# Local Null Controllability for a Chemotaxis System

Stephen Guffey

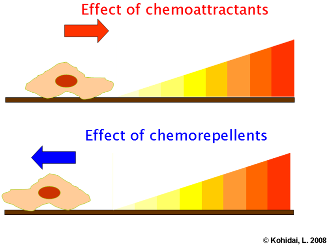
Department of Mathematical Sciences
University of Memphis
**Control in Times of Crisis**

# Outline

1. PDE Analysis
   - Introduction and Background
   - Existence Results
2. Control Theoretic Analysis
   - Local Null Controllability of Heat
   - Null Controllability of Linear System
3. Future Work and Conclusions

# Chemotaxis

We consider a system involving chemotaxis, the process of motile organisms moving in the direction of an increasing (or decreasing) chemical gradient. This gradient is created by concentrations of substances known as chemoattractants (chemorepellents).



# Model for Bacterial Infection

We consider a model for a radially-symmetric wound with bacterial infection, modelled after Schugart et al. (2008)[1]:

$$\begin{cases} w_t &= w_{xx} - \lambda_1 n w - \lambda_2 b w - \lambda_3 w \\ n_t &= n_{xx} - \lambda_6 (n c_x)_x + \lambda_7 b n (1 - n) \\ b_t &= \lambda_8 b(1-b) - b \dfrac{w}{\lambda_9 + w} \dfrac{\lambda_{10} + \lambda_{11} n}{\lambda_{12} b + 1} \\ c_t &= \lambda_{15} c_{xx} + \lambda_{13} b - \lambda_{14} c \end{cases} \tag{1}$$

in $(0, T] \times \Omega = (0, T] \times (0, 1)$. Boundary conditions are taken to be zero Neumann conditions.

Remarks:

- The system has a high nonlinear coupling.
- Bacterial equation is degenerate.
- The term $(nc_x)_x$ in the $n$ equation represents chemotactic response.

[1] R. Schugart, A. Friedman, R. Zhou, C Sen, *Wound angiogenesis as a function of tissue oxygen tension: A mathematical model.* PNAS **105**, 2628–2633.

# Classical Keller-Segel Model

One prominent mathematical model to describe this phenomena was introduced in 1970 by Keller and Segel[2].

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)), & \text{in } (0,T] \times \Omega \\ v_t = \Delta v - v + u, & \text{in } (0,T] \times \Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{in } (0,T] \times \partial\Omega \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where $\chi(v)$ has several proposed forms, such as polynomial, logarithmic, or rational-type functions. Here $u = u(t,x)$ is the cell density at position $x$ and time $t$, and $v = v(t,x)$ is the density of the chemoattractant.

[2] E.F. Keller & L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*. J. Theor. Bio. **26** (1970), 399–415.

# Known Results for Classical K-S Chemotaxis

- (Osaki-Yagi) (2001) Suppose $\Omega \subset \mathbb{R}$ is a domain, and that $u_0 \in C\left(\overline{\Omega}\right)$ and $v_0 \in \cup_{q>1} W^{1,q}(\Omega)$ are given non-negative functions. Then $(u, v)$ is global and bounded in the sense there exists a $C > 0$ (depending on the initial conditions) such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t > 0.$$

- (Nagai-Senba-Yoshida)(1997) Let $n = 2$. If $\int_\Omega u_0 dx < 4\pi$, then $(u, v)$ exists globally and there exists a $C > 0$ (depending on the initial conditions) such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t > 0.$$

  . If, moreover, $\Omega$ is a disk and $(u_0, v_0)$ is radially symmetric, then the same conclusion holds under the assumption that $\int_\Omega u_0 dx < 8\pi$.
- When in $\mathbb{R}^2$, Herrero and Velázquez (Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1997) constructed radially-symmetric solutions (depending on particular radially-symmetric initial data) that blow up in finite time.

# Local Existence

We consider the system (1) with Neumann boundary conditions.

## Proposition 1 (Guffey)

For all $T_0 > 0$ there exists an $R_1 > 0$ such that for all initial data $X_0 = [w_0, n_0, c_0, b_0]^T \in [H^1(\Omega)]^4$ with $\|X_0\| < R_1$ there is a solution

$$\begin{aligned} X &= [w, n, c, b]^T \\ &\in [H^1(0, T_0; L^2(0,1)) \cap L^2(0, T_0; H^2(0,1)]^3 \times H^1(0, T_0; H^1(0,1)) \end{aligned}$$

to (1) with zero Neumann conditions.

Mechanics of proof:

- Introduce operators $A_i$ incorporating the action of the Laplacian with associated boundary conditions, writing the system as an abstract ODE system. The $A_i$'s are generators of analytic semigroups.
- Use maximal regularity for

$$y_t = \Delta y + f, y(0) = y_0$$

to obtain the mapping

$$f \mapsto A_i \int_0^t e^{A_i(t-s)} f(s) ds$$

is bounded from $L^2(0, T; L^2(\Omega))$ to $L^2(0, T; L^2(\Omega))$ [a]. It should be noted that $A_i \int_0^t e^{A_i(t-s)} f(s) ds \sim \frac{1}{t}$ and hence is singular along $t = 0$.

- Show the nonlinear terms define locally Lipschitz operators from the the ball into the space.
- Choose small enough initial conditions to apply Contraction Mapping Principle.

---

[a] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser Verlag. 1995.

# Numerical Simulations

For suitably chosen parameter values and initial conditions, the solutions to the system appear to be well behaved. Numerical approximations were developed using a finite difference scheme for the following initial conditions:

$$
\begin{aligned}
w_0(x) &= 1 \\
n_0(x) &= x^2 e^{-(1-x)^2} \\
b_0(x) &= (1-x)^2 e^{-x^2} \\
c_0(x) &= (1-x)^2 e^{-x^2}
\end{aligned}
$$

# Positivity of Solutions

Suppose we restrict to initial conditions corresponding to concentration variables

$$\mathcal{K} = \{f \in H^1(0,1) : 0 \leq f(x) \leq 1 \text{ for all } x \in (0,1)\}.$$

## Theorem 2.1 (Guffey)

*Suppose $w_0, n_0, b_0, c_0 \in \mathcal{K}$. Then the local solutions to (1) satisfy $w(t,x), n(t,x), b(t,x), c(t,x) \geq 0$ for all $x \in (0,1), 0 \leq t \leq T_0$.*

Idea: Introduce $C^{1,1}(\mathbb{R})$ function

$$H(s) = \begin{cases} \frac{1}{2}s^2, & -\infty < s < 0 \\ 0, & 0 \leq s < \infty \end{cases}$$

This is used to define the test function $\Psi(t) = \int_\Omega H(u(t,x))\,\mathrm{d}x$, used to achieve a Grönwall inequality $\Psi'(t) \leq a(t)\Psi(t)$. From this and the fact that $\Psi(0) = 0$ we obtain $u(t,x) \geq 0$ .

# Existence on the Cone on Positive I.C.'s

Again consider system (1) with zero Neumann conditions. Let

$$\mathcal{K} = \{f \in H^1(0,1) : 0 \leq f(x) \leq 1 \text{ for all } x \in (0,1)\}.$$

Then we have the following

## Proposition 2 (Guffey)

For all initial data $X_0 = [w_0, n_0, c_0, b_0]^T \in [\mathcal{K}^4[H^1(\Omega)]^4$ there is a $T_0 > 0$ such that there is a **nonnegative solution**

$$\begin{aligned} X &= [w, n, c, b]^T \\ &\in [H^1(0, T_0; L^2(0,1)) \cap L^2(0, T_0; H^2(0,1)]^3 \times H^1(0, T_0; H^1(0,1)) \end{aligned}$$

to (1) with zero Neumann conditions.

# Local Null Control Problem

Recall our system, now with controllers $u_i, i \in 1, 2, 4$:

$$\begin{cases} w_t &= w_{xx} - \lambda_1 nw - \lambda_2 bw - \lambda_3 w + \chi_{\omega_1} u_1 \\ n_t &= n_{xx} - \lambda_6 (nc_x)_x + \lambda_7 bn(1-n) + \chi_{\omega_2} u_2 \\ b_t &= \lambda_8 b(1-b) - b\dfrac{w}{\lambda_9 + w}\dfrac{\lambda_{10} + \lambda_{11} n}{\lambda_{12} b + 1} + u_3 \\ c_t &= \lambda_{15} c_{xx} + \lambda_{13} b - \lambda_{14} c + \chi_{\omega_4} u_4. \end{cases} \tag{3}$$

Here the sets $\omega_i \subset\subset (0, 1)$ and $u_i \in L^2(0, T; L^2(0, 1))$ for each $i$. We want the following problem: Given $T > 0, \omega_i$ for $i = 1, 2, 4$ are there controllers $u_i$ such that $w(T, x) = n(T, x) = b(T, x) = c(T, x) = 0$ in $L^2(0, 1)$?

# Linear Controllability

We consider the controlled linear part of our system. Let $\omega$ be a smooth subdomain, compactly supported in $(0,1)$. Consider

$$\begin{cases} w_t & -w_{xx} + \lambda_3 w = \alpha_1 \chi_{\omega_1} u_1 \\ n_t & -\lambda_5 n_{xx} = \alpha_2 \chi_{\omega_2} u_2 \\ b_t & -\lambda_8 b = \alpha_3 u_3 \\ c_t & +\lambda_{13} c_{xx} - \lambda_{14} b + \lambda_{15} c = \alpha_4 \chi_{\omega_4} u_4 \end{cases} \tag{4}$$

with associated boundary conditions and initial conditions. Studying the controllability of this linear problem will provide crucial information for the control-to-state map, used in the nonlinear problem.

We have the following result

## Theorem 3.1 (Guffey)

*For given $T > 0$, the linear system*

$$\begin{cases} w_t & -w_{xx} + \lambda_3 w = \chi_1 u_1 \\ n_t & -\lambda_5 n_{xx} = \chi_2 u_2 \\ b_t & -\lambda_8 b = u_3 \\ c_t & +\lambda_{13} c_{xx} - \lambda_{14} b + \lambda_{15} c = 0. \end{cases} \tag{5}$$

*in $(0, T] \times (0, 1)$ with associated boundary and initial conditions is null controllable with $\omega_i \subset\subset \Omega$ for $i = 1, 2$. with controls in $\left[L^2(0, T; L^2(\Omega))\right]^3 \times \{0\}$.*

# Localized Control of Heat

Consider the following heat equation. Let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$ with $C^2$ boundary $\Gamma$. Let $\omega \subset \Omega$ be open with compact closure in $\Omega$ be sufficiently smooth.

$$\begin{cases} y_t = \Delta y + \chi_\omega u, & \text{in } \Omega \times (0,T) \\ \dfrac{\partial y}{\partial \nu} = 0, & \text{on } \Gamma \times (0,T) \\ y(0,x) = y_0(x), & \text{in } \Omega. \end{cases} \tag{6}$$

where $u$ is the control in $L^2(0,T;L^2(\Omega))$, $y_0$ is a given function from $L^2(\Omega)$, $\nu$ denotes the outward normal vector of $\Omega$ and $\chi_\omega$ denotes the characteristic function on $\omega$.

# Dual Problem to (4)

The dual problem to (5) is the following **inverse problem**: let $\Omega$ be a bounded, open subset of $\mathbb{R}^n$ with $C^2$ boundary $\Gamma$. Let $\omega \subset \Omega$ be open with compact closure in $\Omega$ and suppose $\omega$ is sufficiently smooth. Consider the backward heat equation with final time condition:

$$\begin{cases} \phi_t + \Delta\phi = 0, & \text{in } \Omega \times (0,T) \\ \dfrac{\partial \phi}{\partial \nu} = 0, & \text{on } \Gamma \times (0,T) \\ \phi(T,x) = \phi^T(x), & \text{in } \Omega. \end{cases} \tag{7}$$

We wish to obtain the following observability inequality:

$$\int_{\Omega} |\phi(x,0)|^2 dx \leq C \int_0^T \int_{\omega} |\phi(x,t)|^2 dx dt, \tag{8}$$

This inequality (in the terms of inverse problems) tells us we can reconstruct the initial conditions from the restricted observation on $\omega \times (0,T)$. **Though this inequality appears like an energy estimate, the restriction on the observation makes the problem nontrivial.** To establish this inequality we require Carleman estimates.

# Carleman Estimates: A brief history

To this end, we will require the use of a more generalized type of energy method that relies on the use of Carleman estimates.

- The technique was first introduced in Carleman, T. (1939). *Sur un Probléme d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes.*
- The method was revived, in part, by inclusion in the Hörmander texts *The Analysis of Linear Partial Differential Operators, I-IV* (1983,1985). Hörmander derived results for variable $C_0^\infty$ coefficients.
- The method was further developed and adopted to control theory for parabolic phenomena in recent years by Yu. Immanuvilov, A. Fursikov, E. Fernández-Cara , S. Guerrero, E. Zuazua, *etc*.
- Of particular interest is the exposition on Carleman estimates for parabolic problems found in E. Fernández-Cara and S. Guerrero "Global Carleman Inequalities for Parabolic Systems and Applications to Controllability." (2006).

# Carleman Estimates:Basics

The general localized Carleman inequality has the form

$$\int_0^T \int_\Omega \rho^2 |\phi|^2 dxdt \leq C_T \int_0^T \int_\omega \rho^2 |\phi|^2 dxdt, \quad C_T \to \infty \text{ as } T \to 0$$

where $\rho = \rho(x,t)$ is a continuous, positive weight function vanishing strongly at 0, $T$. We use the above inequality and properties of the weights to obtain estimate of the form

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\phi|^2 dxdt \leq C \int_0^T \int_\omega |\phi|^2 dxdt.$$

This estimate coupled with an energy estimate of the solution lead to to observability inequality. In the case of problem (6), this becomes

$$\int_\Omega |\phi(x,0)|^2 dx \leq C \int_0^T \int_\omega |\phi(x,t)|^2 dxdt,$$

whereby we gain a bound on the initial conditions that are unknown in the inverse problem.

The global Carleman inequality for these problems usually start with invoking the following lemma, found in "Global Carleman Inequalities and for Parabolic System and Applications to Controllability":

### Lemma 4.1

*There exists constants* $\lambda_1 = C(\omega, \Omega) > 1, s_1 = C(\omega, \Omega)(T + T^2)$ *and* $C_1(\omega, \Omega)$ *such that, for any* $\lambda \geq \lambda_1, s \geq s_1$ *the following inequality holds:*

$$s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|q_t|^2 + |\Delta q|^2) \, dx \, dt + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla|^2 \, dx \, dt$$
$$+ s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt \leq C_1 \left( \iint_Q e^{-2s\alpha} |q_t + \Delta q|^2 \, dx \, dt \right.$$
$$\left. s^3 \lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt \right).$$

for all $q \in C^2(\overline{Q})$ with $q = 0$ on $\Sigma$. Here $\alpha, \xi$ are weight functions which blow up as $t \to 0, T$.

# Observability inequality

Returning to (4), our equivalent condition for null controllability is

$$\int_\Omega |\phi_1(x,0)|^2 \, dx + \int_\Omega |\phi_2(x,0)|^2 \, dx + \int_\Omega |\phi_3(x,0)|^2 \, dx + \int_\Omega |\phi_4(x,0)|^2 \, dx$$
$$\leq C \left( \int_0^T \int_{\omega_1} |\phi_1(x,t)|^2 dx \, dt + \int_0^T \int_{\omega_2} |\phi_2(x,t)|^2 \, dx \, dt + \int_0^T \int_\Omega |\phi_3(x,t)|^2 \, dx \, dt \right)$$

We note that the first use of the Carleman estimate is to decouple the last off-diagonal terms in the matrix representation of our linear system. Further use is to eliminate the need for the controller on $c$, corresponding to setting $\alpha_4 = 0$.

# Nonlinear Problem

With the linear system controlled, we can turn our attention to the nonlinear system (3). The approach is similar to the arguments found in M. E. Bradley (1998) "Local Controllabililty of a Nonlinear Shallow Spherical Shell." The solution to the nonlinear problem (given by the existence results) can be written as the abstract vector problem

$$\begin{aligned} y(t) &= e^{At}y_0 + \int_0^t e^{A(t-s)}Bu(s)\mathrm{d}s + \int_0^t e^{A(t-s)}F(y(u(s)))\mathrm{d}s \\ &:= e^{At}y_0 + \mathcal{M}_t(u, y(u)), \end{aligned}$$

where $A$ is the generator of a $c_0$ semigroup and $B$ is the restriction of the controllers.

# Method for Nonlinear Problem

The control-to-state operator $\mathcal{M}_t : L^2(0, T; L^2(\omega)) \to X$, where $X$ is the range of the semigroup at time $T$. To have local null controllability, for us, means we which to find a control $u$ so that

$$0 = e^{AT} y_0 + \mathcal{M}_t(u, y(u)),$$

which means we want $\mathcal{M}_t$ to be a homeomorphism, locally.
We show via Inverse Function Theorem, where the linear control problem is used to show that the Fréchet derviative $D\mathcal{M}_t$ is boundedly invertible near the origin.

# Conclusion

I have attained the following results:

1. Our system (1) with associated boundary conditions is well-posed for small time (or small initial conditions).
2. The system admits positive solutions for positive initial conditions.
3. The linearized system (4) is null controllable with 3 controls, 2 of which can be localized. This number is optimal in the sense that no reduction of controllers is possible with this linearization.
4. (In progress) The nonlinear system (3) is null controllable with 3 controllers with 2 localized controllers.

# Future Work

We have the following directions for future research.

- (PDE Problem) Study stability of system to equilibrium states.
- (Control Theory Problem 1) Apply techniques on nonlinear functional analysis to gain local control of nonlinear problem.
- (Control Theory Problem 2) Determine the relationship between the null-controllers and the time for null controllability; *i.e.*, determine the size of the controllers as $T \to 0$.
- (Numerical Problem 1) Study the convergence and stability of the difference scheme.
- (Numerical Problem 2) Develop more robust methods. In particular derivation of a positive, mass conserving scheme would be beneficial. Operator splitting (such as Strang splitting) are useful for isolating the reaction-diffusion-advection processes.

# References


- P. Albano & P. Cannarsa, *Lectures on Carleman Estimates for Elliptic and Parabolic Operators with Applications*.
- N. Bellomo, A. Belloquid, Y. Tao, M Winkler, *Towards a mathematical thoery of Keller-Segel models of pattern formation in biological tissues*. Mathematical Models and Methods in Applied Sciences. **25** (2015), 1663-1763.
- X. Cao, *Global bounded solutions of the higher-dimensional Keller–Segel system under smallness conditions in optimal spaces*. Discrete Contin. Dynam. Syst. Ser. A **35** (2015), 1891–1904.
- E. Fernádez-Cara & S. Guerrero, *Global Carleman Estimates for Parabolic Systems and Applications to Controllability*. J. Control Optim. **45** (2006), 1395-1446.
- E. Feireisl, P. Laurenc˜ot and H. Petzeltovo, *On convergence to equilibria for the Keller–Segel chemotaxis model*. J. Differential Equations **236** (2010), 551–569.
- M. A. Herrero and J. J. L. Vel´azquez, *A blow-up mechanism for a chemotaxis model*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24** (1997), 633–683.
- E.F. Keller & L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*. J. Theor. Bio. **26** (1970), 399-415.
- S. Kesavan, *Topics in Functional Analysis*. John Wiley & Sons. 1989.

# References 2


- A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser Verlag. (1995).
- T. Nagai, T. Senba and K. Yoshida, *Global existence of solutions to the parabolic systems of chemotaxis*. RIMS Kokyuroku **1009** (1997), 22–28.
- K. Osaki, T. Tsujikawa, A. Yagi and M. Mimura, *Exponential attractor for a chemotaxis-growth system of equations*. Nonlinear Anal. **51** (2002) 119–144.
- K. Osaki and A. Yagi, *Finite-dimensional attractor for one-dimensional Keller–Segel equations*. Funkcial. Ekvac. **44** (2001), 441–469.
- A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag. 1983.
- R. Schugart, A. Friedman, R. Zhou, C Sen, *Wound angiogenesis as a function of tissue oxygen tension: A mathetical model.* PNAS **105**, 2628-2633.
- M. Winkler, *Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source*. Comm. Partial Differential Equations **35** (2010), 1516– 1537.
- M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system*. J. Math. Pures Appl. **100** (2013), 748–767.
- A. Yagi, *Abstract Parabolic Evolution Equations and their Applications*. Springer Monographs in Mathematics. 2010.
- P. Zheng, C. Mu, R. Willie, X. Hu, *Global asymptotic stability of steady states in a chemotaxis-growth system with singular sensitivity*. Computers & Mathematics with applications. **75** (2018) 1667-1675.