Rapid stabilization of an unstable heat equation under boundary disturbance

Webinar: *Control in Time of Crisis*.

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Table of contents

1. Problem statement
2. Feedback design
3. Well-possedness of the closed-loop system
4. Numerical Simulations
5. Conclusions

Control system and problem statement

Let us to consider the following unstable heat equation with Neumann control,

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) = u(t) + d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1}$$

where, $T > 0$, $L > 0$, the state z, $a \in C([0,L])$, the time-dependent function u is a boundary control and d is an unknown disturbance.

Our goal is to design a feedback control which achieves a rapid exponential stabilization in presence of an unknown boundary disturbance. That is, the decay rate of the system in closed loop is exponential with a rate arbitrarily large.

Literature overview

Many control approaches have been developed to deal with the stabilization problem in presence of uncertainties in the context of PDE. Among them:

- **Adaptive control**:
  1. Wei Guo and Bao-Zhu Guo. “Adaptive Output Feedback Stabilization for One-Dimensional Wave Equation with Corrupted Observation by Harmonic Disturbance”. In: *SIAM Journal on Control and Optimization* 51.2 (2013)
  2. Tarek Ahmed-Ali et al. “Adaptive boundary observer for parabolic PDEs subject to domain and boundary parameter uncertainties”. In: *Automatica* 72 (2016). ISSN: 0005-1098
- **Active disturbance rejection control (ADRC)**:
  3. Dong Zhang, Shu-Xia Tang, and Scott J. Moura. “State and Disturbance Estimator for Unstable Reaction-Advection-Diffusion PDE with Boundary Disturbance”. In: *2019 Proceedings of the Conference on Control and its Applications (CT)*
  4. Hongyinping Feng, Cheng-Zhong Xu, and Peng-Fei Yao. “Observers and Disturbance Rejection Control for a Heat Equation”. In: *IEEE Transactions on Automatic Control* 65.11 (2020)

Feedback design

The main idea behind the feedback design is to split the control as follows

$$u = u_1 + u_2. \tag{2}$$

- u_1 deals with the instability caused by the a.
- u_2 deals with the boundary disturbance d.

Feedback design: u_1

Let us consider the following Backstepping transformation,

$$y(t,x) = z(t,x) - \int_0^x k(x,s;\omega) z(t,s)\,\mathrm{d}s, \tag{3}$$

where the kernel k is solution to

$$\begin{cases} k_{xx} - k_{ss} = (a(s) + \omega)k & (x,s) \in \Omega, \\ k_s(x,0) = 0, & x \in (0,L), \\ k(x,x) = -\frac{1}{2}\int_0^x a(s) + \omega\,\mathrm{d}s, & x \in (0,L), \end{cases} \tag{4}$$

where, $\Omega = \{(x,s) :\ 0 \leq s \leq x \leq L\}$, and ω is a positive constant which can be chose large.

- M. Krstic and A. Smyshlyaev. *Boundary Control of PDEs: A Course on Backstepping Designs*. Advances in Design and Control. Society for Industrial and Applied Mathematic, 2008

This transformation maps the following system

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) - k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s = 0, & t \in (0,\infty), \\ z(0,x) = z_0, & x \in (0,L), \end{cases} \tag{5}$$

into this **exponentially stable target system**

$$\begin{cases} y_t - y_{xx} = -\omega y, & (t,x) \in (0,\infty) \times (0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) = 0, & t \in (0,\infty), \\ y(0,x) = y_0, & x \in (0,L). \end{cases} \tag{6}$$

Then, we choose u_1 as follows

u_1 design

$$u_1(t) = k(L,L)z(t,L) + \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s. \tag{7}$$

Feedback design: u_2

We plug-in u_1 into to the control system (1) and using the Backstepping transformation, we get the following system

$$\begin{cases} y_t - y_{xx} = -\omega y, & (t,x) \in (0,\infty) \times (0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) = u_2(t) + d(t), & t \in (0,\infty), \\ y(0,x) = y_0, & x \in (0,L). \end{cases} \tag{8}$$

multiplying (8) by y and performing an integration by parts, it holds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y\|^2_{L^2(0,L)} + \omega\|y\|^2_{L^2(0,L)} = -\|y_x\|^2_{L^2(0,L)} + (u_2(t) + d(t))y(t,L). \tag{9}$$

Feedback design: u_2

u_2 design

$$u_2(t) = -Dsign(y(t,L)). \tag{10}$$

where $D > 0$ is a constant.

Where

$$\begin{aligned} sign &: \mathbf{R} \longrightarrow 2^{\mathbf{R}}, \\ p &\longmapsto sign(p) = \begin{cases} \frac{p}{|p|}, & \text{if } p \neq 0, \\ [-1,1], & \text{if } p = 0, \end{cases} \end{aligned} \tag{11}$$

- **A1** There exist a positive constant D, such that $|d(t)| \leq D$, for all $t \geq 0$.

with this we obtain that (thanks to $psign(p) = |p|$)

$$-\|y_x\|^2_{L^2(0,L)} - Dsign(y(t,L))y(t,L) + d(t)y(t,L) \leq 0, \ \forall t \geq 0. \tag{12}$$

therefore the solution y, satisfies that $\|y(t,\cdot)\|^2_{L^2(0,L)} \leq e^{-2\omega t}\|y_0\|^2_{L^2(0,L)}$

Feedback design

Feedback law u in variable z

$$u(t) = k(L,L)z(t,L) + \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s - D\mathit{sign}\left(z(t,L) + \int_0^L k(L,s)z(t,s)\,\mathrm{d}s\right) \tag{13}$$

Then the closed-loop system is given by

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) - k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,\mathrm{d}s & \\ +D\mathit{sign}\left(z(t,L) - \int_0^L k(L,s)z(t,s)\,\mathrm{d}s\right) \ni d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{14}$$

Well-possedness of the closed-loop system

To begin with, we apply the backtstepping transformation to previous closed-loop system to get

$$\begin{cases} y_t - y_{xx} = -\omega y, & (t,x) \in (0,\infty) \times (0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) + Dsign(y(t,L) \ni d(t), & t \in (0,\infty), \\ y(0,x) = y_0, & x \in (0,L). \end{cases} \tag{15}$$

Let us consider the following change of variable

$$w(t,x) = y(t,x) - \phi(x)d(t) \tag{16}$$

where $\phi : [0,L] \to \mathrm{R}$ is a function smooth enough such that $\phi'(0) = \phi(L) = 0$ and $\phi'(L) = 1$. Then, w satisfies the following differential inclusion

$$\begin{cases} w_t - w_{xx} = -\omega w + f, & (t,x) \in (0,\infty) \times (0,L), \\ w_x(t,0) = 0, & t \in (0,\infty), \\ w_x(t,L) + Dsign(w(t,L)) \ni 0, & t \in (0,\infty), \\ w(0,x) = w_0(x), & x \in (0,L), \end{cases} \tag{17}$$

where $f = -\phi\dot{d} + \phi'' d + \omega\phi d$.

Let us introduce the following operator

$$\mathcal{A} : D(\mathcal{A}) \subset L^2(0,L) \longrightarrow L^2(0,L), \tag{18}$$

$$p \longmapsto \mathcal{A}p = -p'' + \omega p, \tag{19}$$

where $\omega > 0$ and the domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \left\{ p \in L^2(0,L)/Ap \in L^2 : p'(0) = 0,\; p'(L) + Dsign(p(L)) \ni 0 \right\}. \tag{20}$$

Remark

The operator $\mathcal{A}$ is not linear.

Now, the differential inclusion (17) can be written in a operator form as follows,

$$\begin{cases} w_t + \mathcal{A}w = f, \quad t \in (0,\infty), \\ w(0) = w_0. \end{cases} \tag{21}$$

Our first task is to prove the well-posednes of (21). To do that, we prove the following result

Proposition

The operator $\mathcal{A}$ is a maximal monotone operator.

Proof proposition 3.1

By Minty's theorem, $\mathcal{A}$ is a maximal monotone operator if and only if

1. $\mathcal{A}$ is monotone, i.e for all $u, v \in D(\mathcal{A})$, $(\mathcal{A}(u-v), u-v)_{L^2(0,L)} \geq 0$.
2. For any $f \in L^2(0,L)$ there exist $p \in D(\mathcal{A})$ such that

$$p + \mathcal{A}p = f,$$

almost everywhere $x \in (0,L)$.

In order to prove that operator $\mathcal{I} + \mathcal{A}$ has full rank.

Let us consider the Hilbert space $H^1(0, L)$ and the following functional, $\mathcal{J} : H^1(0, L) \longrightarrow \mathbf{R}$, defined by

$$\mathcal{J}(p) = \frac{1}{2}\int_0^L (p')^2 + (\omega + 1)p^2 - fp\,\mathrm{d}x + \varphi_\lambda(p(L)), \tag{22}$$

where,

- $\varphi_\lambda : \mathbf{R} \to \mathbf{R}$ is the Moreau Regularization of the function

$$\varphi : \mathbf{R} \to \mathbf{R} \quad \varphi = D|x|.$$

Besides,

- $\alpha(x) = (\partial\varphi)(x) = Dsign(x)$,
- $J_\lambda = (I + \lambda\alpha)^{-1}$, the resolvent of α.
- $\alpha_\lambda :\ \mathbf{R} \to \mathbf{R}$, $\alpha_\lambda = \frac{1}{\lambda}(I - J_\lambda)$. (The Yosida Approximation of α)

By the Moreau Theorem, φ_λ is a convex, differentiable function and

$$\varphi_\lambda(x) = \frac{\lambda}{2}|\alpha_\lambda|^2 + \varphi(J_\lambda(x)), \qquad \varphi'_\lambda(x) = \alpha_\lambda(x).$$

Lemma

For all $\lambda > 0$ and for all $f \in L^2(0,L)$, there exist a minimizer m_λ of $\mathcal{J}$ such that $m_\lambda \in H^2(0,L)$ and

$$\begin{cases} m_\lambda + \mathcal{A}m_\lambda = f \quad x \in (0,L), \; a.e. \\ m_\lambda'(0) = 0 \quad m_\lambda'(L) + \alpha_\lambda(m_\lambda(L)) = 0. \end{cases} \tag{23}$$

Moreover, the minimizer m_λ satisfies the following inequalities. There exist positives constants C_i, $i \in \{1,2,3\}$ such that, for any $\lambda > 0$

$$\|m_\lambda\|_{H^1(0,L)} \leq C_1 \|f\|_{L^2(0,L)}, \tag{24}$$

$$|\alpha_\lambda(m_\lambda(L))| \leq C_2 \|f\|_{L^2(0,L)}, \tag{25}$$

$$\|m_\lambda\|_{H^2(0,L)} \leq C_3 \|f\|_{L^2(0,L)}. \tag{26}$$

Finally, it can be prove that $(m_\lambda)_{\lambda>0}$ converges to m solution to

$$\begin{cases} m + \mathcal{A}m = f \quad x \in (0,L), \; a.e. \\ m'(0) = 0 \quad m'(L) + D\text{sign}(m(L)) = 0 \end{cases} \tag{27}$$

when $\lambda \to 0^+$. Therefore $\mathcal{A}$ is a maximal monotone operator.

- **A2** Let us assume that $d \in W^{2,1}(0,\infty)$ and $d(0) = 0$.

This implies that $f \in W^{1,1}(0,\infty; L^2(0,L))$, then by following theorem

Theorem (Kato, Theorem 4.1 in [2])

] Let $\mathcal{B}$ be a maximal monotone operator, $z_0 \in D(\mathcal{A})$ and $g \in W^{1,1}(0,\infty; L^2(0,L))$. Then there exisit a unique $z \in W^{1,1}(0,\infty; L^2(0,L))$ such that

1. $z(0) = z_0$,
2. $z_t + \mathcal{B}z \ni g \quad a.e \quad t \in (0,\infty)$,
3. $z(t) \in D(\mathcal{A}) \quad a.e \quad t \in (0,\infty)$.

Thus, if $w_0 \in D(\mathcal{A})$, there is an unique solution $w \in W^{1,1}(0,\infty; L^2(0,L))$ to

$$\begin{cases} w_t + \mathcal{A}w = f, & t \in (0,\infty), \\ w(0) = w_0 \end{cases} \tag{28}$$

[2] R. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Vol. 49. Mathematical Surveys and Monographs. American Mathematical Society, 1997.

By the change of variable $w(t,x) = y(t,x) - \phi(x)d(t)$.

It holds that, if $y_0 \in D(\mathcal{A})$, there exists a unique solution $y \in W^{1,1}(0,\infty; L^2(0,L))$

$$\begin{cases} y_t - y_{xx} = -\omega y, & (t,x) \in (0,\infty) \times (0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) + Dsign(y(t,L) \ni d(t), & t \in (0,\infty), \\ y(0,x) = y_0, & x \in (0,L). \end{cases} \tag{29}$$

Now, using the inverse of the backstepping transformation given by

$$z(t,x) = y(t,x) - \int_0^x l(x,s) y(t,s)\, \mathrm{d}s \tag{30}$$

where the kernel $l(x,s)$, is solution to

$$\begin{cases} l_{xx} - l_{ss} = -(a(s) + \omega) l & (x,s) \in \Omega, \\ l_s(x,0) = 0, & x \in (0,L), \\ l(x,x) = -\frac{1}{2} \int_0^x a(s) + \omega\, \mathrm{d}s, & x \in (0,L), \end{cases} \tag{31}$$

We conclude that, if

$$\left\{ z_0 \in H^2(0,L), \text{ such that } y_0'(0) = 0 \text{ and } y_0'(L) + D\text{sign}(y_0(L)) \ni 0 \right\}, \tag{32}$$

where $y_0(x) = z_0(x) + \int_0^x k(x,s) z_0(s)\,\mathrm{d}s$.

There exists an unique $z \in W^{1,1}(0,\infty; L^2(0,L))$ solution to the closed-loop system

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) - k(L,L) z(t,L) - \int\limits_0^L k_x(L,s) z(t,s)\,\mathrm{d}s & \\ +Dsign\left(z(t,L) - \int\limits_0^L k(L,s) z(t,s)\,\mathrm{d}s \right) \ni d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{33}$$

Finally, the exponential decay follows from the continuity of the backstepping transformation and its inverse.

That is, exist C_1, and C_2, positive constants such that

$$\|z(t,\cdot)\|^2_{L^2(0,L)} \leq C_1\|y(t,\cdot)\|^2_{L^2(0,L)} \leq C_1 e^{-2\omega t}\|y_0\|^2_{L^2(0,L)} \leq C_1 C_2 e^{-2\omega t}\|z_0\|^2_{L^2(0,L)} \tag{34}$$

Uncontrolled disturbed system

$$\begin{cases} y_t - y_{xx} = -\omega y, \\ y_x(t,0) = 0, \\ y_x(t,L) = d(t), \\ y(0,x) = y_0. \end{cases} \tag{35}$$

Here we have choose as disturbance, $d(t) = 2\sin(2t)$, decay parameter $\omega = 1$ and initial condition $y_0 = 2(1 - \frac{x^2}{2})$.

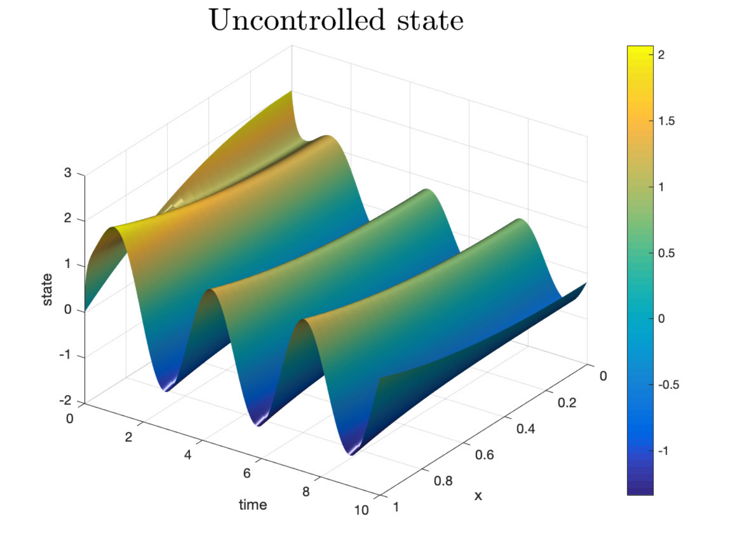

Figure 1: Uncontrolled state

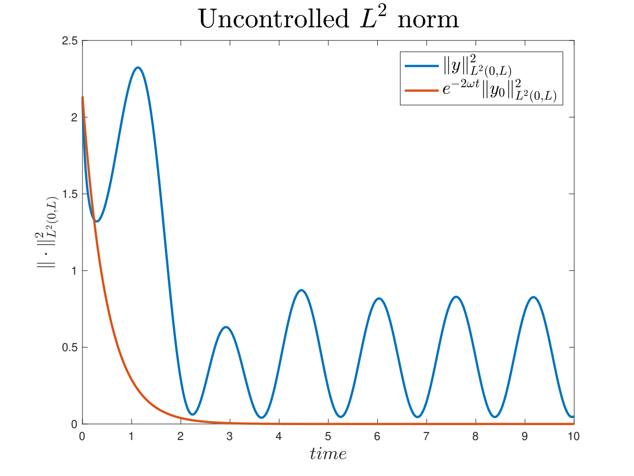

Figure 2: Uncontrolled $L^2(0, L)$ norm

Controlled disturbed system

$$\begin{cases} y_t - y_{xx} = -\omega y, \\ y_x(t,0) = 0, \\ y_x(t,L) + D sign(y(t,L)) \ni d(t) \\ y(0,x) = y_0, \end{cases} \quad (36)$$

Simulations parameters, $d(t) = 2\sin(2t)$, $\omega = 1$ and $D = 2$, $y_0 = 2(1 - \frac{x^2}{2})$.

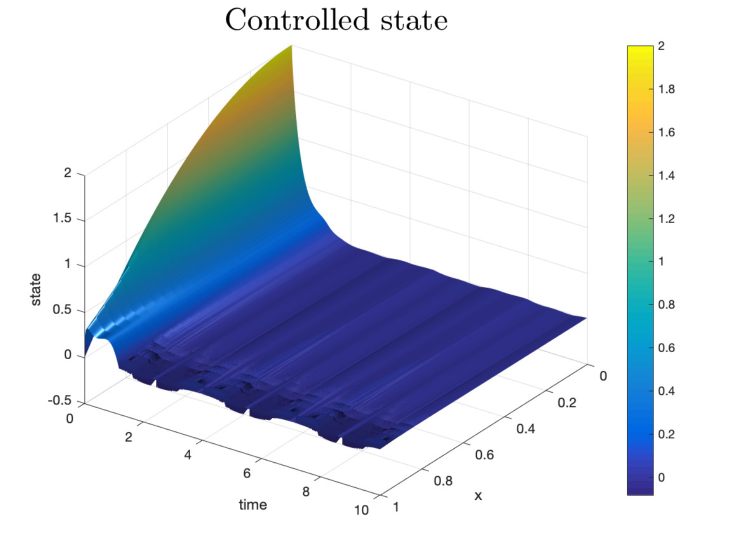

Figure 3: controlled state

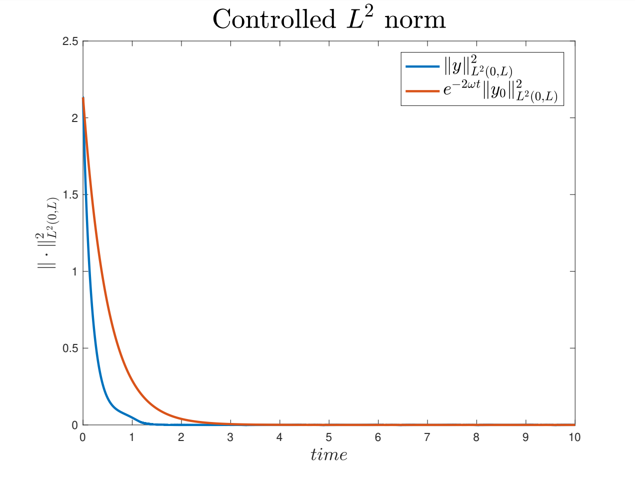

Figure 4: controlled $L^2(0, L)$ norm

Conclusions

- In this talk the rapid stabilization problem for an unstable heat equation under boundary disturbances, was addressed.
- The control design rely on a combination of the backstepping method and the suitable use of the multivalued operator $\text{sign}(\cdot)$.
- The well-possedness of the closed-loop system was proved by using the theory of maximal monotone operators.
- Numerical simulations was performed in order to illustrate the theoretical results.

Thanks for your attention!