Control in Times of Crisis

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Remarks on the Cauchy Problem for the Laplacian and Lagrangian Control of the Euler Equation

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- Joint work with Thierry Horsin (Conservatoire National des Arts & Métiers, Paris, France),

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Today's talk

Well-posedness

Lagrangian control

Cauchy problem in a rectangle

Cauchy problem in a ball

Langrangian control: idea of proof

Well-posedness

- Given two topological spaces X and Y, and a mapping $F : X \longrightarrow Y$, the problem

$$\text{given } b \in Y, \quad \text{find } u \in X, \quad F(u) = b \tag{1}$$

is said to be **well-posed in (X, Y) in the sense of J. Hadamard** if the following conditions are satisfied

(i) for any $b \in Y$ the solution u exists and is unique;

(ii) the mapping $b \mapsto u$ is continuous from $Y \longrightarrow X$.

- Problem **(1)** is said to be **ill-posed in (X, Y)** if either, or both, of the above conditions is not satisfied.

Well-posedness

- J. Hadamard has given the following example of ill-posedness: find a function $u : (0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$(2) \quad \Delta u = 0 \quad \text{in } (0, \infty) \times \mathbb{R}, \quad u(x, 0) = \varepsilon \sin(kx) \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 0) = 0.$$

- Then the unique solution is

$$u(x, y) = \varepsilon \sin(kx) \cosh(ky)$$

  and for k large and $y > 0$, the factor $\varepsilon \cosh(ky)$ may be as large as one may wish.

- And if we take $X := C_b^2([0, \infty) \times \mathbb{R})$ and $Y := C_b(\mathbb{R}) \times C_b(\mathbb{R})$, then there is no solution at all.

- This is a typical ill-posed problem, and usually one of the main difficulties is the choice of the spaces X and Y.

Well-posedness

- A more common example is the classical Dirichlet problem: $\Omega \subset \mathbb{R}^N$ being a smooth bounded domain and $N \geq 2$ an integer, let $X := C^2(\overline{\Omega})$ and $f \in C(\overline{\Omega})$. Then

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

  is an ill-posed problem in (X, Y).

- For instance, let Ω be the unit ball of $\mathbb{R}^N$, choose $\zeta \in C_c^\infty(\mathbb{R})$ such that

  $1_{|t|\leq 1/2} \leq \zeta(t) \leq 1_{[|t|\leq 3/4]}$, and set $\varphi(x) := x_1 x_2 \zeta(|x|)$, $\varphi_k(x) := \varphi(2^k x)$.

  and set (for $k \geq 1$ integer)

$$\varphi(x) := x_1 x_2 \zeta(|x|), \qquad \varphi_k(x) := \varphi(2^k x).$$

Well-posedness

- Then if

$$u(x) := \sum_{k=1}^{\infty} \frac{2^{-2k}}{k} \varphi_k(x), \quad \text{and} \quad f(x) := \sum_{k=1}^{\infty} \frac{1}{k} (-\Delta\varphi)(2^k x)$$

we have

$$-\Delta u = f, \qquad f \in C(\overline{\Omega}), \qquad u \in C^1(\overline{\Omega}), \quad \text{but } u \notin C^2(\overline{\Omega}).$$

- Here $\partial_{12} u$ has a singularity at the origin.
- However **(3)** is well-posed in (X, Y) if one takes for instance

$$X := C^{2,\alpha}(\overline{\Omega}), \qquad Y := C^{0,\alpha}(\overline{\Omega}),$$

or

$$X := H_0^1(\Omega), \qquad Y := H^{-1}(\Omega).$$

Well-posedness

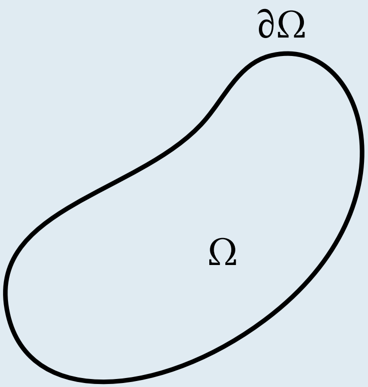
- As a matter of fact, most problems of the type **(1)** are in the same situation: according to how appropriately the spaces X and Y are chosen, **(1)** may be a well-posed or an ill-posed problem in (X, Y)...

- In this talk we want to study whether one may find appropriate spaces so that the Cauchy problem for the Laplace operator is well-posed.

Well-posedness

- Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, with $N \geq 2$, such that the boundary $\partial\Omega$ is connected.
- We shall denote by $\Lambda : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ the Steklov-Poincaré operator defined by

$$\Lambda f := \frac{\partial u}{\partial \boldsymbol{n}} \qquad \text{where} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$



- For a given $(f, g) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$, the Cauchy problem

$$(4) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in H^1(\Omega)$ if, and only if, one has $g = \Lambda f$.

Well-posedness

- In other terms, the Cauchy problem **(4)** is well posed in $(H^1(\Omega), \mathbb{H})$, where

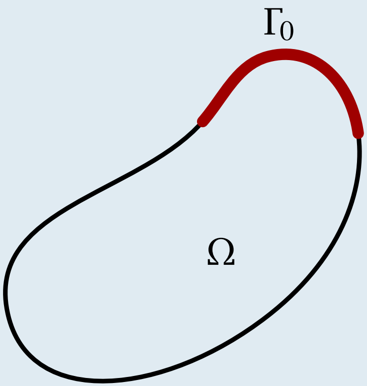
$$\mathbb{H} := \left\{ (f, \Lambda f) \; ; \; f \in H^{1/2}(\partial\Omega) \right\} = G(\Lambda) \subset H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega).$$

(We denote by $G(\Lambda)$ the graph of the operator Λ).

- But when the Cauchy data (f, g) is given on a portion $\Gamma_0 \subset \partial\Omega$, the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_0 \end{cases} \tag{5}$$

is more difficult to analyze.



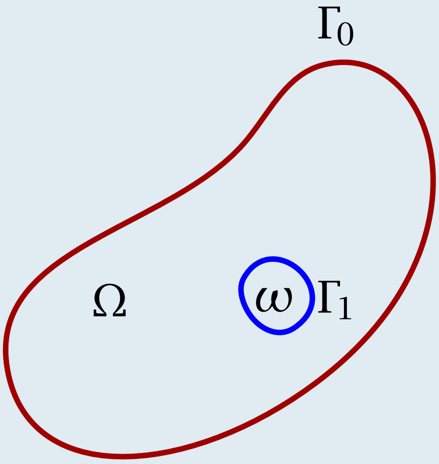
- The difficulty arises from the fact that it is not clear how one should « extend » f and g to the whole boundary $\partial\Omega$.

Well-posedness

- A situation where the problem can be studied is the following. Let Ω_0 be a smooth domain, with $\Gamma_0 := \partial\Omega_0$ connected, and $\omega \subset\subset \Omega_0$ a subdomain with $\Gamma_1 := \partial\omega$ connected.
- Now set $\Omega := \Omega_0 \setminus \omega$, so that $\partial\Omega = \Gamma_0 \cup \Gamma_1$.
- Then one can characterize the subspace $\mathbb{H} \subset H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ for which the problem

$$
\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g & \text{on } \Gamma_0 \end{cases} \tag{6}
$$

is well-posed.

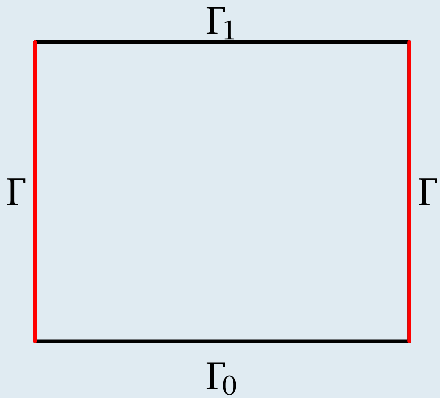


Well-posedness

- Let $\Omega := (0,\pi)\times(0,\ell)$ for some $\ell > 0$ and set

$$\Gamma_0 := [0,\pi]\times\{0\}, \qquad \Gamma_1 := [0,\pi]\times\{\ell\}, \qquad \Gamma := \{0\}\times[0,\ell]\bigcup\{\pi\}\times[0,\ell]$$

- We wish to solve in a unique and continuous manner the equation



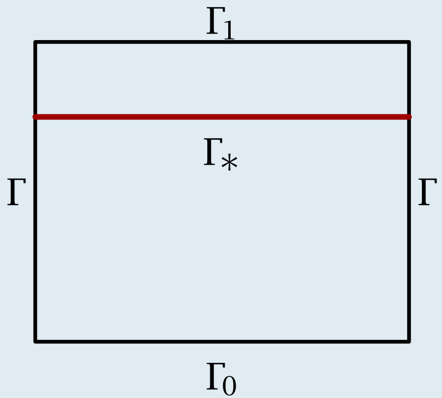
$$(7)\qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u(x,0) = f_0(x) & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}}(x,0) = g_0(x) & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}}(\sigma) = 0 & \text{on } \Gamma. \end{cases}$$

- In particular we would like to characterize the space $\mathbb{H} \subset H^{1/2}(\Gamma_0)\times H^{-1/2}(\Gamma_0)$ such that for $(f_0,g_0)\in\mathbb{H}$ the above equation has a unique solution and

$$\|u\|_{H^1} \le c\,\|(f_0,g_0)\|_{\mathbb{H}}\,.$$

Well-posedness

- Once the above analysis is carried over, a related problem is the following.
- Let $\Gamma_* := [0,\pi] \times \{\ell_*\}$ for some $0 < \ell_* < \ell$, and $g_* \in H^{-1/2}(\Gamma_*)$ be given with $\langle g_*, 1\rangle_{H^{-1/2},H^{1/2}} = 0$.



- The question is: find (f_0, g_0) so that the solution of the equation

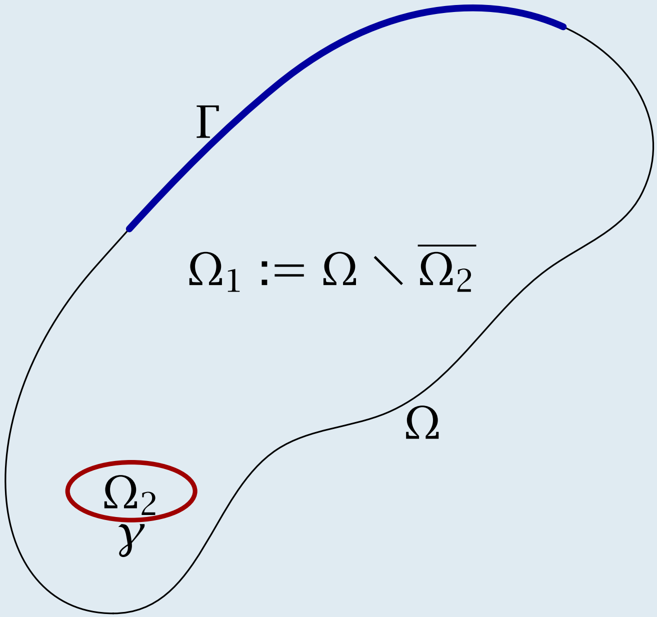
$$
(8) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u(x,0) = f_0(x) & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}}(x,0) = g_0(x) & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}}(\sigma) = 0 & \text{on } \Gamma. \end{cases}
$$

exists and moreover

$$
\frac{\partial u}{\partial \boldsymbol{n}} = g_* \quad \text{or} \quad \left\| \frac{\partial u}{\partial \boldsymbol{n}} - g_* \right\|_{H^{-1/2}(\Gamma_*)} \leq \varepsilon \;\; \text{for some } \varepsilon > 0.
$$

Lagrangian control

- Consider a domain $\Omega \subset \mathbb{R}^2$, and a subdomain $\Omega_2 \subset\subset \Omega$ such that $\gamma := \partial\Omega_2$ is a Jordan curve. Denote by Γ a part of the boundary $\partial\Omega$.



Let $h \in H^{-1/2}(\gamma)$ with $\langle h, 1\rangle = 0$.
We wish to find $v \in H^{-1/2}(\partial\Omega)$ such that if Ψ solves

$$\begin{cases} \Delta\Psi = 0 & \text{in } \Omega \\ \dfrac{\partial\Psi}{\partial\boldsymbol{n}} = v & \text{on } \Gamma \\ \dfrac{\partial\Psi}{\partial\boldsymbol{n}} = 0 & \text{on } \partial\Omega\setminus\Gamma \end{cases}$$

then we have

$$\frac{\partial\Psi}{\partial\boldsymbol{n}} = h \quad \text{on } \gamma.$$

Lagrangian control

- Indeed this is not possible in general... For example if γ is smooth, then $\partial\Psi/\partial\boldsymbol{n}$ is as smooth as γ, and so not any h can be attained.

- Then we can ask whether this is possible approximately: that is for a given $\varepsilon > 0$, find $v \in H^{-1/2}(\Gamma)$ such that

$$\left\| \frac{\partial \Psi}{\partial \boldsymbol{n}} - h \right\|_{H^{-1/2}(\gamma)} \leq \varepsilon. \tag{9}$$

- Before any precise statement, let us explain the motivation of this question. Assume that $h \in C^1([0,T]; H^{-1/2}(\gamma))$ and that we can construct a function $\Psi \in C^1([0,T]; H^1(\Omega))$ such that for each $t \in [0,T]$ the above inequality is satisfied.

- Then if we set

$$\boldsymbol{u}(t,x) := \nabla\Psi(t,x), \qquad p(t,x) := -\partial_t \Psi(t,x) - \frac{1}{2}|\nabla\Psi(t,x))|^2, \tag{10}$$

the pair $(\boldsymbol{u}, p)$ is a solution of the following Euler equation:

Lagrangian control

$$(11)\qquad \begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = 0 & \text{in } (0,T)\times\Omega \\ \operatorname{div}(\boldsymbol{u}) = 0 & \text{in } (0,T)\times\Omega \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0 & \text{in } \Omega \\ \boldsymbol{u}(t,\sigma)\cdot\boldsymbol{n}(\sigma) = 0 & \text{on } (0,T)\times(\partial\Omega\setminus\Gamma) \\ \boldsymbol{u}(t,\sigma)\cdot\boldsymbol{n}(\sigma) = v(t,\sigma) & \text{on } (0,T)\times\Gamma \end{cases}$$

and moreover on γ we have

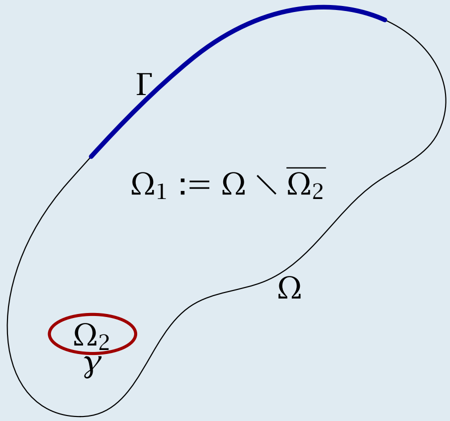
$$\sup_{0\le t\le T} \|\boldsymbol{u}(t,\cdot)\cdot\boldsymbol{n} - h(t,\cdot)\|_{H^{-1/2}(\gamma)} \le \varepsilon.$$

- Such a construction of Ψ can be interpreted as a control problem for the Euler equation from the part of boundary denoted Γ. This is called the Lagrangian control of Euler equation, and has been studied by Thierry Horsin and Olivier Glass.
- We denote by $H^{1/2}_m(\gamma) \subset L^2_m(\gamma) \subset H^{-1/2}_m(\gamma) = \left(H^{1/2}_m(\gamma)\right)'$ the Sobolev spaces of functions (or distributions) on γ with mean value zero.

Lagrangian control

- Define the operator $\Lambda_\gamma : H_m^{-1/2}(\Gamma) \longrightarrow H_m^{-1/2}(\gamma)$ as follows: for $v \in H_m^{-1/2}(\partial\Omega)$ with $\mathrm{supp}(v) \subset \Gamma$ we solve:

$$
\begin{cases}
\Delta\Psi = 0 & \text{in } \Omega \\
\dfrac{\partial\Psi}{\partial \boldsymbol{n}} = v & \text{on } \Gamma \\
\dfrac{\partial\Psi}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \setminus \Gamma
\end{cases} \tag{12}
$$



then we set

$$
\Lambda_\gamma(v) := \frac{\partial\Psi}{\partial \boldsymbol{n}} \quad \text{on } \gamma. \tag{13}
$$

1 Theorem. *The image of Λ_γ is dense in $H_m^{-1/2}(\gamma)$.*

Cauchy problem in a rectangle

- In order to study **(7)**, for $g_1 \in H^{-1/2}(\Gamma_1)$ and $f_0 \in H^{1/2}(\Gamma_0)$ given, we solve

$$
(14) \qquad \begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v(x,0) = f_0(x) & \text{on } \Gamma_0 \\ \dfrac{\partial v}{\partial \boldsymbol{n}}(x,\ell) = g_1(x) & \text{on } \Gamma_1 \\ \dfrac{\partial v}{\partial \boldsymbol{n}}(\sigma) = 0 & \text{on } \Gamma. \end{cases}
$$

- Solving the Cauchy problem **(7)** is equivalent to find $g_1 \in H^{-1/2}(\Gamma_1)$ such that the corresponding solution v of **(14)** satisfies

$$
g_0 = \frac{\partial v}{\partial \boldsymbol{n}} \quad \text{on } \Gamma_0.
$$

- In order to solve **(14)** and express $\partial v/\partial \boldsymbol{n}$ we shall write the unique solution v of **(14)** as a series, using Steklov eigenfunctions $(\psi_{j,k})_{k\geq 0}$ for $j = 0$ or $j = 1$.

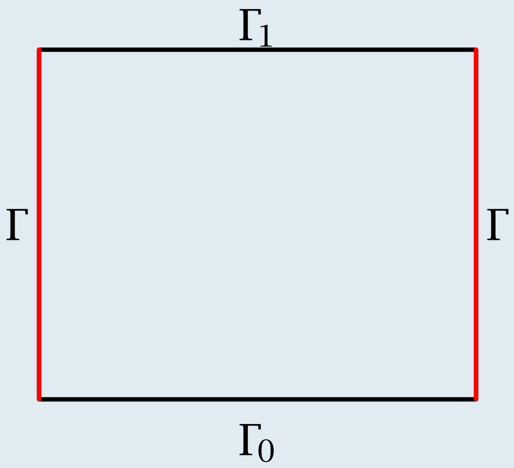
Cauchy problem in a rectangle

- The Steklov eigenfunctions $(\psi_{0,k})_k$ are defined by:

$$\begin{cases} -\Delta\psi_{0,k} = 0 & \text{in } \Omega \\ \dfrac{\partial\psi_{0,k}}{\partial \boldsymbol{n}} = \mu_{0,k}\psi_{0,k} & \text{on } \Gamma_0 \\ \dfrac{\partial\psi_{0,k}}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma_1 \\ \dfrac{\partial\psi_{0,k}}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma. \end{cases}$$

And the Steklov eigenfunctions $(\psi_{1,k})_k$ are defined by

$$\begin{cases} -\Delta\psi_{1,k} = 0 & \text{in } \Omega \\ \dfrac{\partial\psi_{1,k}}{\partial \boldsymbol{n}} = \mu_{1,k}\psi_{1,k} & \text{on } \Gamma_1 \\ \psi_{1,k} = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial\psi_{1,k}}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma \end{cases}$$



Cauchy problem in a rectangle

- One can see easily that with the normalization

$$\int_{\Gamma_j} \psi_{j,k}(\sigma)\, \psi_{j,n}(\sigma)\, d\sigma = \delta_{kn},$$

we have

$$\psi_{0,0}(x,y) := \frac{1}{\sqrt{\pi}}, \qquad \psi_{1,0}(x,y) := \frac{1}{\ell\sqrt{\pi}}\, y,$$

and for $k \geq 1$,

$$\psi_{0,k}(x,y) := \sqrt{\frac{2}{\pi}} \cos(kx) \frac{\cosh(k(\ell - y))}{\cosh(k\ell)}, \tag{15}$$

while

$$\psi_{1,k}(x,y) := \sqrt{\frac{2}{\pi}} \cos(kx) \frac{\sinh(ky)}{\sinh(k\ell)}. \tag{16}$$

Cauchy problem in a rectangle

- The Steklov eigenvalues are given by $\mu_{0,0} := 0$, and $\mu_{1,0} := 1/\ell$, while for $k \geq 1$

$$\mu_{0,k} := k\tanh(k\ell) \qquad \text{and} \qquad \mu_{1,k} := k\,\mathrm{cotanh}(k\ell). \tag{17}$$

- We denote

$$f_{0,k} := \int_0^\pi f_0(x)\psi_{0,k}(x,0)\,dx, \qquad g_{1,k} := \int_0^\pi g_1(x)\psi_{1,k}(x,\ell)\,dx,$$

so that

$$f_0 \in H^{1/2}(\Gamma_0) \iff |f_{0,0}|^2 + \sum_{k\geq 1} \mu_{0,k}|f_{0,k}|^2 < \infty,$$

and

$$g_1 \in H^{-1/2}(\Gamma_1) \iff |g_{1,0}|^2 + \sum_{k\geq 1} \frac{1}{\mu_{1,k}}|g_{1,k}|^2 < \infty.$$

Cauchy problem in a rectangle

- Then the solution v of **(14)** is given by

$$v = \ell\, g_{1,0}\, \psi_{1,0} + \sum_{k\geq 1} \frac{\tanh(k\ell)}{k} g_{1,k}\, \psi_{1,k} + \sum_{k\geq 0} f_{0,k}\, \psi_{0,k}\,,$$

and moreover on Γ_0 we have

$$\frac{\partial v}{\partial \boldsymbol{n}} = \frac{-1}{\sqrt{\pi}} g_{1,0} + \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} \left[k \tanh(k\ell)\, f_{0,k} - \frac{1}{\cosh(k\ell)} g_{1,k} \right] \cos(kx).$$

- Now it is clear that solving the Cauchy problem **(7)** is equivalent to find $g_1 \in H^{-1/2}(\Gamma_1)$ such that on Γ_0 we have

$$g_0 = \frac{\partial v}{\partial \boldsymbol{n}} = \frac{-1}{\sqrt{\pi}} g_{1,0} + \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} \left[k \tanh(k\ell)\, f_{0,k} - \frac{1}{\cosh(k\ell)} g_{1,k} \right] \cos(kx).$$

Cauchy problem in a rectangle

- This means that one should have $g_{1,0} = -g_{0,0}$ and for all $k \geq 1$

(18)
$$g_{1,k} = k \sinh(k\ell) \left[f_{0,k} - \frac{\operatorname{cotanh}(k\ell)}{k} g_{0,k} \right].$$

- Since $g_1 \in H^{-1/2}(\Gamma_1)$, we should have

$$\sum_{k \geq 1} \frac{1}{k} |g_{1,k}|^2 < \infty.$$

- Now, since $g_{1,k}$ is given by **(18)**, it is possible to characterize all the pairs (f_0, g_0) for which one can solve equation **(7)** in $H^1(\Omega)$.

Cauchy problem in a rectangle

Thus we have the following result:

2 Theorem. *The Cauchy problem* **(7)** *has a unique solution* $u \in H^1(\Omega)$ *if and only if the Cauchy boundary data* $(f_0, g_0) \in \mathbb{H}(\ell)$ *where* $\mathbb{H}(\ell) \subset H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *is defined by*

$$\mathbb{H}(\ell) := \left\{ (f_0, g_0) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \; ; \; \|(f_0, g_0)\|^2_{\mathbb{H}(\ell)} < \infty \right\}, \tag{19}$$

with

$$\|(f_0, g_0)\|^2_{\mathbb{H}(\ell)} := |f_{0,0}|^2 + |g_{0,0}|^2 + \sum_{k=1}^{\infty} k|f_{0,k}|^2 + \sum_{k=1}^{\infty} k \sinh^2(k\ell) \left(f_{0,k} - \frac{\operatorname{cotanh}(k\ell)}{k} g_{0,k} \right)^2 .$$

Moreover for two positive constants c_1, c_2 *we have*

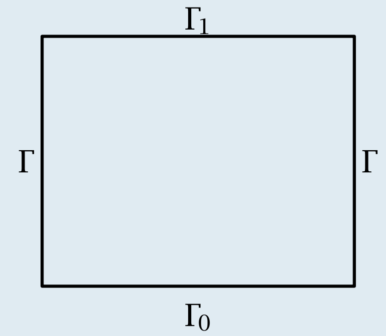
$$c_1 \|(f_0, g_0)\|^2_* \le \|u\|^2_{H^1(\Omega)} \le c_2 \|(f_0, g_0)\|^2_* .$$

Cauchy problem in a rectangle

- Here one sees the kind of instability one has to tackle: if $g_1 \in H^{-1/2}(\Gamma_1)$ is given and we wish to find f_0, g_0 on Γ_0 such that the solution u of **(7)** satisfies

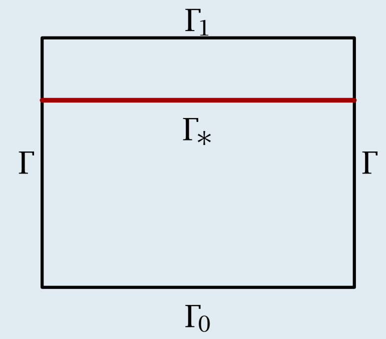
$$\frac{\partial u}{\partial \boldsymbol{n}} = g_1 \quad \text{on } \Gamma_1,$$

then an error of order ε on f_0, g_0 on their k-th component is translated into an error of order $\varepsilon \sinh(k\ell)$ on g_1.



Cauchy problem in a rectangle

- An analogous analysis can be made when, for a given $g_* \in H^{-1/2}(\Gamma_*)$, one wishes to determine f_0, g_0 such that $\partial u/\partial \boldsymbol{n}$ on Γ_* is close or equal to g_*.



- Other types of boundary conditions on Γ can be considered and, for each of them, one can characterize the class of those pairs (f_0, g_0) for which there exists $u \in H^1(\Omega)$ such that $\Delta u = 0$ in Ω and $u = f_0$ and $\partial u/\partial \boldsymbol{n} = g_0$ on Γ_0. The intersection of two classes, associated to two different boundary conditions on Γ, is reduced to $\{0\}$. $\square$

Cauchy problem in a ball

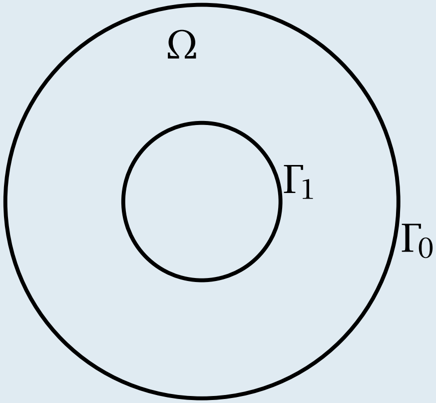
- Here we study the Cauchy problem **(6)** in the special case of an annular domain.

- Let $0 < R_1 < R_0$ and $\Omega := \left\{x \in \mathbb{R}^2 \,;\, R_1 < |x| < R_0\right\}$, and denote

$$\Gamma_j := [\,|x| = R_j\,] \quad \text{for} \quad j = 0, 1.$$

- For $(f_0, g_0) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ consider the Cauchy problem

$$\text{(20)} \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f_0 & \text{sur } \Gamma_0 \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_0 & \text{on } \Gamma_0 \end{cases}$$



Cauchy problem in a ball

3 Theorem. *Setting*

(21) $$\alpha := \frac{R_1}{R_0} \in (0,1),$$

and for $k \in \mathbb{Z}$

$$f_{0,k} := \frac{1}{2\pi}\int_0^{2\pi} f_0(\theta)\mathrm{e}^{-\mathrm{i}k\theta}d\theta, \qquad g_{0,k} := \frac{1}{2\pi}\int_0^{2\pi} g_0(\theta)\mathrm{e}^{-\mathrm{i}k\theta}d\theta,$$

define the subspace $\mathbb{H}(\alpha) \subset H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ *by*

(22) $$\mathbb{H}(\alpha) := \left\{(f_0, g_0) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)\ ;\ \|(f_0,g_0)\|^2_{\mathbb{H}(\alpha)} < \infty\right\}$$

where

$$\|(f_0,g_0)\|^2_{\mathbb{H}(\alpha)} := \|f_0\|^2_{H^{1/2}(\Gamma_0)} + \sum_{k\in\mathbb{Z}} \alpha^{-2k}(1+|k|)\left(f_{0,k} - \frac{1}{1+|k|}g_{0,k}\right)^2 < \infty.$$

Then **(20)** *has a unique solution if, and only if,* $(f_0, g_0) \in \mathbb{H}(\alpha)$ *and moreover for two constants* $c_1, c_2 > 0$ *we have*

$$c_1 \|(f_0,g_0)\|^2_{\mathbb{H}(\alpha)} \le \|u\|^2_{H^1(\Omega)} \le c_2 \|(f_0,g_0)\|^2_{\mathbb{H}(\alpha)}.$$

Cauchy problem in a ball

- The analysis is carried out using the following Steklov-Poincaré operators.

- For $j = 0$ or $j = 1$, let $\Lambda_j : H^{1/2}(\Gamma_j) \longrightarrow H^{-1/2}(\Gamma_j)$ be defined as being

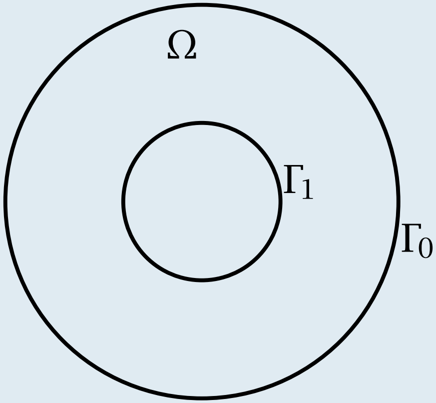
$$\text{for } f_j \in H^{1/2}(\Gamma_j), \qquad \Lambda_j u_j := \frac{\partial u_j}{\partial \boldsymbol{n}} \text{ on } \Gamma_j, \tag{23}$$

where $u_j \in H^1(\Omega)$ is solution to

$$\begin{cases} -\Delta u_j = 0 & \text{in } \Omega \\ u_j = 0 & \text{on } \Gamma_{1-j} \\ u_j = f_j & \text{on } \Gamma_j. \end{cases} \tag{24}$$

- Define also $\Lambda_{10} : H^{1/2}(\Gamma_1) \longrightarrow H^{-1/2}(\Gamma_0)$ as being

$$\Lambda_{10} f_1 := \frac{\partial u_1}{\partial \boldsymbol{n}} \text{ sur } \Gamma_0. \tag{25}$$



Cauchy problem in a ball

- Writing a possible solution to **(20)** in the form
$$u = u_0 + u_1$$
where u_j satisfies **(24)**, solving the Cauchy problem **(20)** is equivalent to find $f_1 \in H^{1/2}(\Gamma_1)$ such that
$$g_0 = \frac{\partial u}{\partial \boldsymbol{n}} = \Lambda_0 f_0 + \Lambda_{10} f_1.$$
- In other terms, the solution of **(20)** exists if, and only if, one can find $f_1 \in H^{1/2}(\Gamma_1)$ such that

(26) $$\Lambda_{10} f_1 = \Lambda_0 f_0 - g_0.$$

- So the compatibility condition on (f_0, g_0) is
$$\Lambda_0 f_0 - g_0 \in R(\Lambda_{10}).$$

Cauchy problem in a ball

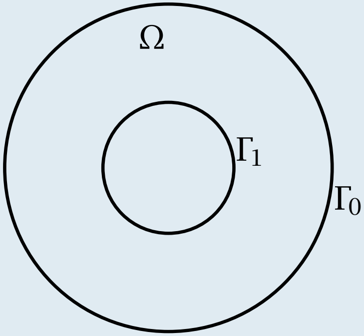
- Thus we need to characterize the range $R(\Lambda_{10})$ of Λ_{10} in $H^{-1/2}(\Gamma_0)$.
- For the annular domain Ω this can be done using a spectral analysis of the operators Λ_j, and a diagonal representation of the operator Λ_{10}, yielding the compatibility condition on f_0, g_0:

$$\sum_{k\in\mathbb{Z}} \alpha^{-2k}(1+|k|)\left(f_{0,k} - \frac{1}{1+|k|} g_{0,k}\right)^2 < \infty.$$

- The Cauchy problem

$$(27) \qquad \begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = f_1 & \text{sur } \Gamma_1 \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g_1 & \text{on } \Gamma_1 \end{cases}$$

for $(f_1, g_1) \in H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_1)$ can be studied in the same manner, yielding an analogous compatibility condition. ■



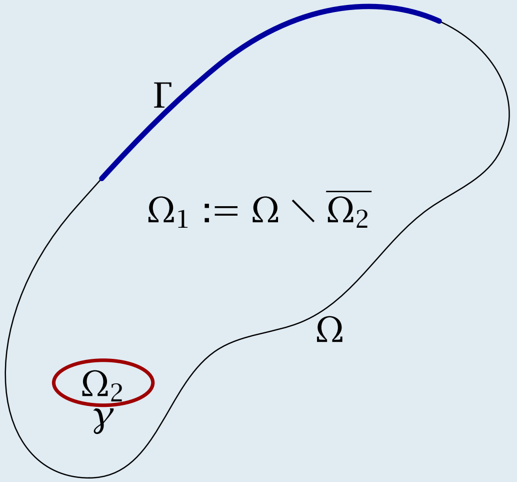
Langrangian control: idea of proof

- Again we introduce two Poincaré-Steklov operators as follows.
- For $\psi \in H_m^{-1/2}(\gamma)$ we solve

(28) $$\begin{cases} -\Delta\xi = 0 & \text{in } \Omega_1 \\ \dfrac{\partial\xi}{\partial\boldsymbol{n}} = \psi & \text{on } \gamma \\ \dfrac{\partial\xi}{\partial\boldsymbol{n}} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\gamma \xi(\sigma)\, d\sigma = 0 \end{cases}$$

- Then we define $\Lambda_1 : H_m^{-1/2}(\gamma) \longrightarrow H_m^{1/2}(\gamma)$ by

(29) $$\Lambda_1\psi := \xi_{|\gamma}.$$



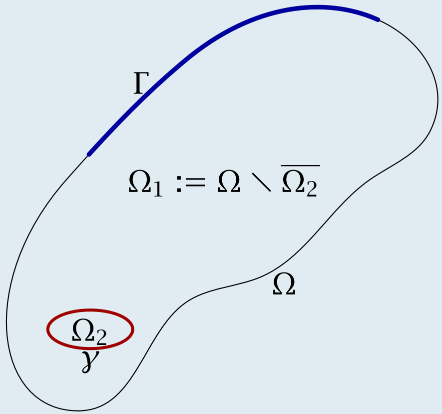
Langrangian control: idea of proof

- The second operator is defined by solving for $\varphi \in H_m^{1/2}(\gamma)$ the equation

(30) $$\begin{cases} -\Delta\zeta = 0 & \text{in } \Omega_2 \\ \zeta = \varphi & \text{on } \gamma \end{cases}$$

and then $\Lambda_2 := H_m^{1/2}(\gamma) \longrightarrow H_m^{-1/2}(\gamma)$ with

(31) $$\Lambda_2\varphi := \frac{\partial\zeta}{\partial \boldsymbol{n}} \quad \text{on } \gamma.$$



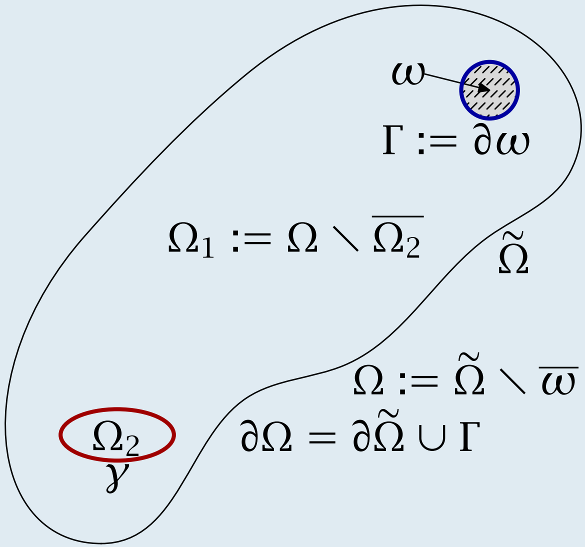
4 Lemma. *Define the operator T_{12} by*

$$T_{12}\varphi := \varphi + \Lambda_1\Lambda_2\varphi, \quad \text{for } \varphi \in H_m^{1/2}(\gamma).$$

Then $T_{12} : H_m^{1/2}(\gamma) \longrightarrow H_m^{1/2}(\gamma)$ is a homeomorphism.

Langrangian control: idea of proof

- For the proof of **Theorem 1**, the operator Λ_γ being defined by **(13)**, since $\overline{R(\Lambda_\gamma)} = N(\Lambda_\gamma^*)^\perp$ we have to show that $N(\Lambda_\gamma^*) = \{0\}$.
- The explicit determination of $\Lambda_\gamma^* : H_m^{-1/2}(\gamma) \longrightarrow H^{-1/2}(\Gamma)$ is not easy. We are going to establish an implicit representation formula for Λ_γ^*.



We proceed in two steps. In the first step we assume that the domain Ω and the piece of the boundary Γ, where the control is imposed, are as in the opposite figure.

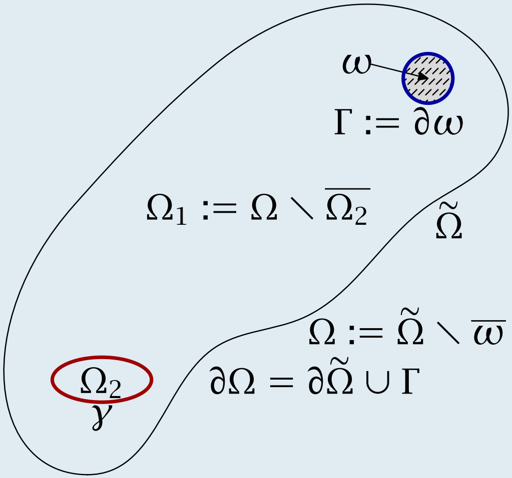
Langrangian control: idea of proof

- For $\varphi \in H_m^{1/2}(\gamma)$ and $v \in H_m^{-1/2}(\Gamma)$ we show first that

$$\langle \Lambda_\gamma v, \varphi \rangle = -\langle \Lambda_\gamma v, \Lambda_1 \Lambda_2 \varphi \rangle + \langle v, \xi \rangle,$$

where $\xi := \xi(-\Lambda_2 \varphi)$ satisfies **(32)** with $\psi := -\Lambda_2 \varphi \in H_m^{-1/2}(\gamma)$.

$$(32) \quad \begin{cases} -\Delta \xi = 0 & \text{in } \Omega_1 \\ \dfrac{\partial \xi}{\partial \boldsymbol{n}} = \psi & \text{on } \gamma \\ \dfrac{\partial \xi}{\partial \boldsymbol{n}} = 0 & \text{on } \partial\Omega \\ \displaystyle\int_\gamma \xi(\sigma)\, d\sigma = 0 & \end{cases}$$



- Therefore we have

$$\langle v, \Lambda_\gamma^* (I + \Lambda_1 \Lambda_2) \varphi \rangle = \langle v, \xi \rangle.$$

Langrangian control: idea of proof

- If for some $\tilde{\varphi} \in H_m^{1/2}(\gamma)$ we have $\Lambda_\gamma^* \tilde{\varphi} = 0$, then this is equivalent to say that letting φ be such that
$$(I + \Lambda_1 \Lambda_2)\varphi = \tilde{\varphi}$$
then for all $v \in H_m^{-1/2}(\Gamma)$ we have
$$\langle v, \xi \rangle = \langle v, \Lambda_\gamma^* \tilde{\varphi} \rangle = 0.$$
- This implies that $\xi = 0$ on Γ and since ξ satisfies **(32)** and $\partial \xi / \partial \boldsymbol{n} = 0$ on Γ, thanks to the unique continuation principle, we have $\xi \equiv 0$ in Ω_1.
- Also we have $\psi \equiv 0$ on γ, that is $\Lambda_2 \varphi = 0$.
- Then this means that if ζ is the solution of $-\Delta \zeta = 0$ in Ω_2 and $\zeta = \varphi$ on γ, we have
$$\Lambda_2 \varphi = \frac{\partial \zeta}{\partial \boldsymbol{n}} = 0 \quad \text{on } \gamma.$$
This implies that ζ is a constant in Ω_2, that is φ is a constant on γ and finally, since $\int_\gamma \varphi(\sigma)\, d\sigma = 0$, we have $\varphi \equiv 0$ and also $\tilde{\varphi} = 0$. $\square$

Langrangian control: idea of proof

- In a second step, we treat the general case with the help of this figure