset R^3$-smooth as needed.

$$E_0(t) \equiv \int_\Omega [|u_t(t)|^2 + |\nabla u(t)|^2] dx$$

$$E_1(t) \equiv \int_\Omega [|u_{tt}(t)|^2 + |\nabla u_t(t)|^2 + |\Delta u(t)|^2] dx$$

Theorem (([Kaltenbacher&Lasiecka, 2009], MathNachrichten))

Let $b > 0$ and $E_0(0) + E_1(0)$ be sufficiently small.
Then there exists a solution u on $[0, T]$, which is unique and satisfies

$$E_0(t) + E_1(t) \leq M, t \in (0, T)$$

- M-T-G eq.- Cattaneo Law, "finite" speed of propagation

Let Ω be a bounded domain of $\mathbb{R}^n$ with regular boundary Γ

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = 2k \frac{d^2}{dt^2} u^2 + c^2 \frac{d}{dt} |\nabla u|^2 \tag{2}$$

$$u = 0 \ on \ \Gamma = \partial\Omega \tag{3}$$

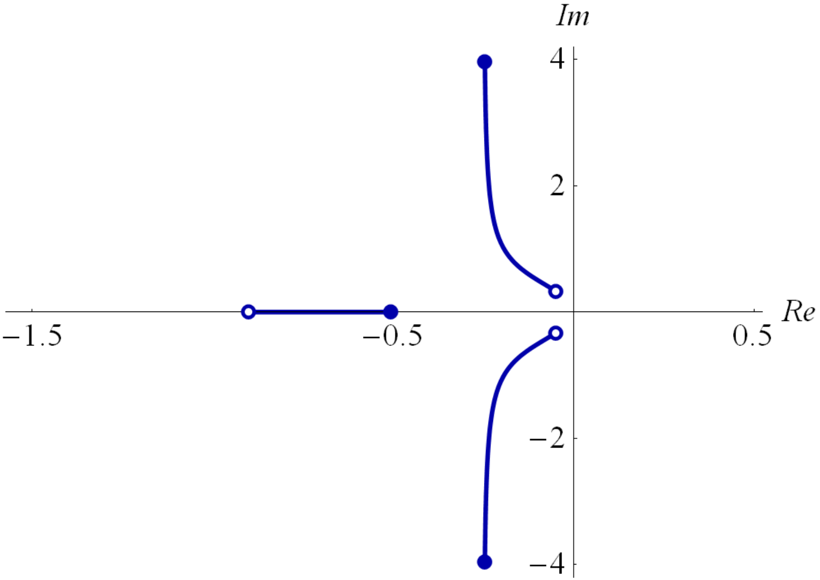
where in a physical context of the acoustic waves

- the variable u denotes a scalar acoustic velocity potential $\vec{v} = -\nabla u$ with $\vec{v}$ denoting the acoustic particle velocity.
- c^2 denotes the speed of sound ,
- τ denotes thermal relaxation resulting from **replacing Fourier's law by the Maxwell Cattaneo law**.
- The coefficient $b \equiv \delta + \tau c^2$ where δ is the diffusivity of the sound.
- The coefficient $\alpha > 0$ describes natural damping effects associated with an acoustic environment.

The presence of the **third time derivative is typical in Extended Irreversible Thermodynamics (EIT)** *a theory originally proposed to remove the unpleasant property of propagation of heat and velocity signals with an infinite velocity when Fourier-Navier-Stokes equations are used* **.** The guiding idea behind is that physical quantities such as thermodynamic fluxes typically given by constitutive relations, in EIT theory are governed by evolution equations with a suitable relaxation time τ.

- $\tau = 0$. This is **Parabolic like** Problem.
  Kuznetsov eq (Westervalt without the blue term) .
- $\tau > 0$. This is **Hyperbolic like** Problem.
  Moore-Thompson-Gibson equation.

:: **Case $\tau > 0$ first introduced by "Professor Stokes" in 1851** .



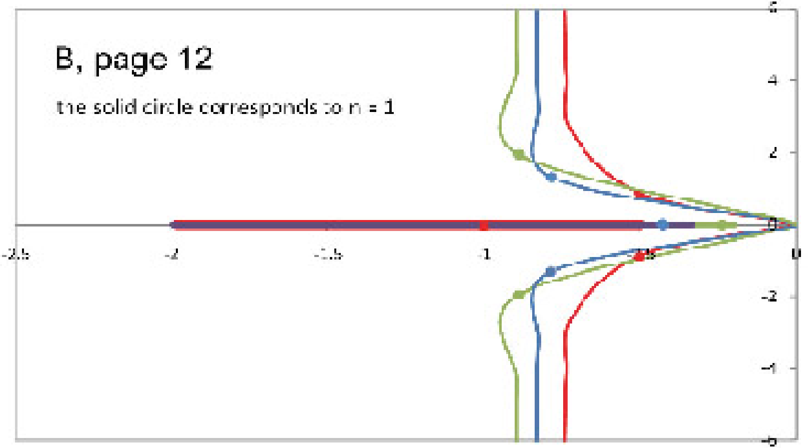
Im
4
2
−2
−4
Re
−1.5
−0.5
0.5

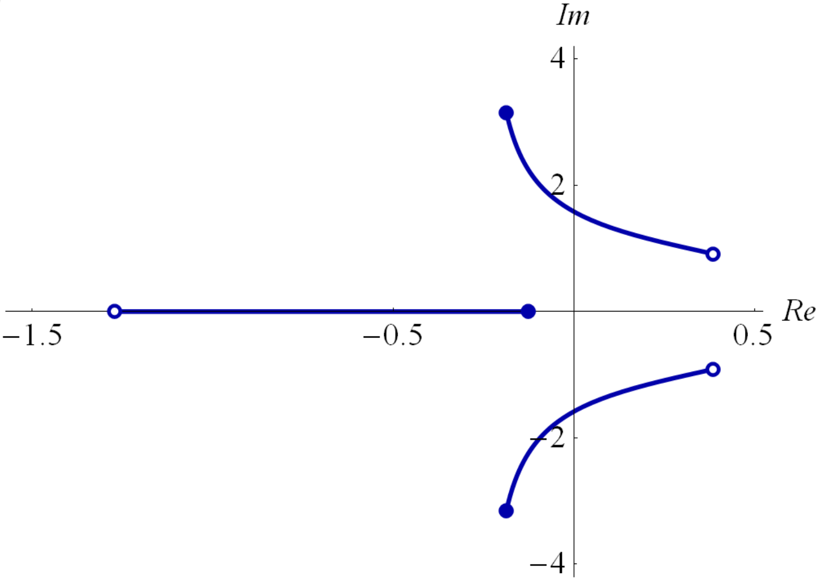
$\tau = c = 1$

$n = 0.032$ or $0.034, 1, 2, 3, \dots 100$

$\alpha = 2$

$b = 2$	$b = 3$	$b = 5$
$\gamma = 1.5$	$\gamma = 1.66$	$\gamma = 0.8$





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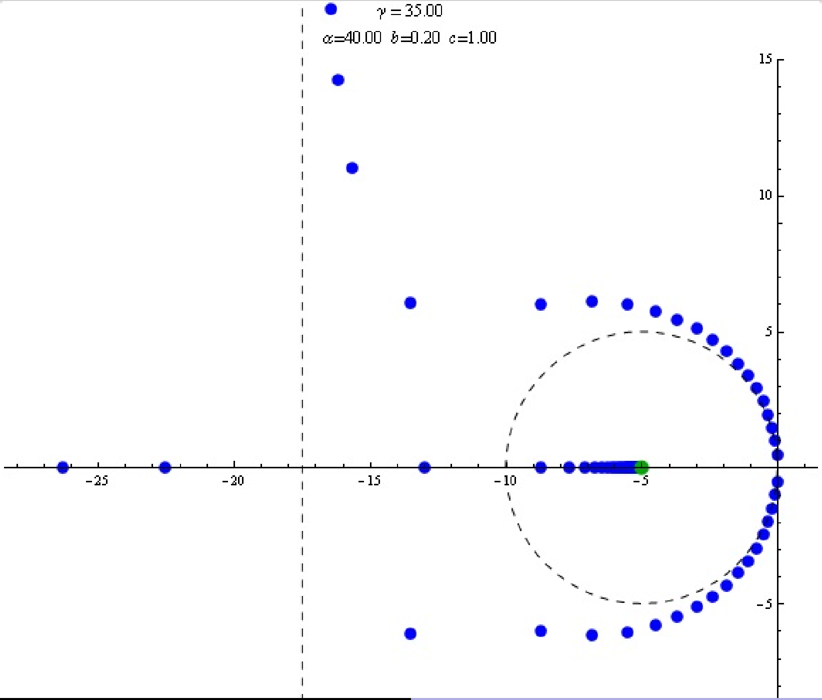

$\gamma = 35.00$
α=40.00 b=0.20 c=1.00
15
10
5
-5
-25
-20
-15
-10
-5

Abstract formulations

$\tau = 0$- **Westervalt/ Kuznetsov**

$$(\alpha - 4ku)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 4ku_t^2 \tag{4}$$

$$H \equiv D(\mathcal{A}^{1/2}) \times \mathcal{H}, \qquad H_1 \equiv D(\mathcal{A}) \times \mathcal{H}$$

$\tau > 0$- **MGT**

$$\tau u_{ttt} + (\alpha - 4ku)u_{tt} + c^2\mathcal{A}u + b\mathcal{A}u_t = 4ku_t^2 \tag{5}$$

$$H \equiv D(\mathcal{A}^{1/2}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}, \qquad H_1 \equiv D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}$$

$\mathcal{A}$ corresponds to negative Laplasjan with zero boundary conditions [Neuman/Dirichlet]

Theorem (THM 1: Linear stability)

Let $k = 0$.

1. $\tau = 0$. e^{At} *is exponentially stable iff* $\alpha > 0, b > 0$.
2. $\tau > 0$. e^{At} *is exponentially stable iff* $\gamma \equiv \alpha - \frac{\tau c^2}{b} > 0$.

Theorem (THM 2: Global solutions for nonlinear system.)

1. *When* $\tau = 0$, $\alpha > 0$ *there exists unique* **global** *solution provided the initial data are small with respect to* $D(\mathcal{A}) \times D(\mathcal{A}^{1/2})$. $W_p^2 \times W_p^1$ *for* $p > 2$.
2. $\tau \geq 0$, $\gamma > 0$, *there exists unique* **global** *solution provided the initial data are small with respect to* $D(\mathcal{A}) \times D(\mathcal{A}^{1/2}) \times \mathcal{H}$.

McDevitt, Marchand, Triggiani, 2012 MMAS.
Kaltenbacher, IL, 2012, *MathNach*
Kaltenbacher, IL. M. Pospiesz , 2013 *MMMAS*
Meyer, Wilke, 2013, *AMO*

Parameter of stability: $\gamma \equiv \alpha - \frac{\tau c^2}{b}$

Energy functions

$$E_0(t) \equiv ||\mathcal{A}^{1/2}u(t)||^2 + ||\mathcal{A}^{1/2}u_t(t)||^2 + ||u_{tt}(t)||^2$$

$$E_1(t) \equiv ||\mathcal{A}u(t)||^2 + E_0(t)$$

$\gamma = \alpha - \frac{\tau c^2}{b}$, For $\gamma > 0$ and k =0, $E_i(t) \leq Ce^{-\omega t}$

Owing to **exponential stability with** $\gamma > 0$ of the linearization

Proof of global wellposedness of the **nonlinear problem** for small initial data is based on "barrier's method " used in hyperbolic quasilinear theory. Technical tools: a string of suitable estimates developed for the linearization. .

Stability for Critical JMGT

JMGT is not stable in the critical case $\gamma = \alpha - \tau \frac{c^2}{b} = 0$,

What if $\gamma = 0$

When $\gamma = 0$ then $E(t) \sim \textit{const}$ for linear model.
Need to stabilize. Two options:

- **Memory** damping: $\int_0^t g(t-s)\mathcal{A}[au(s) + bu_t(s)]ds$.
  Filippo Del' Oro, Vittorino Pata, Xiaojun Wang, IL
- **Boundary** damping on a part of the boundary. with M. Bongarti.

Asymptotic Analysis when $\tau \to 0$ Why do we need to be careful with the density argument in the nonlinear enviroment?

Convergence of $U^\tau \to U^{\tau=0}$ when $\tau \to 0$???? Where???

- Wellposedness and regularity results for the JMGT (nonlinear) require some type of **smallness of the initial data** .
- How small ??? So far:

  $\longrightarrow$ wellposedness in $\mathbb{H}_1 \longleftrightarrow$ data small in $\mathbb{H}_1$

  $\longrightarrow$ wellposedness in $\mathbb{H}_2 \longleftrightarrow$ data small in $\mathbb{H}_2$.
- In order to show convergence of the **nonlinear** semigroups in the phase space $\mathbb{H}_1$, one needs estimates in a higher topology: $\mathbb{H}_2$.
- Although $\mathbb{H}_2$ is dense in $\mathbb{H}_1$, if wellposedness in each space is **tied to smallness in that space**, one cannot use density.

Weak versus strong limit

Singular generator A^τ:

$$A^\tau = \frac{-1}{\tau} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ c^2\mathcal{A} & \beta\mathcal{A} & \alpha I \end{pmatrix}$$

- B. Kaltenbacher and V. Nikolic: MMAS 2020, 2019) **Weak** convergence established.
- **Strong Convergence - Open Problem.**
- **Why OPEN?** Singular generator and need to handle quasilinearity at various topological levels.

To settle the problem three ingredients:

- (1) control of **singularity** in τ of u_{ttt},
- (2) **Tightness [reduction] of the "smallness"** to the base energy reflected in all uniform estimates of the energy.
- (3) **Invariance** of the "tightness" on the dynamics.

Strong convergence

Theorem (Bongarti, Charoenphon, Lasiecka, JEE, 2020)

a) Rate of Convergence: Let $T > 0$ and let $U_0 \in \mathbb{H}_2 \sim H^2 \times H^2 \times L_2$ with $\|U_0\|_{\mathbb{H}_0^\tau} \leq \rho$ sufficiently small. Then there exists a τ-independent constant C_T such that

$$\left\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\right\|^2_{H^2 \times H^1} \leqslant C_T \tau \|U_0\|_{H^2 \times H^2 \times H^1}$$

uniformly (in t) for $t \in [0, T]$.

b) Strong Convergence: Let $U_0 \in \mathbb{H}_1 = H^2 \times H^1 \times L_2$ with $\|U_0\|_{\mathbb{H}_0^\tau} \leq \rho$ for ρ as above. Then the following strong convergence

$$\left\|P(U^\tau(t, U_0)) - U^0(t, PU_0)\right\|_{H^2 \times H^1} \to 0 \quad \text{as} \quad \tau \to 0$$

holds uniformly on $[0, \infty)$.

PART II- View from the boundary

$\tau > 0$- **MGT,** $U = (u, u_t, u_{tt})$ **. Questions asked: Hidden regularity of the maps**

- $$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f \tag{6}$$
  $$u = 0 \text{ on } \Sigma \tag{7}$$
  $$U_0 \in H_1 \equiv H^2 \times H^1 \times L_2$$
  $$(f, U_0) \to (\frac{\partial}{\partial \nu} u)$$

- and

  $$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f \tag{8}$$
  $$u = g \text{ on } \Sigma \tag{9}$$
  $$U_0 \in H_{-1} \equiv L_2 \times H^{-1} \times H^{-2}$$
  $$(f, U_0, g) \to (U, \frac{\partial}{\partial \nu} u)$$

Brutal force-Interior regularity + Trace Thm. .

$$U_0 \in H_1, f \in L_2(0, T; L_2) \to U(t) \in C(0, T; H_1), H_1 = H^2 \times H^1 \times L_2$$

Trace theorem implies

$$\frac{\partial}{\partial \nu} u(t) \in H^{1/2}(\Gamma)$$

What to expect? Can we do better? **Hidden Regularity?**

Wave equation:

$$w_{tt} - \Delta w = f, W(0) = [w_0, w_1], w = 0 \ on \ \Sigma$$

$(f, W_0) \in L_1(L_2) \times H^1 \times L_2 \to W \in C(0, T; H^1 \times L_2)$
Brutal force: $(\frac{\partial}{\partial \nu} u) \in ? = C(0, T; H^{-1/2}(\Gamma))$
However-hidden regularity

Theorem (IL,Lions,Trigggiani,1986)

$(f, W_0) \in L_1(L_2) \times H^1 \times L_2 \to \frac{\partial}{\partial \nu} w \in L_2(\Sigma)$
Gain of $1/2$ *derivative.*

Theorem

$(f, W_0, g) \in L_1(L_2) \times H^1 \times L_2 \times H^1(\Sigma) \to W \in C(H^1), \frac{\partial}{\partial \nu} w \in L_2(\Sigma)$
Gain of $1/2$ *derivative.*

Theorem

$g \in (L_2)(\Sigma) \to w \in C(0, T; L_2(\Omega)), \frac{\partial}{\partial \nu} w \in H^{-1}(\Sigma)$
Gain of $1/2$ *derivative.*

It is not so "hidden" . I.L. Triggiani, 1983

$$g \in (L_2)(\Sigma) \to w \in C(0,T;L_2(\Omega)), \frac{\partial}{\partial \nu} w \in H^{-1}(\Sigma)$$

proof:

$$w_{tt} = \Delta w, \quad w = g \in L_2(\Sigma), W(0) = 0$$

$t \to s = \alpha + i\beta, y \to i\eta$, x-normal, y - tangential .

$$\hat{w}_{xx}(x,s,\eta) = [s^2+\eta^2]\hat{w}(x,s,\eta)$$

$$\hat{w}(0,s,\eta) = \hat{g}(s,\eta)$$

$$\hat{w}(x,s,\eta) = \hat{g}(s,\eta)e^{-Re\sqrt{s^2+\eta^2}x} \in L_2(Q)$$

$$\frac{\partial}{\partial \nu}\hat{w} = \hat{w}_x(0,s,\eta) = -\underbrace{\sqrt{s^2+\eta^2}}_{OP^1}\hat{g}(s,\eta) = OP^1 \cdot L_2(\Sigma) \in H^{-1}(\Sigma)$$

Neumann problem-Lopatinski Condition violated.

- $$w_{tt} = \Delta w, \frac{\partial}{\partial \nu} w = g \in L_2(\Sigma), W(0) = 0$$

  $t \to s = \alpha + i\beta, y \to i\eta$, x-normal, y - tangential . $\hat{w}$ = FT

- $$\hat{w}_{xx}(x, s, \eta) = [s^2 + \eta^2]\hat{w}(x, s, \eta)$$
  $$\hat{w}_x(0, s, \eta) = \hat{g}(s, \eta)$$
  $$\hat{w}(x, s, \eta) = \frac{1}{-\sqrt{s^2 + \eta^2}}\hat{g}(s, \eta)e^{-\sqrt{s^2+\eta^2}x}$$

- $$\hat{w}(0, s, \eta) = -\frac{1}{\sqrt{s^2 + \eta^2}}\hat{g}(s, \eta) = OP^{-1/2} \cdot L_2(\Sigma) \in H^{1/2}(\Sigma)$$
  $$s^2 + \eta^2 = \alpha^2 + \eta^2 - \beta^2 + 2i\alpha\beta$$

  In the characteristic sector $\sqrt{s^2 + \eta^2} \sim \beta$ gains **only** $1/2$ tangential derivative .

  $$w|_{\Sigma} \in H^{1/2}(\Sigma)$$

.

Neumann Continued

In general [geometry + commutators] IL. Trig, 1989, Tataru,1993

Theorem

Assume $\Box w = 0$, $\frac{\partial}{\partial \nu} w = g \in L_2(\Sigma), W(0) = 0$ *. Then*

$$w|_\Sigma \in H^{1/3}(\Sigma), w \in H^{2/3}(Q)$$

NOT $H^1(Q)$!!!!

There is hidden gain but only a fraction. Loss of $1/3$ derivative.
Solutions are $H^1(Q)$ for one dimensional domains only
Counterexample in higher dimensions;J.Rauch 1978

Methods to exhibit hidden regularity

$$||W||_{C(H^1\times L_2)} + |\frac{\partial}{\partial \nu} w|_{L_2(\Sigma)} \leq C[||W_0||_{H^1\times L_2} + ||f||_{L_1(L_2)} + |g|_{H^1(\Sigma)}]$$

- PDO calculus
- Multipliers applied to smooth solutions
- Duality-Transpositions
- Semigroup methods
- First Order Hyperbolic Systems [Kreiss, Majda, Osher, Sakamoto] [I.L + R.T , AMO, 1983]

MGT- Strange things are happening on the boundary with L_2 boundary data

. **We are looking at**

- $u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = f$
- $u = g \in L_2(\Sigma), f \in L_1(L_2), U(0) = 0$

Questions:

- Definition of a solution in $L_2(\Omega) \times H^{-1}(\Omega) \times [D(A)]'$.
- Regularity of the solution. $U = [u, u_t, u_{tt}] \in$?????.

Introduce the notation: $\mathcal{A}$ is the Laplasjan with zero BC . The generator

$$-A \equiv \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ c^2 \mathcal{A} & b\mathcal{A} & \alpha I \end{bmatrix}$$

We can write

$$-A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & b\mathcal{A} & 0 \end{bmatrix} + P = A_0 + P$$

where P is of a lower order.

Considering principal part only we are led to consider:
Simplified boundary model driven by boundary Dirichlet data g .

$$w_{ttt} = \Delta w_t \tag{10}$$
$$w(0) = w_t(0) = w_{tt}(0) = 0 \tag{11}$$
$$w = g \ on\ \Sigma \tag{12}$$

A Favorite MGT Toy does not work. Change of variables :
$z = w_t + \frac{c^2\tau}{b} w$ leads $[c = 0]$ to $z = w_t$. $z(0) = z_t(0) = 0$.

$$z_{tt} = \Delta z, Z(0) = 0$$

$$z|_\Gamma = w_t|_\Gamma = g_t, on\ \Sigma$$

.

- Take $g \equiv 1 \in C^\infty(\Sigma) \to g_t \equiv 0.$ $\Rightarrow z \equiv 0$ and $w = \int_0^t z(s)ds \equiv 0$.
- CONTRADICTS $w = g = 1$ on the boundary.
- WHAT is GOING WRONG? The correct solution of this problem [in -say- 1 d case] is $w = char(x - t)$ and $w_t = \delta(x - t)$. But $\delta(0) \neq 0$.

Brings up the issue of a correct model and def of weak solutions

MGT- Hyperbolic 1 order system-non symmetric and characteristic

- $t \to i\beta, y \to i\eta, x = normal$
- $P_0(x, \beta, \eta) \equiv i\beta^3 u + i\tau[D_x^2 - |\eta|^2]u$
- $\begin{cases} u_1 = (|\eta|^2 + \beta^2)u \\ u_2 = D_x\sqrt{|\eta|^2 + \beta^2}u \\ u_3 = D_x^2 u \end{cases}$

Solving on the boundary:

$$D_x \begin{pmatrix} u_1 \\ u_2 \\ 0 \cdot u_3 \end{pmatrix} = \begin{pmatrix} \sqrt{|\eta|^2 + \beta^2}u_2 \\ \sqrt{|\eta|^2 + \beta^2}u_3 \\ \frac{(i\beta)^3 + i\beta|\eta|^2}{|\eta|^2 + \beta^2}u_1 - i\beta u_3 \end{pmatrix}$$

leads to a **characteristic problem on the boundary** of the first order hyperbolic system. M. Eller 2020.

MGT as a hyperbolic first order system on $H^2 \times H^1 \times L_2$
Since MGT is a third order hyperbolic system, one could expect [Sakamoto, Kreiss,Majda]. $U \equiv [u, u_t, u_{tt}]$.

$$||U||_{C(H^2\times H^1\times L_2)} + |\frac{\partial}{\partial\nu}u|_{H^1(\Sigma)} + |(\frac{\partial}{\partial\nu})^2 u|_{L_2(\Sigma)}$$
$$\leq C[||U_0||_{H^2\times H^1\times L_2} + ||f||_{L_2(L_2)} + |g|_{H^2(\Sigma)}]$$

Such result expected for any 3-rd order hyperbolic system which satisfies strong Lopatinski condition [Dirichlet trace] and has **non characteristic** boundary.
However, MGT has a characteristic boundary due to the absence of D_n^3

Theorem (Bucci-Eller, 2020-smooth solutions)

$$||U||_{C(H^2\times H^1\times L_2)} + |\frac{\partial}{\partial\nu}u|_{H^1(\Sigma} \leq C[||U_0||_{H^2\times H^1\times L_2} + ||f||_{L_2(L_2)} + |g|_{H^2(\Sigma)}]$$

No estimate for $|(\frac{\partial}{\partial\nu})^2 u|_{L_2(\Sigma)}$**.** Proof based on the verification of Kreiss - Sakamoto condition.

MGT with Rough boundary Data

Theorem (For simplified model)

Assume $g \in L_2(\Sigma)$. $U(0) = 0$.

- $u(t) = \int_0^t \mathcal{A}S(t-s)Dg(s)ds \in C(L_2)$
- $u_t(t) = \int_0^t \mathcal{A}C(t-s)Dg(s)ds \in C(H^{-1})$
- $u_{tt}(t) = \mathcal{A}\int_0^t \mathcal{A}S(t-s)Dg(s)ds + \mathcal{A}Dg(t) \in C([D(\mathcal{A})]') + L_2([D(\mathcal{A}^{3/4+}]')$

Key observation: $\frac{d}{dt}\Delta w \neq \Delta w_t$ unless g is smooth and **compatible at the origin**.

Notation:

- $C(t)$ cosine operator. $C_{tt}(t) = \mathcal{A}C(t)$
- $S(t) = \mathcal{A}^{-1}\frac{d}{dt}C(t)$.
- D is a harmonic Dirichlet extension of boundary conditions.

The solution to the boundary problem is defined by singular integrals:

$$U(t) = e^{At}U(0) + \begin{bmatrix} 0 \\ 0 \\ b\mathcal{A}Dg(t) \end{bmatrix}$$

$$-A\int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ 0 \\ b\mathcal{A}Dg(s) \end{bmatrix} ds - \int_0^t e^{A(t-s)} \begin{bmatrix} 0 \\ 0 \\ c^2\mathcal{A}Dg(s) \end{bmatrix}$$

The model is obtained by homogenization of the boundary data and using compatibility conditions. $\mathcal{A} = A_D$, D is Dirichlet harmonic extension.

$g \to U$ bounded operator $L_2(\Sigma) \to L_2(D(A^2)')$.

- **Main Goal** - $L_2(\Omega)$ Regularity of u and $H^{-1}(\Sigma)$ regularity of $\frac{\partial}{\partial \nu}u$:

Theorem

Let $U(0) \in X = L_2(\Omega) \times H^{-1}(\Omega) \times [D(A)]'$ *and* $g \in L_2(\Sigma)$. *Then there exists a unique solution*

$$U \in C(L_2) \times C(H^{-1}) \times [L_2(D(\mathcal{A}^{3/4+})') + C(D(\mathcal{A})')]$$

$$\frac{\partial}{\partial \nu} u \in H^{-1}(\Sigma)$$

References: Triggiani 2018 [SOTA] ,Bucci-Pandolfi, 2019 [JEE], Bongarti-IL-Triggiani 2021.

TOOLS for hidden regularity.

Lemma

- $g \to \int_0^t \mathcal{A}S(t-s)Dg(s)ds$ *is bounded* $L_2(\Sigma) \to C(L_2(\Omega))$
- $g \to \int_0^t \mathcal{A}C(t-s)Dg(s)ds$ *is bounded* $L_2(\Sigma) \to C(H^{-1}(\Omega))$
- $\frac{\partial}{\partial \nu} S(t) = D^* \mathcal{A}S(t) : L_2(\Omega) \to L_2(\Sigma)$.*Gains* $1/2$ *derivative* .
- $\frac{\partial}{\partial \nu} C(t) = D^* \mathcal{A}C(t) : H_0^1(\Omega) \to L_2(\Sigma)$.*Gains* $1/2$ *derivative.*

Conclusion. Correct definition leads to recovery of **hidden regularity** for MGT.

Control Problem

$$\begin{aligned} \tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) &= k D_t^2 u^2 + \gamma_1 D_t^2 |\nabla(\int_0^t u \, d\tau)|^2 \text{ in } (0, T) \times \\ \frac{\partial u}{\partial n} + u = g & \quad \text{on } (0, T) \times \Gamma_0 \quad \ldots \text{boundary excitatio} \\ D_t u + \frac{\partial u}{\partial n} = 0 & \quad \text{on } (0, T) \times \Gamma \quad \ldots \text{absorbing boundary} \end{aligned}$$

$\gamma = \alpha - \frac{\tau}{c^2 b} \geq 0$**-including the critical case**

.

Optimal Boundary Control Problem

Past results: Finite horizon and smooth controls

$$\min_{g \in G^{ad}} J(g,u) \text{ s.t.}$$

$$\begin{aligned}
\tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) &= kD_t^2 u^2 + \gamma D_t^2 |\nabla(\int_0^t u \, d\tau)|^2 \text{ in } (0,T) \times \Omega \\
\frac{\partial u}{\partial n} = g & \quad \text{on } (0,T) \times \Gamma \quad \ldots \text{boundary excitation} \\
D_t u + c\frac{\partial u}{\partial n} = 0 & \quad \text{on } (0,T) \times \hat{\Gamma} \quad \ldots \text{absorbing boundary}
\end{aligned}$$

$$J(g,u) = \frac{1}{2}\int_0^T ||u - u_d||_{L_2}^2 + \frac{1}{2}\int_0^T |g|_G^2$$

(tracking with u_d ...desired pressure distribution, C.Clason, B. Kaltenbacher

Smooth Controls

$$
\begin{aligned}
G^{ad} &= \{g \in G \,|\, ||g||_G \leq K \text{ and } g(0,\cdot) = \frac{\partial u_0}{\partial n} \text{ on } \Gamma\} \\
||g||_G &:= ||g||_{H^2(0,T;H^{-1/2}\partial\Omega)} + ||g||_{H^1(0,T;H^{1/2}\partial\Omega)} \\
||u||_U : &= \max\{||u||_{C(0,T;H^2(\Omega))}, ||u||_{C^1(0,T;H^1(\Omega))}, ||D_t^2 u||_{L^2(0,T;H^1(\Omega))}\}
\end{aligned}
$$

For such control problem ($\tau = 0$)-open loop solution via Maximum Principle.

- Kaltenbacher, Clason, JMAA 2009, EECT 2015

GOAL : Non-smooth controls, $T = \infty$, and $\tau \geq 0$

$$\min_{g \in L_2(L_2)} J(g,u) \text{ s.t.}$$

$$\tau D_t^3 u + D_t^2 u - c^2 \Delta u - b\Delta(D_t u) = 0$$

$$\frac{\partial u}{\partial n} = g \qquad \text{on } (0,T) \times \Gamma_0 \quad \text{control}$$

$$D_t u + \frac{\partial u}{\partial n} = 0 \qquad \text{on } (0,T) \times \Gamma \quad \text{absorbing BC}$$

$$J(g,u) = \frac{1}{2} \int_0^\infty ||u - u_d||^2_{L_2(\Omega)} + \frac{1}{2} \int_0^\infty |g|^2_{L_2(\Gamma)}$$

Stability-critical case

Theorem (M. Bongarti, I.L 2020)

Assumptions

- *Let* $\gamma \geq 0$. *Consider* $g = -u$ *and nonlinear dynamics.*
- *Geometric condition on* Γ_0.
- *Initial data:* $U(0) = [u(0), u_t(0), u_{tt}(0)] \in H \equiv H^2(\Omega) \times H^1(\Omega) \times L_2(\Omega)$ *subject to compatibility conditions.*

$$D_t u(0) + \frac{\partial u(0)}{\partial n} = 0 \text{ on } \Gamma, \frac{\partial u(0)}{\partial n} + u(0) = 0 \text{ on } \Gamma_0$$

Then, *there exists* $r > 0$ *such that if* $||U(0)||_H \leq r$ *then there exists a unique solution such* $U(t) = [u(t), u_t(t), u_{tt}(t)] \in C([0, \infty); H)$ *such that*

$$||U(t)||_H \leq C(||U(0)||_H) e^{-\omega t}, t > 0$$

Remark: The value of C does not depend on $\gamma \geq 0$. Thus the result is **valid with** $\gamma = 0$ **which is a critical case.**In the linear case $H = H^1 \times H^1 \times L_2$.

Input-output dynamics.

- $y(t) = (u(t), u_t(t), u_{tt}(t)) = e^{\mathcal{A}t} y_0 + B_1 g(t) + (Lg)(t)$
  $L(g) \equiv \int_0^t e^{\mathcal{A}(t-s} B_0 g(s) + \mathcal{A} \int_0^t e^{\mathcal{A}(t-s)} B_1 g(s) ds$
- control operators $B_i \in L(L_2(\Gamma_0) \to [D(\mathcal{A})]'), i = 0, 1$, are given by

$$B_0 = \begin{pmatrix} 0 \\ 0 \\ \tau^{-1} c^2 \mathcal{A} N_0 \end{pmatrix}, \qquad B_1 = bc^{-2} B_0 \tag{13}$$

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -\tau^{-1} c^2 \mathcal{A} & -\tau^{-1} \left[b\mathcal{A} + c\mathcal{A} N_1 N_1^* \mathcal{A} \right] & -\tau^{-1} \left[\alpha I + \frac{b}{c} \mathcal{A} N_1 N_1^* \mathcal{A} \right] \end{pmatrix}$$

- Let $g \in L_2(0, \infty; U)$, $U = L_2(\Gamma)$, $Y = H^1 \times H^1 \times L_2$. Minimize

$$J(g) = J(g, y(g)) = \int_0^\infty ||R(y - y_d)||_Y^2 + \int_0^\infty |g|_U^2$$

Note: **Operators** B_0, B_1 **are boundary operators-uncloseable.** Very **rough** dynamics in $[D(A^2)]'$

When $\tau > 0$-hyperbolic case. -F. Bucci , I.L *Optimization*, 2019.
Finite time horizon.

Theorem (F. Bucci, IL , Optimization, 2019)

- *There exists unique optimal control* $g^* \in L_2(U)$.
- $y^* \in C([0,T];[D(A^{2*})]')$, $g^* \in C([0,T];U)$. $Ry^* \in C(0,T;Y)$

In order to extend to **infinite time horizon**,

- Needs uniform **boundary stability** of the linear dynamics. Theorem [Bongarti, IL 2020].
- Analysis of boundary dynamics.
- Analysis of Riccati ARE with **unbounded** coefficients.

Theorem

1. **Partial Regularity** *For any* $y_0 \in [D(A^{*2})]'$, $\exists$ *unique optimal* $g^* \in C([0,\infty; U]$: $Ry^* \in C[0,\infty; Y]$.
2. **Riccati Equation**. $\exists$ *a selfadjoint positive operator* P *on* $L(Y)$ *s.t.* ::
   - $A^*PA \in L(Y), B_1^*A^*P \in L(Y; C(U))$

   which satisfies the nonstandard Riccati equation: for all $y, \hat{y} \in Y$

$$((Ay, P\hat{y})_Y + (Py, A\hat{y})_Y + (Ry, R\hat{y})_Y =$$

$$(B_1^*R^*Ry + K_{B_0,B_1}Py, [I + B_1^*R^*RB_1]^{-1}[B_1^*R^*R\hat{y} + K_{B_0,B_1}P\hat{y})_U$$

$$K_{B_0,B_1} \equiv B_0^* + B_1^*A^*$$

3. **Feedback synthesis**: *The optimal control* $\hat{g}_{g_0}$ *satisfies* $\forall t > 0$

$$g^*(t) = -G^{-1}[B_0^* + B_1^*A^*]Py^*(t)$$

where $G \equiv I - [B_0^* + B_1^*A^*]PB_1$ *is* **bounded invertible** *on* U .

Conclusions:

- We solved the original HIFU problem with L_2 **rough** controls. The observed quantities Ry are in $C(Y)$. Rough dynamics.
- Existence of solutions to **non-standard** Riccati equation
- **Feedback synthesis** -on line control - achieved.

Open Problems

There are several open problems triggered by the work presented.

- Application of infinite horizon feedback control to **nonlinear system**. It is anticipated that local theory for small initial data should emerge. Such feedback should provide stabilizing effect on nonlinear dynamics.
- Extension of the theory to more **general observation** operator. The structure of the problem is important. It is anticipated that some smoothing effect of the observation will be necessary. In addition, the structure of the problem interaction between control and observation must be carefully chosen, so that the optimal L_2 solution does exist.

References

- M.Bongarti, I. Lasiecka. Boundary stabilization of the linear MGT equation with Feedback Neumann control. Invited paper for a special volume dedicated to Z. Nashed. 2021, CRC.
- D. G. Crighton, Model equations of nonlinear acoustics, *Ann. Rev. Fluid Mech.*, **11**, 1979
- I. Lasiecka, Pandolfi, R. Triggiani, A singular control approach to highly damped second-order abstract eq. *Applied Mathematics and Optimization*, 36, 67-107, 1997.
- C. Cattaneo, On a form of heat equation which eliminates the paradox of instantaneous propagation, *C.R. Acad. Sci. Paris*, 431-433, 1958.
- B. Kaltenbacher, I. Lasiecka and M. Pospieszalska, Well-posedness and decay of the energy in MGT equation arising in high intensity ultrasound, *Math. Models Methods Appl. Sci.*, **22**, (2012), no. 11.

- C.Clason, B.Kaltenbacher. Avoiding degeneracy in the Westervelt equation by state constrained optimal control, *Evol. Equ. Control Theory* **2** (2015), no.2, 281-300.
- C.Clason, B.Kaltenbacher, S.Veljoviic , Boundary Optimal Control of the Westervelt and the Kuznetsov equations, *J. Math. Anal. Appl.* **356** (2009), 738-751.
- P.Dell'Oro, I. Lasiecka, V. Pata. The MGT equation in the critical case, *JDE*, v 261, 4188-4222, 2016
- Dell'Oro, Filippo; Lasiecka, Irena; Pata, Vittorino. A note on the MGT equation with memory of type II. J. Evol. Equ. 20 (2020)
- Dell'Oro, Filippo; Pata, Vittorino On a fourth-order equation of MGT type. Milan J. Math. 85 (2017), no. 2, 215-234.
- Dell'Oro, Filippo; Pata, Vittorino On the MGT equation and its relation to linear viscoelasticity. Appl. Math. Optim. 76 (2017), no. 3, 641-655.

- P. Jordan,Second-sound phenomena in inviscid, thermally relaxing gases, DCDS, vol 19, 2014.
- P.Jordan, Nonlinear acoustic phenomenon in viscous thermally relaxing fluids:shock bifurcation and the emergence of diffusive solitons. The Journal of Acoustical Society of America, vol 124(4), 2008.
- B. Straughan, Heat waves, Applied Mathematical Sciences, 177. Springer-Verlag, 2011
- Straughan, B. Jordan-Cattaneo waves: analogues of compressible flow. Wave Motion 98 (2020), 102637, 13 pp. 35
- Christov, Ivan C. On a C-integrable equation for second sound propagation in heated dielectrics. Evol. Equ. Control Theory 8 (2019), no. 1, 57- 72. 8
- Y.Rabotnov, Elements of hereditary solid mechanics, MIT Publishers. Moscow, 1980.
- K.Naugolnychh,L.Ostrovsky, O.Shaposhnikov, O. Hamilton, Nonlinear Wave Processes in Acoustics, 2000.

- I. Lasiecka . Global solvability of MGT equation with memory arising in nonlinear acoustics, *J. Evolution Eq*, vol 17, pp 411–41, 2017.
- I. Lasiecka and X. Wang, Intrinsic Decay Rate Estimates for Semilinear Abstract Second Order Equations with Memory, *New Prospects in Evolution Equations. Springer INdAM Series*, Vol 10, (2014), pp 271-303.
- I. Lasiecka and X. Wang, Moore-Gibson-Thompson equation with memory, part I: exponential decay of energy, *ZAMP* 2015. .
- B. Kaltenbacher, I. Lasiecka and R. Marchand, well-posedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound, *Control Cybernet.*, **40**, (2011), no. 4, 971-988.
- I.Lasiecka, X Wang, MGT equation with memory: General decay of energy, *JDE*, 2015.

- R. Marchand, T. McDevitt and R. Triggiani, An abstract semigroup approach to MGT equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability, *Math. Methods Appl. Sci.*, **35**, (2012), no. 15, 1896-1929.
- A. Caixeta, I. Lasiecka, V. Cavalcanti, Global attractors for the third order in time nonlinear systems, *JDE*, 2016.
- F. Bucci, L. Pandolfi On the regularity of solutions to the Moore-Gibson-Thompson equation: a perspective via wave equations with memory, J. Evol. Equ. 20 (2020), no. 3, 837- 867.
- M. Bongarti, S. Charphoen, I. Lasiecka; Singular thermal relaxation limit for MGT equation arising in propagation of acoustic waves. *SOTA* , 2019.
- R. Triggiani, Sharp interior and boundary regularity of the SMGTJ equation with Dirichlet and Neumann control. pp 379-425, SOTA ,2018
- F. Bucci and M. Eller, The Cauchy Dirichlet problem for the MGT equation, Arxiv 2020.

- B.Kaltenbacher; V. Nikolic. Vanishing relaxation time limit of the Jordan-Moore-Gibson-Thompson wave equation with Neumann and absorbing boundary conditions. Pure Appl. Funct. Anal. 5 (2020), no. 1, 1-26
- B. Kaltenbacher; V. Nikolic, . The Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. Math. Models Methods Appl. Sci. 29 (2019), no. 13, 2523- 2556
- V.Nikolic and B.Said-Houari, On the JMGT wave equation in hereditary fluids with quadratic gradient nonlinearity, J. Math.Fluid Mech. 2021.
- Stokes, An examination of the possible effect of the radiation of heat on the propagation of sound. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science, Nr 4, pp 305-317, 1851.
- M.Bongarti, I.Lasiecka and R.Triggiani. The SMGT equation from the boundary: regularity and stabilization. 2021. Preprint.