Solution Concepts for Optimal Feedback Control of Nonlinear Partial Differential Equations

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Closed loop optimal control

$$\begin{cases} \min_{u(\cdot)\in U} J(u(\cdot),x) := \int_0^\infty \ell(y(t)) + \frac{\gamma}{2}|u(t)|^2\, dt \\ \text{subject to} \quad \dot{y}(t) = f(y(t)) + Bu(t)\,, \quad y(0) = x \end{cases}$$

$\ell(0) = f(0) = 0,$ optimal value function

$$V(x) := \min_{u(\cdot)\in U} J(u(\cdot),x)$$

Hamilton-Jacobi-Bellman equation

$$\min_{u\in U}\{\nabla V(x)^\top (f(x) + Bu) + \ell(x) + \frac{\gamma}{2}|u|^2\} = 0\,, \quad V(0) = 0\,,$$

if $U \equiv$ linear space,

$$u^*(x) = -\frac{1}{\gamma}B^\top \nabla V(x)\,,$$

then

$$\nabla V(x)^\top f(x) - \frac{1}{2\gamma}\nabla V(x)^\top BB^\top \nabla V(x) + \ell(x) = 0\,.$$

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if $U \equiv$ linear space, and $f(y) = Ay$, $\ell(y) = \frac{1}{2}|y|^2$

$$u^*(x) = -\frac{1}{\gamma}B^\top\nabla V(x),$$

then

$$\Pi A + A^\top\Pi - \gamma^{-1}\Pi BB^\top\Pi + D^\top D = 0$$

Before we start

IS HJB WORTH THE EFFORT ?

compare Riccati – and if yes, how to get it ?

- solve it directly (curse of dimensionality)
- solve using tensor calculus (TT-rank)
- Training neural networks
- Taylor expansion
- interpolate from open loop data
- Hopf formulas
- max-plus algebra techniques
- ...

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Chapter 0: Worthwhile – Yes

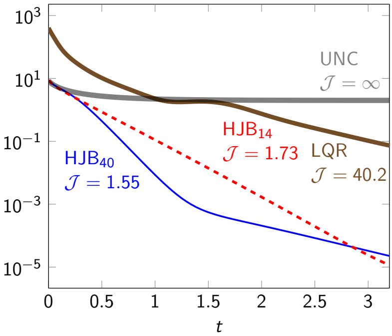
Optimal HJB-based feedback stabilization of the Newell-Whitehead equation

$$
\begin{aligned}
y_t &= \nu \Delta y + y(1-y^2) + \chi_\omega(x) u(t) && \text{in } (-1,1)\times(0,\infty),\\
y_x(-1,t) &= y_x(1,t) = 0 && \text{for } t \geq 0,\\
y(x,0) &= y_0(x) && \text{in } (-1,1),
\end{aligned}
$$

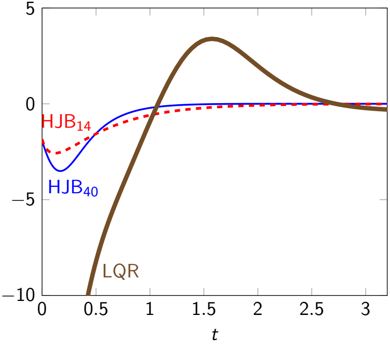
Note: 0 is unstable, ± 1 are stable equilibria

describes excitable systems such as neurons or axons, relates to Schlögel model, describing Rayleigh-Benard convection.

Newell-Whitehead Equation $d = 40$



states



controls

Chapter 1: Towards neural network based optimal feedback control

$$(P_\gamma^{y_0}) \quad \begin{cases} \inf\limits_{y\in W_\infty,\, u\in L^2(I;\mathbb{R}^m)} \dfrac{1}{2}\displaystyle\int_0^\infty \left(|Dy(t)|^2 + \gamma|u(t)|^2\right)\,\mathrm{d}t \\ \dot{y} = f(y) + Bu, \quad y(0) = y_0, \end{cases}$$

where

$$W_\infty = \{\, y \in L^2(0,\infty;\mathbb{R}^n) \mid \dot{y} \in L^2(0,\infty;\mathbb{R}^n)\,\}, \quad B \in \mathbb{R}^{n\times m}.$$

our interest: optimal feedback stabilization

$u^*(t) = F^*(y^*(t)) = -\frac{1}{\gamma}B^\top \nabla V(y^*(t))$

for all y_0 in a compact set $Y_0 \subset \mathbb{R}^n$ containing 0.

assume: $sup_{y_0\in Y_0}\|y^*(y_0)\|_{W_\infty} \leq M_0$

The learning problem

$$(\mathcal{P}_{y_0}) \quad \begin{cases} \displaystyle \min_{\substack{\mathcal{F}\in\mathcal{H},\\ y\in W_\infty}} J(y,\mathcal{F}) = \frac{1}{2}\int_0^\infty \left(|Dy(t)|^2 + \gamma|\mathcal{F}(y)(t)|^2\right) \,\mathrm{d}t \\ \displaystyle \frac{d}{dt}y = f(y) + B\mathcal{F}(y), \quad y(0) = y_0, \end{cases}$$

where

$$\mathcal{H} = \left\{\mathcal{F}(y)(t) = F(y(t)) : F \in \mathrm{Lip}\left(\bar{B}_{2M_0}(0);\mathbb{R}^m\right),\ F(0) = 0\right\}.$$

Yes, but . . . .
Bellman principle implies learn along: $S = \{y(t) : t \in (0,\infty)\}$. Thus

$$\begin{cases} \displaystyle \min_{\mathcal{F}\in\mathcal{H},\, y(y_0^i)\in W_\infty} \frac{1}{m}\sum_{i=1}^m J(y(y_0^i),\mathcal{F}) \\ \displaystyle s.t. \quad \frac{d}{dt}y(y_0^i) = f(y(y_0^i)) + B\mathcal{F}(y(y_0^i)), \quad y(0) = y_0^i, \end{cases}$$

The learning problem

$$
(\mathcal{P}) \quad \begin{cases} \displaystyle\min_{\substack{\mathcal{F}\in\mathcal{H}, \\ \mathrm{y}\in L^\infty_\mu(Y_0;W_\infty)}} \quad j(\mathrm{y},\mathcal{F}) = \int_{Y_0} J(\mathrm{y}(y_0)\mathcal{F}(\mathrm{y}(y_0)))\, \mathrm{d}\mu(y_0), \\ \dfrac{d}{dt}\mathrm{y}(y_0) = f(\mathrm{y}(y_0)) + B\mathcal{F}(\mathrm{y}(y_0)), \quad \text{for } \mu\text{-a.e.} y_0 \in Y_0, \\ \|\mathrm{y}\|_{L^\infty_\mu(Y_0;W_\infty)} \leq M_0 \end{cases}
$$

where $(Y_0, \mathcal{A}, \mu)$ is a complete probability space.

Proposition

$(\mathcal{P})$ *admits a solution and we have equivalence to* μ*-a.e. solutions of* $(\mathcal{P}_0)$ *on* Y_0.

Recap on neural networks

$$f_{i,\theta}(x) = \sigma(W_i x + b_i) \qquad \forall x \in \mathbb{R}^{N_{i-1}}, \; i = 1, \dots, L-1$$
$$f_{L,\theta}(x) = W_L x + b_L \qquad \forall x \in \mathbb{R}^{N_{L-1}}$$

$\sigma \in C^1(\mathbb{R}, \mathbb{R})$ activation function

$$\theta = (W_1, b_1, \dots, W_L, b_L)$$
$$\mathcal{R} = \otimes_{i=1}^{L} \left(\mathbb{R}^{N_i \times N_{i-1}} \times \mathbb{R}^{N_i} \right),$$

which is uniquely determined by its *architecture*

$$\mathrm{arch}(\mathcal{R}) = (N_0, N_1, \dots, N_L) \in \mathbb{N}^{L+1},$$
$$f_{L,\theta} \circ f_{L-1,\theta} \circ \cdots \circ f_{1,\theta}(x)$$

$$F_\theta(x) = f_{L,\theta} \circ f_{L-1,\theta} \circ \cdots \circ f_{1,\theta}(x) - f_{L,\theta} \circ f_{L-1,\theta} \circ \cdots \circ f_{1,\theta}(0) \quad \forall x \in \mathbb{R}^n$$

Approximation by neural networks

Theorem

Let $\eta_1 > 0, \eta_2 > 0$, and assume that the activation function σ is not a polynomial. Then for each $\epsilon > 0$ there exist $L_\varepsilon \in \mathbb{N}$, $\mathrm{arch}(\mathcal{R}_\varepsilon) \in \mathbb{N}^{L_\varepsilon+1}$, and a neural network

$$\theta_\varepsilon = (W_1^\varepsilon, b_1^\varepsilon, \ldots, W_{L_\varepsilon}^\varepsilon, b_{L_\varepsilon}^\varepsilon) \in \mathcal{R}_\varepsilon$$

such that $\|W_1^\varepsilon\|_\infty \leq \eta_1$, $|b_i^\varepsilon|_\infty \leq \eta_2$, $i = 1, \ldots, L_\varepsilon$, as well as

$$|F^*(x) - F_{\theta_\varepsilon}(x)| + \|DF^*(x) - DF_{\theta_\varepsilon}(x)\| \leq \varepsilon$$

for all $|x| \leq 2M_0$.

Thus, approximate $\mathcal{F}$ by $\mathcal{F}_{\theta_\varepsilon}$!

Approximation by neural networks:cont

Theorem

Under appropriate stabilizability conditions on the linearized control system, there exist $\varepsilon_1 > 0$, c *such that for* $\varepsilon \in (0, \varepsilon_1)$, $y_0 \in Y_0$

$$\dot{\mathrm{y}}_\varepsilon = \mathrm{f}(\mathrm{y}_\epsilon) + \mathcal{B}\mathcal{F}_{\theta_\epsilon}(\mathrm{y}_\epsilon), \quad y_\epsilon(0) = y_0,$$

admits a unique solution $y_\varepsilon = \mathrm{y}_\varepsilon(y_0) \in \mathcal{Y}_{ad}$ *and*

$$\|\mathrm{y}^*(y_0) - \mathrm{y}_\varepsilon(y_0)\|_{W_\infty} \leq c\varepsilon.$$

Moreover $\mathrm{y}_\varepsilon \in L^\infty_\mu(Y_0; W_\infty)$ *with* $\|\mathrm{y}_\varepsilon\|_{L^\infty_\mu(Y_0;W_\infty)} \leq 2M_0$.

Optimal neural network feedback law

$$
(\mathcal{P}_{\mathcal{R}_\varepsilon}) \quad \begin{cases} \displaystyle\min_{\substack{\theta\in\mathcal{R}_{ad,\varepsilon} \\ \mathrm{y}\in L^\infty_\mu(Y_0;W_\infty)}} \quad j(\mathrm{y},\mathcal{F}_\theta)+\mathcal{G}_{\mathcal{R}_\varepsilon}(\theta) \\ \dfrac{d}{dt}\mathrm{y}(y_0)=f(\mathrm{y}(y_0))+B\mathcal{F}_\theta(\mathrm{y}(y_0)), \text{ for } \mu\text{-a.e.} y_0\in Y_0, \\ \|\mathrm{y}\|_{L^\infty_\mu(Y_0;W_\infty)}\leq 2M_0 \end{cases}
$$

$$
\mathcal{R}_{ad,\varepsilon}=\{\theta=(W_1,b_1,..,W_{L_\varepsilon},b_{L_\varepsilon}):\|W_1\|_\infty\leq\eta_1,|b_i|_\infty\leq\eta_2,i=1,..,L_\varepsilon\}\subset\mathcal{R}_\varepsilon
$$

$$
\mathcal{G}_{\mathcal{R}_\varepsilon}(\theta)=I_{\mathcal{R}_{ad,\varepsilon}}(\theta)+\alpha_{\mathcal{R}_\varepsilon}\sum_{i=2}^{L_\varepsilon}\|W_i\|^2
$$

Theorem

There exists $\varepsilon_1>0$ such that $(\mathcal{P}_{\mathcal{R}_\varepsilon})$ admits a global minimizer $(\theta^*_\varepsilon,\mathrm{y}^*_\varepsilon)\in\mathcal{R}_{ad,\varepsilon}\times L^\infty_\mu(Y_0;W_\infty)$ for every $0<\varepsilon\leq\varepsilon_1$.

Convergence

Theorem

$$\textit{If} \qquad 0 < \alpha_{\mathcal{R}_\varepsilon} \le \frac{\varepsilon}{2\sum_{i=2}^{L_\varepsilon} \|W_i^\varepsilon\|^2}, \quad 0 < \varepsilon < \varepsilon_2, \ \textit{then}$$

$$0 \le j(\mathrm{y}_{\theta_\varepsilon^*}, \mathcal{F}^\sigma_{\theta_\varepsilon^*}) + \mathcal{G}_{\mathcal{R}_\varepsilon}(\theta_\varepsilon^*) - j(\mathrm{y}^*, \mathcal{F}^*) \le c\varepsilon$$

for some constant $c > 0$ *independent of* ε. *In particular*

$$j(\mathrm{y}_{\theta_\varepsilon^*}, \mathcal{F}_{\theta_\varepsilon^*}) \to j(\mathrm{y}^*, \mathcal{F}^*) \ \textit{as} \ \varepsilon \to 0.$$

Theorem

Each weak accumulation point $(\widehat{\mathrm{y}}, \widehat{\mathrm{u}})$ *of* $\{\, (\mathrm{y}_{\theta_\varepsilon^*}, \mathcal{F}_{\theta_\varepsilon^*}(\mathrm{y}_{\theta_\varepsilon^*}))\,\}$ *in* $L^2_\mu(Y_0; W_\infty) \times L^2_\mu(Y_0; L^2(I; \mathbb{R}^m))$ *fulfills* $\|\widehat{\mathrm{y}}\|_{L^\infty_\mu(Y_0;W_\infty)} \le 2M_{Y_0}$ *and*

$$\begin{aligned} &\dot{\widehat{\mathrm{y}}}(y_0) = \mathrm{f}(\widehat{\mathrm{y}}(y_0)) + B\widehat{\mathrm{u}}(y_0), \quad \widehat{\mathrm{y}}(y_0)(0) = y_0, \\ &(\widehat{\mathrm{y}}(y_0), \widehat{\mathrm{u}}(y_0)) \in \ \textit{argmin}\ (P_\gamma^{y_0}) \end{aligned}$$

for μ*-a.e.* $y_0 \in Y_0$. *If* $D > 0$ *the convergence is strong.*

Numerical realization

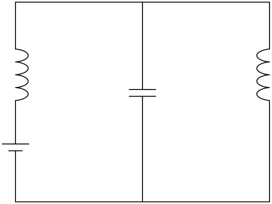
- Replace infinite time horizon by $T > 0$ sufficiently large.
- Fix $L_\varepsilon = 8$, $N_i = 2$ and $\sigma(x) = \max\{|x|^{0.1}x, 0\}$.
- Add residual connections:

$$f_{i,\theta}(x) = \sigma(W_i x + b_i) + x$$

- Assumption: Constraints are inactive, $\alpha_\varepsilon = 0$.

First example: LC-circuit

$$\dot{y} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u,$$
$$y(0) = y_0,$$



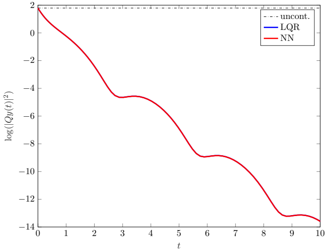
LC-circuit

2 inductors, 1 capacitor,
control voltage u,

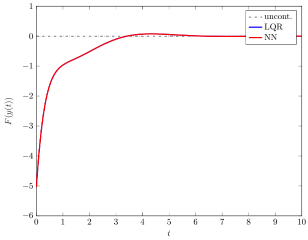
$y = (\phi_1, \phi_2, Q)^t$
magnetic fluxes, charge,

combined magnetic and
electric energy $\frac{1}{2}\,|y(t)|^2$.

Compare: Riccati vs. NN



Optimal state



Feedback control

Results for $y_0 = (-1, 2, 1)^t$.

NN optimal feedback control for a Van der Pol oscillator

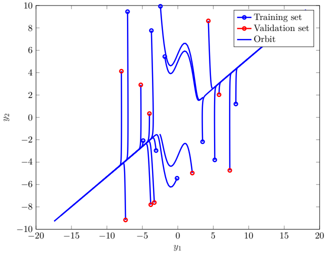
Consider

$$\frac{d^2}{dt^2}y = \frac{3}{2}(1-y^2)\frac{d}{dt}y - y + y^3 + u$$

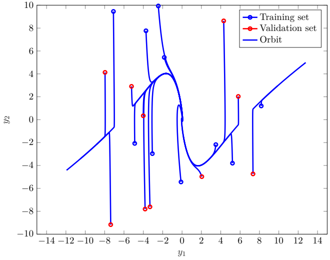
- Finite set of training initial conditions

$$\{y_0^i\}_{i=1}^5 = \{\,\pm(1,0), \pm(0,1), (6,4)\,\}\,,\ \ \mu = \frac{1}{5}\sum_{i=1}^{5}\delta_{y_0^i}$$

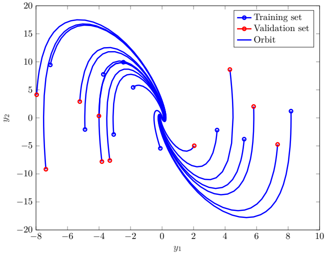
Comparison for Van der Pol



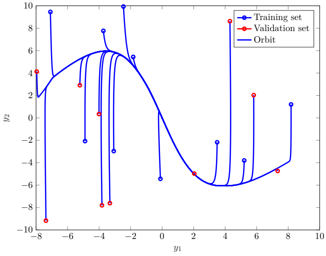
Uncontrolled system



LQR feedback



PSE feedback



Neural network feedback

Viscous Burgers equation

	$\mathcal{Y}_0^1(x) = \cos(2\pi x) cos(\pi x) + 0.5$			$\mathcal{Y}_0^2(x) = \cos(2\pi x) cos(\pi x) + 1.5$		
$\mathcal{F}$	$\|Qy\|_{L^2}$	$\|\mathcal{F}(y)\|_{L^2}$	$J(y, \mathcal{F}(y))$	$\|Qy\|_{L^2}$	$\|\mathcal{F}(y)\|_{L^2}$	$J(y, \mathcal{F}(y))$
uncont.	∞^*	0	$+\infty^*$	∞^*	0	∞^*
LQR	1.0	5.28	1.9	∞	∞	∞
PSE	1.38	4.56	1.99	∞	∞	∞
NN	1.12	4.86	1.81	2.34	11.6	9.47

	$\mathcal{Y}_0^3(x) = -2\,\mathrm{sign}(x)$			$\mathcal{Y}_0^4(x) = 2.5(x-1)^2(x+1)^2$		
$\mathcal{F}$	$\|Qy\|_{L^2}$	$\|\mathcal{F}(y)\|_{L^2}$	$J(y, \mathcal{F}(y))$	$\|Qy\|_{L^2}$	$\|\mathcal{F}(y)\|_{L^2}$	$J(y, \mathcal{F}(y))$
uncont.	∞^*	0	∞^*	∞^*	0	∞^*
LQR	2.67	5.5	5.08	1.85	13.1	10.3
PSE	3.49	7.04	8.57	∞	∞	∞
NN	2.5	1.36	3.23	1.93	11.9	8.94

spectral approximation by 14 dimensional ODE

Chaper 2: Structure exploiting policy iteration

$$\nabla V(x)^\top f(x) - \frac{1}{2\gamma} \nabla V(x)^\top B\, B^\top \nabla V(x) + \ell(x) = 0\,.$$

$$u^*(x) = -\frac{1}{\gamma} B^\top \nabla V(x)^\top.$$

- Solving nonlinear HJB: policy iteration (Howard's alg.), Newton method, Newton-Kleinman iteration for Riccati equations.

Successive Approximation Algorithm

Data: Initialization:*tol*, stabilizing control $u^0(x)$
while *residue* $\geq$ *tol* **do**

1. Solve for $V^{n+1}(x)$

$$(f(x) + Bu^n)^\top \nabla V^{n+1}(x) + \ell(x) + \frac{1}{2\gamma}\|u^n(x)\|^2 = 0\,.$$

2. Update $u^{n+1}(x) = -\frac{1}{\gamma} B^\top \nabla V^{n+1}(x)$.
3. $n = n + 1$.

end
Result: $V^*(x), u^*(x)$

- $u^0(x)$ must be asymptotically stabilizing or
- discounting

Two 'infinities': the dynamical system

Meshfree discretization of dynamical system, e.g. pseudo-spectral collocation based on Chebysheff polynomials

- The state $x(t) = (x_1(t), \dots, x_d(t))^t \in \mathbb{R}^d$.
- The free dynamics $f(x) : \mathbb{R}^d \to \mathbb{R}^d$ are $\mathcal{C}^1$ and separable in every coordinate $f_i(x)$

$$f_i(x) = \sum_{j=1}^{N_f} \prod_{k=1}^{d} \mathcal{F}_{(i,j,k)}(x_k)\,,$$

where $\mathcal{F}(x) : \mathbb{R}^d \to \mathbb{R}^{d \times N_f \times d}$ is a tensor-valued function.

Galerkin Approximation of the GHJB Equation

- Given $u^n(x)$, solve linear Generalized HJB equation

$$(f(x) + Bu^n)^\top \nabla V(x) + \ell(x) + \frac{1}{2\gamma}\|u^n\|^2 = 0\,.$$

- With $\{\phi_j(x)\}_{j=1}^{\infty}$ a complete set of d-dimensional polynomial basis functions, we approximate

$$V(x) \approx \sum_{j=1}^{N} c_j \phi_j(x)$$

- u^n $(n > 0)$ is expressed in the form

$$u^n(x) = -\frac{1}{\gamma} B^\top \nabla V^n(x) = -\frac{1}{\gamma} B^\top \sum_{j=1}^{N} c_j^n \nabla \phi_j(x)\,.$$

- Every term expanded $\rightarrow$ dense linear system for $V^{n+1}(x)$

$$A(c^n)c^{n+1} = b(c^n)\,.$$

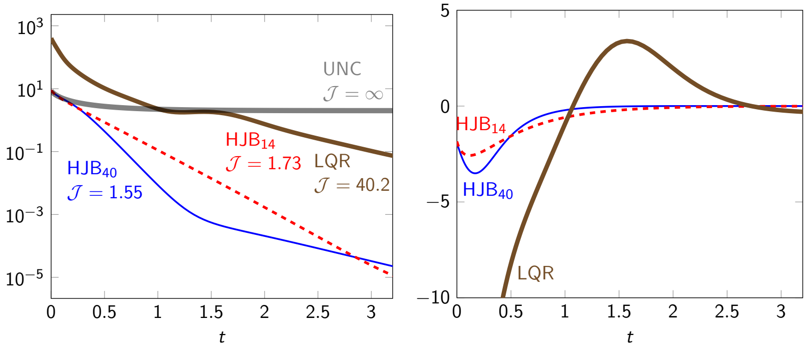
The Ingredients of Policy Iteration

- Meshfree ! eg pseudo-spectral collocation based on Chebysheff polynomials
- separability of f
- Galerkin approximation of GHJB using globally supported polynomials (monomials, Legendre, ...)
- high dimensional integrals: exploit separable structure:

$$\phi_j(x) = \prod_{i=1}^{d} \phi_j^i(x_i) \qquad (\ldots \text{ linear system of order } N^d!)$$

- compressed Tensor-Train decomposition, needs $N\, d\, r^2$ unknowns. $\quad r \sim (log\frac{1}{\varepsilon})^{\frac{7}{2}}$
- linear systems solved by Alternating Linear Scheme (preserves TT structure)

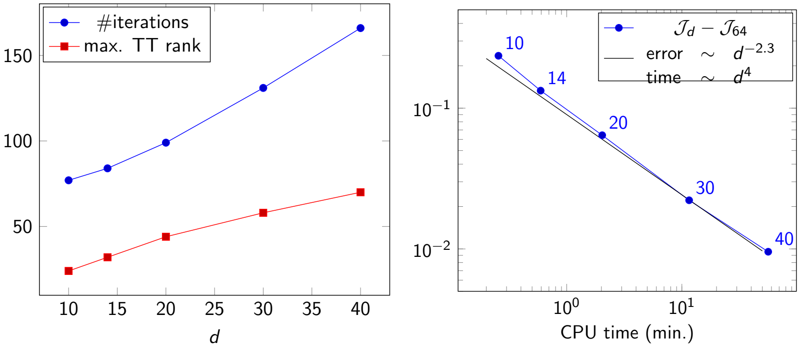
Newell-Whitehead Equation $d = 40$



states controls

tensor train computations with S.Dolgov and D.Kalise

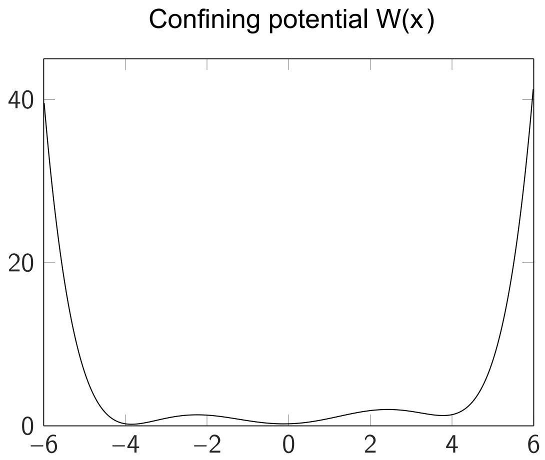
Comments on performance



Legendre polynomials of maximal individual degree 4

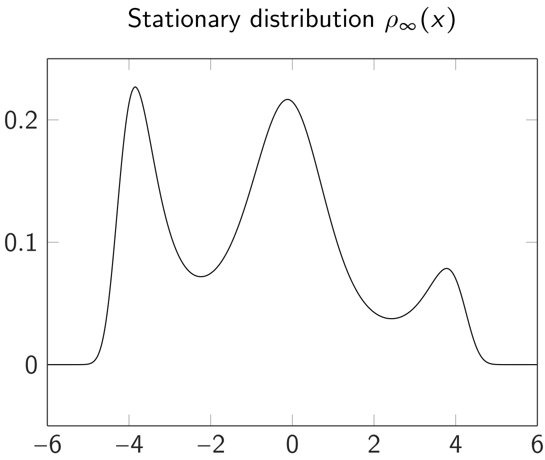
Newell-Whitehead with $\Omega \subset \mathbb{R}^2$: we used d=121.

Chapter 3: Fokker-Planck equation



1D Fokker-Planck equation

Chapter 3: Fokker-Planck equation



1D Fokker-Planck equation

The Fokker-Planck equation

Consider probability distribution function

$$\rho(x,t)\mathrm{d}x = \mathbb{P}[X_t \in [x, x+\mathrm{d}x)]$$

Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla W) && \text{in } \Omega \times (0,\infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla W) \cdot \vec{n} && \text{on } \Gamma \times (0,\infty), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

- $\Omega \subset \mathbb{R}^n$ bounded open set with boundary $\Gamma = \partial\Omega$,
- ρ_0 initial probability distribution with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$.

Control of the Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0,\infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0,\infty), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

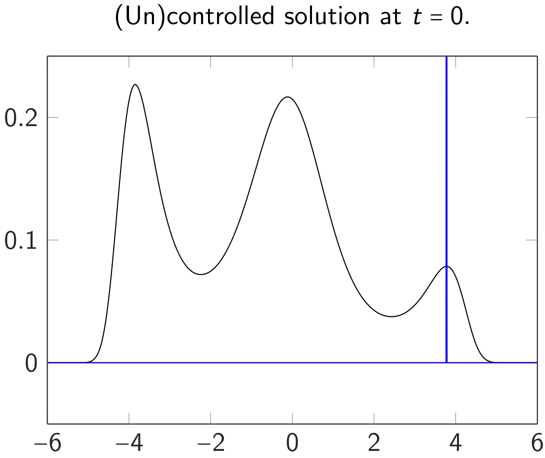
- control the potential $V(x,t) = W(x) + \alpha(x) u(t)$,
- system converges to stationary distribution $\rho_\infty(x)$, (Boltzmann distribution)
- particles have to cross energy barrier between potential wells, $\rightsquigarrow$ may be inadequately slow, $\sim \exp(\Delta W/\nu)$
- what are "good" choices for α?

Shifted state $y := \rho - \rho_\infty$

$$\begin{cases} \dot{y}(t) &= \mathcal{A} y(t) + u(t) \mathcal{N} y(t) + \mathcal{B} u(t), \\ y(0) &= \rho_0 - \rho_\infty, \end{cases}$$

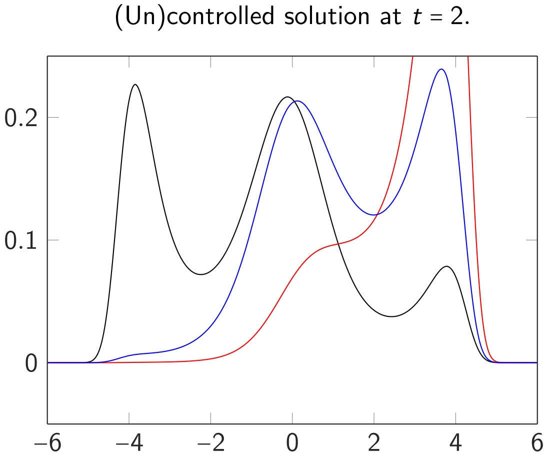
$$\mathcal{A} y = \nu \Delta y + \nabla \cdot (y \nabla W), \quad \mathcal{N} y = \nabla \cdot (y \nabla \alpha), \quad \mathcal{B} u = u \nabla \cdot (\rho_\infty \nabla \alpha)$$

Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

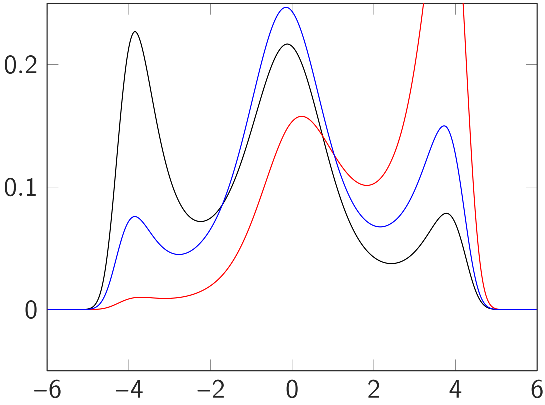
Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Back to the 1D Fokker-Planck equation

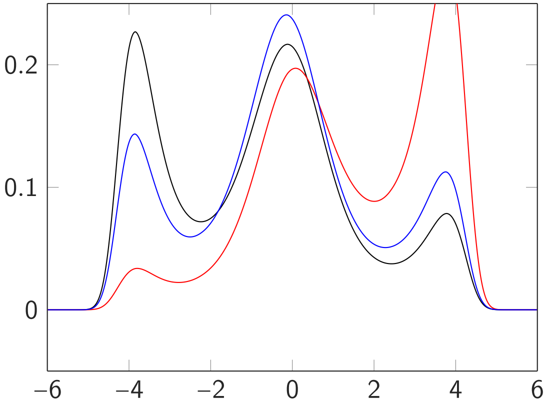
(Un)controlled solution at $t = 4$.



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

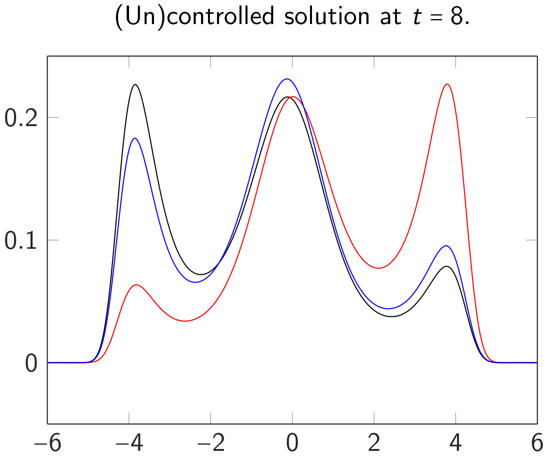
Back to the 1D Fokker-Planck equation

(Un)controlled solution at $t = 6$.



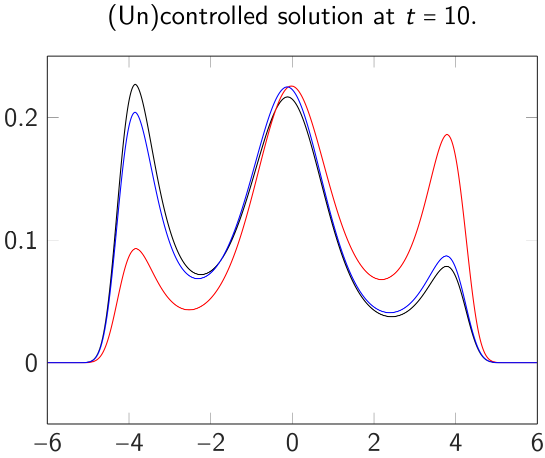
Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

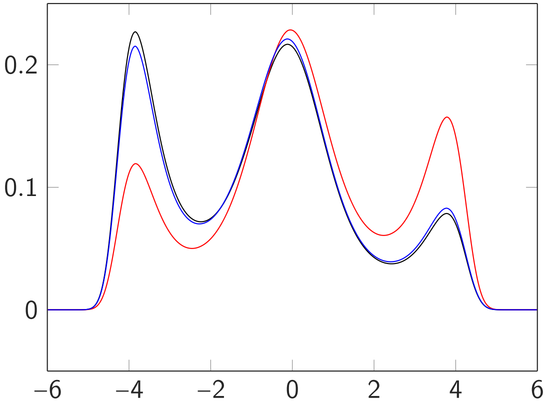
Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Back to the 1D Fokker-Planck equation

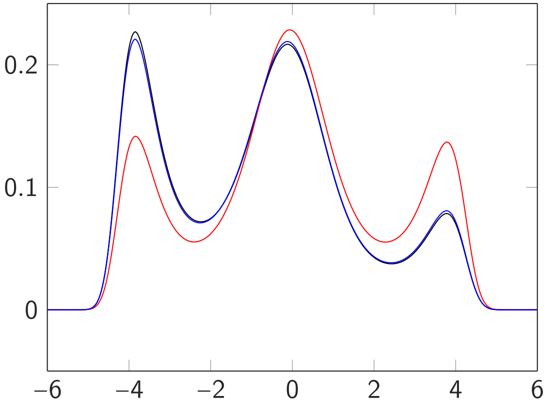
(Un)controlled solution at $t = 12$.



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

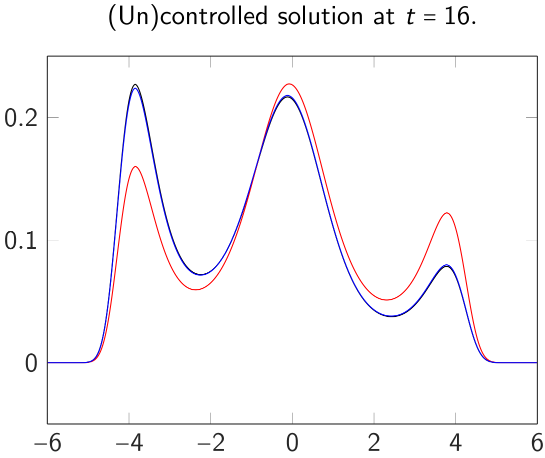
Back to the 1D Fokker-Planck equation

(Un)controlled solution at $t = 14$.



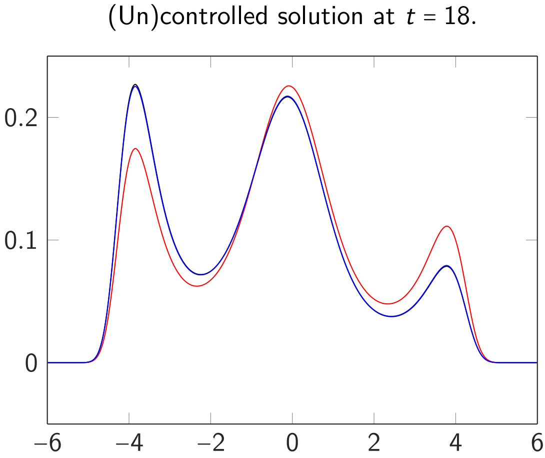
Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

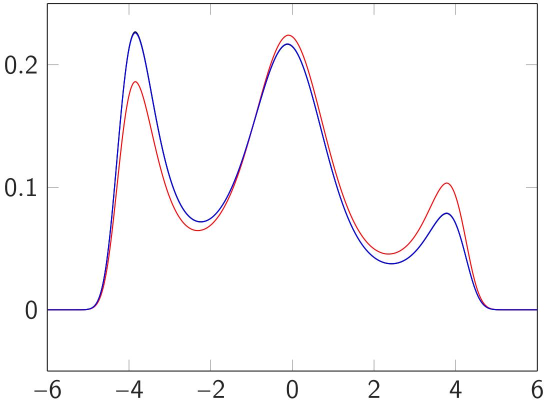
Back to the 1D Fokker-Planck equation



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Back to the 1D Fokker-Planck equation

(Un)controlled solution at $t = 20$.



Fokker-Planck, $n = 1024, r = 10, \gamma = 10^{-5}$.

Optimal Feedback for Bilinear Control Problem

Consider a bilinear control system

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t),$$

- $A, N \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^d$,
- control $u \colon [0, \infty) \to \mathbb{R}$ and
- output $y \colon [0, \infty) \to \mathbb{R}^p$ of the system,
- (A, B) stabilizable.

minimal value functional

$$\mathcal{V}(x_0) = \inf_{u \in L^2(0,\infty)} \frac{1}{2} \int_0^\infty \|x(t)\|^2 \mathrm{d}t + \frac{\gamma}{2} \int_0^\infty u(t)^2 \mathrm{d}t.$$

The Hamilton-Jacobi-Bellman equation

$$\min_{u \in \mathbb{R}} \left[(Ax + (Nx + B)u)^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 + \frac{\gamma}{2} u^2 \right] = 0, \quad \mathcal{V}(0) = 0.$$

Minimization yields Hamilton-Jacobi-Bellman (HJB) equation

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}((Nx + B)^T \nabla \mathcal{V}(x))^2 = 0, \quad \mathcal{V}(0) = 0.$$

Optimal feedback law via solving HJB equation

$$u_{\mathrm{opt}}(x) = -\frac{1}{\gamma}(Nx + B)^T \nabla \mathcal{V}(x).$$

Taylor expansions – basic idea

Assume that $\mathcal{V}$ can be expanded around 0 as follows

$$\mathcal{V}(x) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathbb{R}^d}(x) + \frac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathbb{R}^{d\times d}}(x,x) + \frac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathbb{R}^{d\times d\times d}}(x,x,x) + \dots$$

Feedback law can be determined via

$$u = -\frac{1}{\gamma}\sum_{k=2}^{\infty}\frac{1}{(k-1)!}D^k\mathcal{V}(0)(Nx+B,x,\dots,x)$$

Finite-dimensional case: [Albrekht, Lukes, Cebuhar/Costanza, Krener]

Infinite-dimensional case: [Thevenet/Buchot/Raymond]

Question: Precise structure of $D^k\mathcal{V}(0)$?

Taylor expansions – basic idea

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Question: Precise structure of $D^k\mathcal{V}(0)$?

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}((Nx + B)^T \nabla \mathcal{V}(x))^2 = 0,$$

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}((Nx+B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathbb{R}^d$ yields

$$D^2\mathcal{V}(x)(Ax, z_1) + D\mathcal{V}(x)Az_1 + \langle x, z_1 \rangle$$
$$- \frac{1}{\gamma}\big(D^2\mathcal{V}(x)(Nx+B, z_1) + D\mathcal{V}(x)Nz_1\big)\ \big(D\mathcal{V}(x)(Nx+B)\big) = 0.$$

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}\left((Nx+B)^T \nabla \mathcal{V}(x)\right)^2 = 0,$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathbb{R}^d$ yield

$$\begin{aligned} & D^3\mathcal{V}(x)(Ax, z_1, z_2) + D^2\mathcal{V}(x)(Az_2, z_1) + D^2\mathcal{V}(x)(Az_1, z_2) + \langle z_1, z_2 \rangle \\ & \quad - \frac{1}{\gamma}\Big(D^2\mathcal{V}(x)(Nx+B, z_1) + D\mathcal{V}(x)Nz_1\Big) \cdot \\ & \qquad\qquad \Big(D^2\mathcal{V}(x)(Nx+B, z_2) + D\mathcal{V}(x)Nz_2\Big) \\ & \quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(x)(Nx+B, z_1, z_2) + D^2\mathcal{V}(x)(Nz_2, z_1) + D^2\mathcal{V}(x)(Nz_1, z_2)\Big) \\ & \qquad\qquad \Big(D\mathcal{V}(x)(Nx+B)\Big) = 0. \end{aligned}$$

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}((Nx + B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathbb{R}^d$ yield

$$\begin{aligned}
&\cancel{D^3\mathcal{V}(0)(A0, z_1, z_2)} + D^2\mathcal{V}(0)(Az_2, z_1) + D^2\mathcal{V}(0)(Az_1, z_2) + \langle z_1, z_2 \rangle \\
&\quad - \frac{1}{\gamma}\Big(D^2\mathcal{V}(0)(\cancel{N0} + B, z_1) + \cancel{D\mathcal{V}(0)Nz_1}\Big)\cdot \\
&\qquad\qquad \Big(D^2\mathcal{V}(0)(\cancel{N0} + B, z_2) + \cancel{D\mathcal{V}(0)Nz_2}\Big) \\
&\quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(0)(\cancel{N0} + B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\Big) \\
&\qquad\qquad \Big(\cancel{D\mathcal{V}(0)(N0 + B)}\Big) = 0.
\end{aligned}$$

This is the Riccati equation...

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}\left((Nx+B)^T \nabla \mathcal{V}(x)\right)^2 = 0,$$

$\Rightarrow$ **three differentiations** in directions $z_1, z_2, z_3 \in \mathbb{R}^d$ yield

$$\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3) \\
&\quad - \frac{1}{\gamma}\Big(D^3 V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_2)\Big) \\
&\quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_1)\Big) \\
&\quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\Big)\Big(D^2\mathcal{V}(0)(B, z_3)\Big) \\
&\quad = 0.
\end{aligned}$$

Taylor expansions – Differentiating HJB

Let us return to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|x\|^2 - \frac{1}{2\gamma}((Nx+B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **three differentiations** in directions $z_1, z_2, z_3 \in \mathbb{R}^d$ yield

$$\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3)\\
&\quad - \frac{1}{\gamma}\Big(D^3 V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_2)\Big)\\
&\quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_1)\Big)\\
&\quad - \frac{1}{\gamma}\Big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\Big)\Big(D^2\mathcal{V}(0)(B, z_3)\Big)\\
&\quad = 0.
\end{aligned}$$

☹ Looks complicated ☺ Linear in $D^3\mathcal{V}(0)$

The general structure

Define $A_\Pi = A - \frac{1}{\gamma} BB^* \Pi$. For $k \geq 3$ and $z_1, \dots, z_k \in \mathbb{R}^d$ consider

$$\sum_{i=1}^{k} D^k \mathcal{V}(0)(z_1, ..., z_{i-1}, A_\Pi z_i, z_{i+1}, ..., z_k) = \frac{1}{2\gamma} \mathcal{R}_k(z_1, \dots, z_k) \quad (*),$$

where $\mathcal{R}_k(z_1, \dots, z_k)$ is given by:

$$\begin{aligned}\mathcal{R}_k(z_1, \dots, z_k) = {} & 2k(k-1)\mathrm{Sym}_{1,k-1}\left(\mathcal{C}_1(z_1)\mathcal{G}_{k-1}(z_2, \dots, z_k)\right) \\ & + \sum_{i=2}^{k-2} \binom{k}{i} \mathrm{Sym}_{i,k-i}\Big((\mathcal{C}_i(z_1, \dots, z_i) + i\mathcal{G}_i(z_1, \dots, z_i)) \\ & \times (\mathcal{C}_{k-i}(z_{i+1}, \dots, z_k) + (k-i)\mathcal{G}_{k-i}(z_{i+1}, \dots, z_k))\Big),\end{aligned}$$

where:

$$\mathcal{C}_i(z_1, ..., z_i) = D^{i+1}\mathcal{V}(0)(B, z_1, ..., z_i)$$

$$\mathcal{G}_i(z_1, ..., z_i) = \frac{1}{i}\Big[\sum_{j=1}^{i} D^i \mathcal{V}(0)(z_1, ..., z_{j-1}, N z_j, z_{j+1}, ..., z_i)\Big].$$

A suboptimal feedback law

Consider now the polynomial feedback law

$$u_p(y) = -\sum_{k=2}^{p} \frac{1}{(k-1)!} D^k \mathcal{V}(0)(\mathcal{N} y + \mathcal{B}, y, \ldots, y)$$

and the corresponding (nonlinear) closed-loop system

(CL) $$\dot{y} = \mathcal{A} y - (\mathcal{N} y + \mathcal{B}) u_p(y), \quad y(0) = y_0.$$

Local suboptimality

There exists a constant $C_3 > 0$, C_4 such that: if $\|y_0\|_X \leq C_3$,

- $|\mathcal{V}(y_0) - \mathcal{V}_p(y_0)| \leq C \|y_0\|_X^{2p}$
- $\|\bar{y}(\bar{u}, y_0) - y(u_p, y_0)\|_{W(0,\infty)} \leq C_4 \|y_0\|_X^{p}$
- $\|\bar{u} - u_p\|_{W(0,\infty)} \leq C_4 \|y_0\|_X^{p}$

Tensor calculus

Main numerical task:

$$\text{Solve} \quad \underbrace{\left(\sum_{i=1}^{k} I^{k-i} \otimes A_\Pi^T \otimes I^{i-1}\right)}_{\mathsf{A}} T_k = \underbrace{R_k(T_2, \ldots, T_{k-1})}_{\text{? low rank ?}}.$$

$\mathsf{A} \sim d^k \times d^k$ matrix $\twoheadrightarrow$ $r^k \times r^k$ matrix, by balanced truncation.

Since A is stable: $\mathsf{A}^{-1} = -\int_0^\infty e^{t\mathsf{A}}\, \mathrm{d}t = -\int_0^\infty \bigotimes_{i=1}^{k} e^{tA_\Pi^T}\, \mathrm{d}t.$

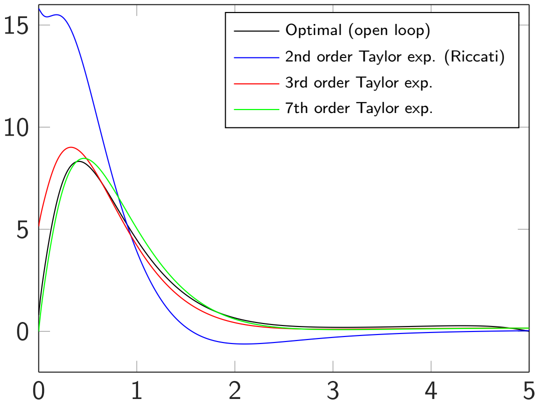
Approximate by quadrature formula

[Grasedyck,Hackbusch,Stenger]

$$\mathsf{A}^{-1} \approx -\sum_{j=-r}^{r} w_j \bigotimes_{i=1}^{k} e^{t_j A_\Pi^T},$$

with suitable quadrature weights w_j and points t_j.

Comparison of control laws $u(t)$



1D Fokker-Planck equation

In place of a table of content, a summary

- HJB equation
- Curse of dimensionality

- Train a neural network to approximate the feedback function
- Policy iteration, exploit separable structure, and tensor calculus
- Taylor expansion, model reduction, and tensor calculus

THE END

THANKS FOR ATTENDING