Output feedback control of an under-actuated and under-observed cascade system of linear Korteweg-de Vries equations
Webinar "Control in Time of Crisis"

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work in collaboration with
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Korteweg-de Vries equations

Soliton on the Scott Russell Aqueduct, Heriot-Watt University, 1995

- Propagation of waters with small amplitude in closed channels, 1895
- Application: spatially periodic cnoïdal waves in vehicle headway distance [L. Hattam (2016)]
- Coupled KdV eqs: oceanic and atmospehric phenomena (atmospheric blockings), atmosphere-ocen interactions, oceanic circulations, hurricanes (Painlevé classification), Hirota-Satsuma model, models of multicomponent KdV equations related to the weak nonlinear dispersion
- Difficult to control and observe all equations

Problem description

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2. Two output feedback control problems. The second one requires a high-gain observer to stabilize, as in the finite dimension for nonlinear systems [H. Khalil (2017)]
   - Are there any solutions for general couplings in one-order and third-order terms and also nonlinearities? Links with internal controllability/observability, e.g. [E. Fernandez-Cara, M. Gonzalez-Burgos, L. De Teresa (2015)]

Cascade System

$$v_t + v_x + v_{xxx} = (A-B)v, \text{ in } (0,+\infty)\times(0,L), \tag{1}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}, \qquad B = \operatorname{diag}(1,1,\ldots,1,-1).$$

*Boundary conditions A **(BC-A)**:*

$$\begin{aligned} v_i(t,0) &= 0, i = 1,\ldots,n-1, \forall t>0, \\ v_n(t,0) &= u(t), \forall t>0, \\ v(t,L) &= 0, v_x(t,L) = 0, \text{ for all } t>0. \end{aligned} \tag{2}$$

*Boundary conditions B **(BC-B)**:*

$$\begin{aligned} v_{i,xx}(t,0) =& 0, i = 1,\ldots,n-1, \forall t>0, \\ v_{n,xx}(t,0) =& u(t), \forall t>0, \\ v(t,L) =& 0, v_x(t,L) = 0, \forall t>0. \end{aligned}$$

Observer

Observation

$$y(t,x) = Cv(t,x), t \geq 0, x \in [0,L];$$
$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$$

⚠What about localized observation in $[L-\epsilon, L]$?

High-gain observer

$$\hat{v}_t(t,x) + \hat{v}_x(t,x) + \hat{v}_{xxx}(t,x) = (A-B)\hat{v}(t,x) - \Theta K\left(y(t,x) - C\hat{v}(t,x)\right)$$

(BC-A): $\hat{v}_i(t,0) = 0, i = 1,\dots,n-1,$
$\hat{v}_n(t,0) = u(t),$
$\hat{v}(t,L) = \hat{v}_x(t,L) = 0,$

(BC-B): $\hat{v}_{i,xx}(t,0) = 0, i = 1,\dots,n-1,$
$\hat{v}_{n,xx}(t,0) = u(t),$
$\hat{v}(t,L) = \hat{v}_x(t,L) = 0.$

Convergence of the observer

Observer parameters:

$$\Theta := \text{diag}\left(\theta, \theta^2, \dots, \theta^n\right); \theta > 0,$$

$K = \begin{pmatrix} k_1 & \cdots k_n \end{pmatrix}^\top$ is such that $A + KC$: Hurwitz.

Theorem

Assume $v^0 \in L^2(0,L)^n, u \in L^2_{loc}(0,\infty)$. Then, for every $\kappa > 0$, there exist $\theta_0 > 0$, such that for every $\theta > \theta_0$, the following holds for all $\hat{v}^0 \in L^2(0,L)^n, t \geq 0$:

$$\|\hat{v}(t,\cdot) - v(t,\cdot)\|_{L^2(0,L)^n} \leq \nu\theta^{n-1}e^{-\kappa t}\|\hat{v}^0(\cdot) - v^0(\cdot)\|_{L^2(0,L)^n}, \tag{3}$$

with $\nu > 0 \sim n, L$.

Sketch of the Proof

Choose Lyapunov function

$$V(t)=\int_0^L \pi(x)\varepsilon^\top(x)P\varepsilon(x)\mathrm{d}x,$$

with $\varepsilon := \Theta^{-1}(\hat{v}-v)$, $P \succ 0$ satisfying

$$P\,(A+KC)+(A+KC)^\top P=-I_n$$

Lemma

There exists $\lambda_0>0$, such that the following assertions hold true.
Assertion 1: For every $\lambda \geq \lambda_0$, there exist $\pi(\cdot)$ in $C^3(0,\infty)$ and $b>0$, such that for all $x \geq 0$

$$\begin{cases} \pi'''(x)+\pi'(x)-2\lambda\pi(x)=-2b\pi(x), \\ \pi''(0)\pi(0)+(\pi'(0))^2+\pi^2(0)\leq 0, \\ \pi(x)>0, \\ \pi'(x)\geq 0. \end{cases} \tag{4}$$

Assertion 2: For every $\lambda \in (0,\lambda_0)$, there exist $\bar{L}, b>0$ and $\pi(\cdot)$ in $C^3(0,\infty)$ satisfying (4) *for all $x \in [0,\bar{L}]$.*

Sketch of the Proof

To prove this Lemma, we use a trick of L. Rosier to solve the characteristic equation of the KdV operator. We obtain $\lambda_0 < 1$, near 1 and $\bar{L} = \arctan \frac{2}{5}$.
In the Lyapunov function V for ε, choose:

- For (BC-A), $\pi = 1$.
- For (BC-B), π satisfies (4) in Lemma and damping term $\theta(A + KC)$, which is controlled by θ, allows to choose λ large enough via θ, leading to the negativity of the time derivative of Lyap. function V.

Output feedback control (backstepping)

Control

$$\text{(BC-A): } u(t) = \int_0^L p(0,y)\hat{v}_n(t,y)\mathrm{d}y,$$

$$\text{(BC-B): } u(t) = -\frac{\omega+1}{3}L\hat{v}_n(t,0) + \int_0^L p_{xx}(0,y)\hat{v}_n(t,y)\mathrm{d}y,$$

$$\begin{cases} p_{xxx} + p_{yyy} + p_x + p_y + (\omega+1)p = 0, & (x,y) \in \Pi \\ p(x,x) = p(x,L) = 0, & x \in [0,L] \\ p_x(x,x) = \frac{\omega+1}{3}(L-x), & x \in [0,L] \end{cases}$$

- Unique solvability of the above equations in $C^3(\Pi); \Pi := \{(x,y); x \in [0,L], y \in [x,L]\}$ (successive approximation).

Stabilization of the closed-loop system

Theorem

For any $\bar{d} > 0$*, there exist* $u(t)$ *and* $\theta_0, \omega_0, \bar{c} > 0$*, st for any* $\theta > \theta_0, \omega > \omega_0$*,*

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}_n\|_{L^2(0,L)} \leq \bar{c}e^{-\bar{d}t} \left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}_n^0\|_{L^2(0,L)}\right).$$

Moreover, we get the following (full state convergence)
a) When ***(BC-A)*** *hold with* $n \geq 2$*, for every* $L > 0$*, there exist* $c, d > 0$*, st*

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq ce^{-dt} \left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}\right),$$

with d *depending on* n*.*
b) When ***(BC-B)*** *hold with* $n = 2$*, for every* $L > 0$*, there exist* $c, d > 0$*, st*

$$\|\hat{v} - v\|_{L^2(0,L)^n} + \|\hat{v}\|_{L^2(0,L)^n} \leq ce^{-dt} \left(\|\hat{v}^0 - v^0\|_{L^2(0,L)^n} + \|\hat{v}^0\|_{L^2(0,L)^n}\right),$$

.
c) When ***(BC-B)*** *hold with* $n > 2$*, there exists* $\bar{L} > 0$ *small, st the previous is guaranteed for all* $L \in (0, \bar{L}]$*.*

Sketch of the proof

Single equation:

$$w_t + w_x + w_{xxx} + \lambda w = 0, \text{in } (0,\infty)\times(0,L), \tag{5}$$

$$w(t,0) = w(t,L) = w_x(t,L) = 0, t>0, \tag{6a}$$
$$w_{xx}(t,0) = w(t,L) = w_x(t,L) = 0, t>0, \tag{6b}$$

Proposition

Consider BC (6a)*. For all* $\lambda > 0$*, it holds* $\forall L > 0$

$$\|w\|_{L^2(0,L)} \leq e^{-\lambda t}\|w^0\|_{L^2(0,L)}, t\geq 0,$$

For BC (6b) *there exists* $\lambda_0 > 0$*, st:*
1) For all $\lambda \geq \lambda_0$*, there exist* $a, b > 0$*, st*

$$\|w\|_{L^2(0,L)} \leq ae^{-bt}\|w^0\|_{L^2(0,L)}, t\geq 0,$$

for every $L > 0$*.*
2) For all $\lambda \in (0,\lambda_0)$*, there exist* $\bar{L}, a, b > 0$ *such that the above is satisfied for all* $L \in (0,\bar{L}]$

Sketch of the proof

Remarks

- In the stabilization of the closed-loop system, high-gain observer design is crucial. Simpler pole-placement observer does not suffice. Observer gain θ compensates for some terms playing the role of nonlinearities in finite dimension.
- For (BC-B), we need a critical damping coefficient for the stabilization as seen in Proposition. In the closed-loop stabilization, the damping coefficient is controlled by

$$\rho := \mathrm{eig}(2I_{n-1} - A_{n-1}^{\top} - A_{n-1}) = 2 - 2\cos\frac{\pi j}{n}, j = 1, \ldots, n-1.$$

where A_{n-1} is the coupling matrix of order $n-1$.

For $n = 2$, the minimum eigenvalue is 2. This is the critical value to obtain sufficient damping. For $n > 2$, the damping is not sufficient and the stabilization holds for small length L of the domain.

Talk Outline

Generalizations and open problems

Generalizations

- Localized observation. Observer design might be possible by use of a composite observer consisting of a localized one in the observed region $[L-\epsilon, L]$ (as the one here) and a boundary one (backstepping) in $[0, L-\epsilon]$.
- General couplings in one-order and third-order (dispersion) terms. Terms of the form $A_1 v_x + A_2 v_{xxx}$, $A_1, A_2 \in \mathbb{R}^{n\times n}$ diagonal with distinct elements and even lower triangular.
- Nonlinearities in zero, one, and third-order coupling terms of triangular form. Solutions to observer problems for nonlinear hyperbolic systems in [C. Kitsos. G. Besançon. C. Prieur (2021)] up to $n = 3$ (three couplings).
- Applicability of these generalizations to nonlinear inverse problems for PDEs, solved by state observers.

An example of hyperbolic system

$$v_t(t,x) + \Lambda(x) v_x(t,x) = M(x) v(t,x),$$
$$\text{Output} \quad y(t,x) = C v(t,x),$$

$$\Lambda(x) = \operatorname{diag}\left(\lambda_1(x), \ldots, \lambda_n(x)\right)$$

⚠ different elements on the diagonal of the diff. operator $(\lambda_1, \ldots, \lambda_n)$

$$M(x) = \begin{pmatrix} m_{1,1}(x) & m_{1,2}(x) & 0 & \cdots & 0 \\ & \ddots & \ddots & & \vdots \\ \vdots & & & & m_{n-1,n}(x) \\ m_{n,1}(x) & \cdots & & & m_{n,n}(x) \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

An example of hyperbolic system

General BC of the following form, not allowing boundary stabilization of the observer error (non dissipative).

BC

$$\begin{pmatrix} v^+(0) \\ v^-(L) \end{pmatrix} = K \begin{pmatrix} v^+(L) \\ v^-(0) \end{pmatrix},$$

Assumptions

“index of strict hyperbolicity”

$$q := \min \left\{ i : \lambda_i \equiv \lambda_j, \forall j = i, i+1, \ldots, n \right\},$$

Define $q_i := \max\left(1, 2q-3-i\right), i = 1, \ldots, n-1, q_n := q_{n-1}$.

Assumption 1 (regularity)

Dynamics are in $C^{q_1}[0, L]$ and IC satisfy compatibility conditions of order q_1.

Assumption 2 (couplings)

$$m_{1,2}(x), m_{2,3}(x), \ldots, m_{n-1,n}(x) \neq 0, \forall x.$$

.

Assumptions

Assumption 3

Periodicity (one version)

$$v(t,0) = v(t,L), \forall t \in [0,+\infty)$$

and $\partial_x^j \lambda_i(0) = \partial_x^j \lambda_i(L), \partial_x^j M(0) = \partial_x^j M(L), j = 0, \ldots, q_1.$

There is an alternative version (for $K \neq I_n$)

State Transformation

Define

$$\mathcal{X} := C^{q_1}[0,L] \times C^{q_2}[0,L] \times \cdots \times C^{q_n}[0,L]$$

There $\exists$ a bounded injective linear transformation

$$\mathcal{T} : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{X}, \|\cdot\|_{\mathcal{X}}),$$

with bounded inverse, st

$$\zeta = \mathcal{T} v;$$
$$\text{with } \zeta_1 = v_1.$$

State Transformation

Target System

$$
(\mathrm{T}) \begin{cases} \zeta_t(t,x) + \lambda_n(x)\zeta_x(t,x) = \bar{M}(x)\zeta(t,x) + \mathcal{M}\zeta_1(t)(x), \\ \begin{pmatrix} \zeta^+(0) \\ \zeta^-(L) \end{pmatrix} = K \begin{pmatrix} \zeta^+(L) \\ \zeta^-(0) \end{pmatrix} + \mathcal{K}_1\zeta_1(0) + \mathcal{K}_2\zeta_1(L), \\ y_\zeta(t,x) = y(t,x) = C\zeta(t,x), \end{cases}
$$

$\mathcal{M} : C^{q-1}[0,L] \to C^0([0,L];\mathbb{R}^n), \mathcal{K}_1, \mathcal{K}_2 : C^{q-1}([0,L];\mathbb{R}) \to \mathbb{R}^n$

State Transformation

Matrix Operator

$$\mathcal{T} := I_n + \tilde{\mathcal{T}};$$

$$\tilde{\mathcal{T}} := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \tau^1_{2,1}(x)\partial_x & 0 & 0 & \cdots & 0 & 0 & 0 \\ \sum_{i=1}^{2}\tau^i_{3,1}(x)\partial^i_x & \tau^1_{3,2}(x)\partial_x & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & \vdots & & \\ \sum_{i=1}^{n-2}\tau^i_{n-1,1}(x)\partial^i_x & \sum_{i=1}^{n-3}\tau^i_{n-1,2}(x)\partial^i_x & \sum_{i=1}^{n-4}\tau^i_{n-1,3}(x)\partial^i_x & \cdots & \tau^1_{n-1,n-2}(x)\partial_x & 0 & 0 \\ \sum_{i=1}^{n-3}\tau^i_{n,1}\partial^i_x & \sum_{i=1}^{n-4}\tau^i_{n,2}\partial^i_x & \sum_{i=1}^{n-5}\tau^i_{n,3}\partial^i_x & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$\tilde{\mathcal{T}}$ solves:

Generalized Sylvester Operator Equation

$$\left[(\lambda_n I_n \partial_x - M)\tilde{\mathcal{T}} + \tilde{\mathcal{T}}((\lambda_n I_n - \Lambda)\partial_x + M)\right]\left(I_n - C^\top C\right)$$
$$+(\lambda_n I_n - \Lambda)\partial_x + (\lambda_1 - \lambda_n)C^\top C\partial_x + M - \bar{M} = 0$$

Further extensions

NONLINEAR systems

- 2×2 quasilinear HYP. systems with lower triangular hyperbolic operator
- 2×2 and 3×3 diffusional Lotka-Volterra systems with distinct diffusivities.

For instance

$$v_t = \Lambda[v_1]v_x + f[v];$$
$$\Lambda[v_1] = \text{diag}\{\lambda_1[v_1], \lambda_2[v_1], \lambda_3[v_1]\},$$
$$f[v] = \begin{pmatrix} f_1[v_1] & f_2[v_1, v_2] & f_3[v_1, v_2, v_3] \end{pmatrix}^T$$
$$y = v_1$$

Assumptions:

- Strong regularity
- Use of spatial derivatives of the observation
- Linear inf. dim. state transformation

Open problems

- Links of internal controllability/observability as in [F. Alabau-Boussouira, J.M. Coron, G. Olive (2017)], [Ch. Zhang (2018)] with control/observer design (algebraic solvability, strong regularity)
- Extensions to nonlinear systems with more than 3 eqs.

Thank you!