Finite-dimensional observer-based control of parabolic PDEs

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Outline

1. Introduction: delays, spatial & modal decomposition
2. Constructive finite-dimensional observer-based control
3. Delayed and sampled-data implementation
4. Predictors and subpredictors
5. Semilinear PDEs

Plan

Effects of delay on stability of PDEs

For **PDEs** arbitrarily **small delays** may **destabilize** the system

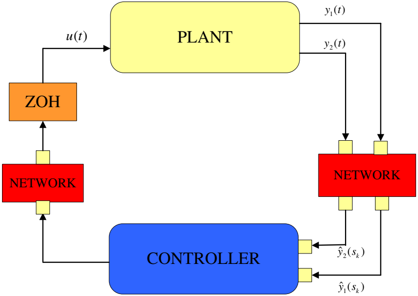
[Datko, SICON'88], [Logemann et al., SICON'96], [Wang, Guo & Krstic, SICON'11]

- The stability of wave eq. is not robust w.r.t. arbitrary small delay:

$$\begin{array}{l} z_{tt}(\xi,t) = z_{\xi\xi}(\xi,t), \quad \xi \in (0,1), \\ z(0,t) = 0, \quad z_\xi(1,t) = -z_t(1,t-h) \end{array}$$

- For $h = 0$ all solutions are zero for $t \geq 2$!
- For arbitrary small $h > 0$ the system has unbounded solutions

Networked control systems are systems, where sensors, controller and actuators *exchange data via communication network*.



Benefits: long distant estimation/control, etc.
Imperfections: variable sampling + delays + ...

Motivation: network-based control of PDEs

- Chemical reactors
- Air-polluted areas
- Multi-agents

Figure 1: 800 drone show in Nanchang: multi-agent deployment

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Objective - robust to input/output delays control of PDEs

Simple Lyapunov functionals for ODEs with tvr delays

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau(t)), \quad \tau(t) \le h.$$

- Employ $x(t-\tau) = x(t) - \int_{t-\tau}^{t} \dot{x}(s)ds \Rightarrow$

$$\dot{x}(t) = (A + A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} \dot{x}(s)ds.$$

- $V_P(x(t)) = x^T(t)Px(t) \Rightarrow$

$$\frac{d}{dt}V_P = 2x^T(t)P[(A + A_1)x(t) - A_1 \int_{t-\tau(t)}^{t} \dot{x}(s)ds].$$

- The simplest LKF $V = V_P + V_R$ [EF & U. Shaked, TAC'02]:

$$V_R(\dot{x}_t) = \int_{t-h}^{t} (s - t + h)\dot{x}^T(s)R\dot{x}(s)ds, \; R > 0.$$

$$\tfrac{d}{dt}V_R(\dot{x}_t) \le h\dot{x}^T(t)R\dot{x}(t) - \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds$$

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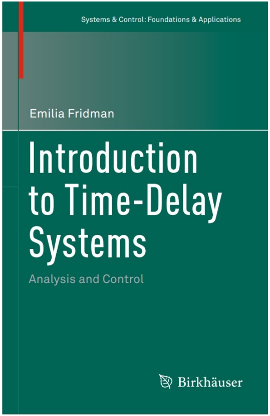
For sawtooth delay $\tau(t) = t - t_k$, Wirtinger-based LKF [K. Liu & EF, Aut'12]:

$$V_R(t, \dot{x}_t) = h^2 \int_{t_k}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \tfrac{\pi^2}{4} \int_{t_k}^{t} [x(s) - x(t_k)]^T R[x(s) - x(t_k)]ds,$$
$$R > 0, \quad t \in [t_k, t_{k+1})$$

- Extension of LKF to Hilbert space: A generates C_0 semigroup + A_1 *bounded* [EF & Y. Orlov, Aut'09]
- For detailed introduction to time-delay & sampled-data & networked control systems see



Spatial decomposition

Introduced in [EF & Blighovsky, Aut '12] for the heat equation

$$z_t(x,t) = z_{xx}(x,t) + \phi\,(z,x,t)\,z(x,t) + \sum_{j=1}^{N} b_j(x)u_j(t), \quad z_x(0,t) = z_x(l,t) = 0$$

with $z\colon [0,l]\times[0,\infty)\to\mathbb{R}$ and $\quad |\phi(z,x,t)| \le q.$

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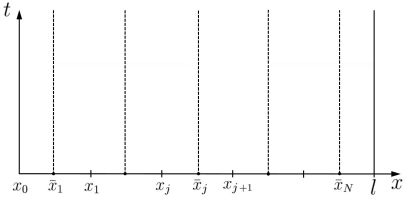

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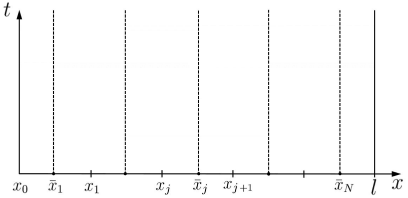

Point measurements:

$$y_j(t) = z(\bar{x}_j, t_k),\ \bar{x}_j = \frac{x_{j-1} + x_j}{2},\ t \in [t_k, t_{k+1})$$

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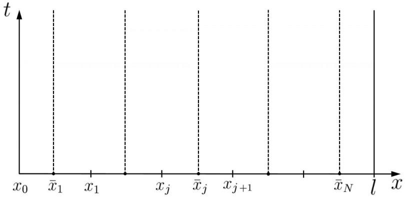

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Static output-feedback: sampled-data via ZOH

$$\begin{aligned} u_j(t) &= -Kz(\bar{x}_j, t_k),\ \ t \in [t_k, t_{k+1}), \\ b_j(x) &= \chi_{[x_j, x_{j+1})}(x). \end{aligned}$$

Spatial decomposition: delayed control via direct L-K

Extensions to delayed and network-based control via direct Lyapunov-Krasovskii (LK):

- Averaged measurements, ND semilinear heat, H_∞ control - [N. Bar Am & EF, Aut '14]
- Event-triggered, 2D under pointlike measurements - [A. Selivanov & EF, Aut '16,18]
- KSE (1D, 2D), KdVB - [W. Kang & EF, Aut '18,19; TAC '22]
- Damped wave, beam - [M. Terushkin & EF, Aut '19; SCL '20]
- Application to deployment of multi-agents - [J. Wei et al Aut'19]; [Terushkin & EF, Aut '21]

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Drawback: many actuators covering (almost) all domain & many sensors.

Challenges:

- Few actuators & sensors
- Boundary control & measurement $\Rightarrow$ direct LK for PDE may not work!

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[Karafyllis & Krstic, Aut'18] introduced sampled-data boundary control for heat eq via modal decomposition - state-feedback

Our objective - finite-dim output-feedback via modal decomposition

Plan

Finite-dim. observer-based control - problem formulation

Rami Katz (PhD, Tel Aviv University)

In [Rami Katz & EF, Aut'20] control of heat eq:

$$\begin{aligned}
&z_t(x,t) = \partial_x\left(p(x)z_x(x,t)\right) + \left(q_c - q(x)\right)z(x,t) + b(x)u(t),\ \ t \geq 0,\\
&z_x(0,t) = z(1,t) = 0;\\
&y(t) = z(0,t).
\end{aligned}$$

- $p \in C^2[0,1],\ \ q \in C^1[0,1]$ satisfying

$$0 < p_* \leq p(x) \leq p^*,\ \ 0 \leq q(x) \leq q^*, \quad x \in [0,1]$$

- $b \in H^1(0,1),\ \ b(1) = 0$
- Non-local actuation and boundary measurement

Finite-dim. observer-based control - problem formulation

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For simplicity, consider $p(x) \equiv 1$, $q(x) \equiv 0$ and $q_c = q$.

Finite-dim. observer-based control - modal decomposition

Sturm-Liouville problem:

$$\phi''(x) + \lambda\phi(x) = 0, \quad 0 < x < 1; \quad \phi'(0) = 0, \quad \phi(1) = 0.$$

- $\rightarrow$ Corresponding eigenvalues $\lambda_1 < \lambda_2 < \ldots$ satisfy $\lim_{n\to\infty} \lambda_n = \infty$.
- $\rightarrow$ Complete and orthonormal (in $L^2(0,1)$) sequence of eigenfunctions.

Here $\lambda_n = \pi^2 \left(n - \frac{1}{2}\right)^2$, $\phi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x), \quad n \geq 1.$

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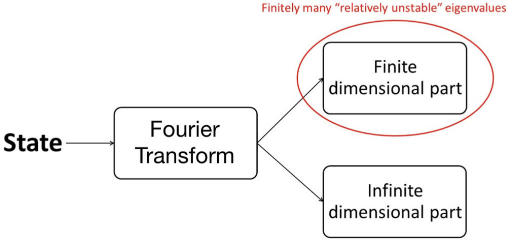
Modal decomposition:

$$z(x,t) = \sum_{n=1}^{\infty} z_n(t)\phi_n(x), \; z_n(t) := \langle z(\cdot,t), \phi_n \rangle, \; t \geq 0.$$

Differentiation of $\langle z(\cdot,t), \phi_n \rangle$ + integration by parts:

$$\begin{aligned} &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), \\ &z_n(0) = \langle z_0, \phi_n \rangle =: z_{0,n}, \quad b_n = \langle b, \phi_n \rangle, \; n = 1, 2, \ldots \end{aligned}$$

Modal decomposition



- Popular in 80s - [Curtain, TAC '82, '92], [Balas, JMAA '88].
- Popular again because of
  - *robustness to sampling/delay*:
    state-feedback [Karafyllis & Krstic, Aut'18],
    finite-dimensional observer [Selivanov & EF, TAC'19]
  - *input delay compensation*:
    state-feedback [Prieur & Trelat, TAC'18; Lhachemi et al, Aut'19]

Works on observer-based control via modal decomposition

- Finite-dimensional observer-based control: bounded control & observation operators
  1. [Curtain, TAC'82] - restrictive assumptions ($b_n = 0, \ n > N_0$).
  2. [Balas, JMAA'88] - qualitative result:
     for large enough "residual mode filter" dimension.
  3. [Harkort & Deutcher, IJC'11] - 1st step to quantitative results:
     conservative estimates on "output filter" and difficult to compute.
- Delayed observer-based control via modal decomposition:
  1. [Katz & Fridman & Selivanov, TAC'21] - PDE observer (separation).

Our goal:
easily verifiable and efficient conditions for finite-dimensional observer-based controller.

Finite-dim. observer-based control - observer design

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), \quad n = 1, 2, ...$$

Let $\delta > 0$ be a desired decay rate. Let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + q < -\delta, \quad n > N_0.$$

N_0 - controller dimension,
$N \geq N_0$ - observer dimension.

- Finite-dimensional observer: $\hat{z}(x,t) := \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x)$

$$\begin{aligned} &\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) - \ell_n \left[\sum_{n=1}^{N} \hat{z}_n(t)\phi_n(0) - y(t)\right], \\ &\hat{z}_n(0) = 0, \quad 1 \leq n \leq N. \end{aligned}$$

Gains selection

Observer and controller gains are designed independently based on N_0 modes:

- Observer: Let

$$A_0 = \mathrm{diag}\left\{-\lambda_1 + q, \dots, -\lambda_{N_0} + q\right\}, \ L_0 = [l_1, \dots, l_{N_0}]^T,$$
$$C_0 = [c_1, \dots, c_{N_0}], \ c_n = \phi_n(0), \quad n \geq 1.$$

Since $c_n \neq 0$ for $1 \leq n \leq N_0$, (A_0, C_0) is observable with L_0 found from

$$P_{\mathrm{o}}(A_0 - L_0 C_0) + (A_0 - L_0 C_0)^T P_{\mathrm{o}} < -2\delta P_{\mathrm{o}}, \quad P_{\mathrm{o}} > 0.$$

Choose $l_n = 0, \ n > N_0$.

- Controller: Assume $b_n = \langle b, \phi_n \rangle \neq 0$ for $1 \leq n \leq N_0$. Let

$$B_0 := \begin{bmatrix} b_1 & \dots & b_{N_0} \end{bmatrix}^T.$$

Then (A_0, B_0) is controllable. Let $K_0 \in \mathbb{R}^{1 \times N_0}$ satisfy

$$P_{\mathrm{c}}(A_0 + B_0 K_0) + (A_0 + B_0 K_0)^T P_{\mathrm{c}} < -2\delta P_{\mathrm{c}}, \quad P_{\mathrm{c}} > 0$$

Control law and estimation error

We propose a N_0-dimensional controller:

$$u(t) = K_0 \hat{z}^{N_0}(t), \quad \hat{z}^{N_0}(t) = [\hat{z}_1(t), \ldots, \hat{z}_{N_0}(t)]^T$$

based on the N-dimensional observer.

Let $e_n(t) = z_n(t) - \hat{z}_n(t),\ 1 \le n \le N$. The error equations can be presented as:

$$\dot{e}_n(t) = (-\lambda_n + q) z_n(t) - l_n \left(\sum_{n=1}^N c_n e_n(t) + \underbrace{\zeta(t)}_{z(0,t) - \sum_{n=1}^N c_n z_n(t)} \right),\ 1 \le n \le N.$$

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Denote

$$\begin{aligned}
& e^{N_0}(t) = [e_1(t), \ldots, e_{N_0}(t)]^T, \\
& e^{N-N_0}(t) = [e_{N_0+1}(t), \ldots, e_N(t)]^T, \\
& \hat{z}^{N-N_0}(t) = [\hat{z}_{N_0+1}(t), \ldots, \hat{z}_N(t)]^T, \\
& \mathcal{L} = \text{col}\left\{ L_0, -L_0, 0_{2(N-N_0)\times 1} \right\}, \\
& \tilde{K} = \left[K_0, \quad 0_{1\times(2N-N_0)} \right], \\
& A_1 = \operatorname{diag}\{-\lambda_{N_0+1} + q, \ldots, -\lambda_N + q\}, \\
& C_1 = [c_{N_0+1}, \ldots, c_N], \quad B_1 = [b_{N_0+1}, \ldots, b_N]^T.
\end{aligned}$$

Finite-dim. observer-based control - closed-loop system

Closed-loop system for $t \geq 0$:

$$\begin{aligned}
\dot{X}(t) &= FX(t) + \mathcal{L}\zeta(t), \\
\dot{z}_n(t) &= (-\lambda_n + q)z_n(t) + b_n\tilde{K}X(t), \quad n > N,
\end{aligned}$$

where

$$X(t) = \mathrm{col}\left\{\hat{z}^{N_0}(t), e^{N_0}(t), \hat{z}^{N-N_0}(t), e^{N-N_0}(t)\right\} \in \mathbb{R}^{2N},$$

$$F = \begin{bmatrix} A_0 + B_0K_0 & L_0C_0 & 0 & L_0C_1 \\ 0 & A_0 - L_0C_0 & 0 & -L_0C_1 \\ B_1K_0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}.$$

We have

$$\begin{aligned}
\zeta^2(t) &= \left[z(0,t) - \sum_{n=1}^{N} \phi_n(0)z_n(t)\right]^2 \\
&\leq \left\| z_x(\cdot,t) - \sum_{n=1}^{N} \phi_n'(\cdot)z_n(t)\right\|^2 = \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t)
\end{aligned}$$

Finite-dim. observer-based control - Stability analysis

For H^1-stability we use

$$V(t) = X^T(t)PX(t) + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t), \quad 0 < P \in \mathbb{R}^{2N \times 2N}.$$

Differentiating along the closed-loop system:

$$\dot{V} + 2\delta V = X^T(t)\left[PF + F^T P + 2\delta P\right] X(t) + 2X^T(t)P\mathcal{L}\zeta(t)$$

$$+2\sum_{n=N+1}^{\infty} \lambda_n(-\lambda_n + q + \delta)z_n^2(t) + \sum_{n=N+1}^{\infty} 2z_n(t)\lambda_n b_n \tilde{K}X(t).$$

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We apply Young's inequality to the cross terms:

$$\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t) b_n \tilde{K}X(t) \leq \frac{1}{\alpha}\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t) + \alpha \left\|b'\right\|_{L^2}^2 \left\|\tilde{K}X(t)\right\|^2.$$

Then

$$2\sum_{n=N+1}^{\infty} \lambda_n \left(-\lambda_n + q + \delta + \frac{1}{2\alpha}\right) z_n^2(t) \leq -2\left(\lambda_{N+1} - q - \delta - \frac{1}{2\alpha}\right) \zeta^2(t)$$

Finite-dim. observer-based control - Stability analysis

Let $\eta(t) = \text{col}\left\{X(t), \zeta(t)\right\}$. The stability analysis leads to

$$\dot{V} + 2\delta V \leq \eta^T(t)\Phi\eta(t) \leq 0$$

provided

$$\Phi = \begin{bmatrix} PF + F^T P + 2\delta P + \alpha \left\|b'\right\|^2 \tilde{K}^T \tilde{K} & P\mathcal{L} \\ * & -2\left(\lambda_{N+1} - q - \delta - \frac{1}{2\alpha}\right) \end{bmatrix} < 0.$$

Can be converted to LMI by Schur complement.

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Observations:

- The LMI dimension grows with N
- $\|P\|$ can grow - may lead to infeasibility for all $N \in \mathbb{N}$

Our contribution:

- Derivation of constructive LMI condition.
- Proof of feasibility for large N
  (based on asymptotic analysis - to bound $\|P\|$).

Finite-dim. observer-based control - Stability analysis

- Summarizing:

Given $\delta > 0$, if there exist $0 < P \in \mathbb{R}^{2N \times 2N}$ and $\alpha > 0$ that satisfy the LMI, then

$$\|z(\cdot,t)\|_{H^1}^2 + \|z(\cdot,t) - \hat{z}(\cdot,t)\|_{H^1}^2 \leq Me^{-2\delta t} \|z_0\|_{H^1}^2 ,$$

with some constant $M > 0$. Moreover, the LMI is always feasible for large enough N.

Finite-dim. observer-based control - Stability analysis

- Summarizing:

  Given $\delta > 0$, if there exist $0 < P \in \mathbb{R}^{2N \times 2N}$ and $\alpha > 0$ that satisfy the LMI, then
  $$\|z(\cdot,t)\|_{H^1}^2 + \|z(\cdot,t) - \hat{z}(\cdot,t)\|_{H^1}^2 \le Me^{-2\delta t} \|z_0\|_{H^1}^2 ,$$
  with some constant $M > 0$. Moreover, the LMI is always feasible for large enough N.

Other cases treated in [Katz & EF, Aut '20] :

$\rightarrow$ Non-local measurement and actuation - L^2 and H^1 stability

$\rightarrow$ Dirichlet actuation and non-local measurement - $H^{-\frac{1}{2}}$ stability $(V = \sum \lambda_n^{-1} z_n^2)$
In this case,
$$|b_n| \approx \sqrt{\lambda_n}$$
which is difficult to compensate in the Lyapunov analysis even for the L^2-norm.

Point measurement & actuation - dynamic extension

[Katz & EF, CDC '20; TAC'22]
Kuramoto-Sivashinsky equation (KSE)

$$\begin{aligned} z_t(x,t) \quad &= -z_{xxxx}(x,t) - \nu z_{xx}(x,t), \quad t \geq 0, \\ &z(0,t) = u(t), \quad z(1,t) = 0, \\ &z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0. \end{aligned}$$

Measurement : $y(t) = z(x_*, t), \quad x_* \in (0,1)$

- Mixed Dirichlet boundary conditions.
- Point measurement and boundary actuation - unbounded operators.

Point measurement & actuation - dynamic extension

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- Mixed Dirichlet boundary conditions.
- Point measurement and boundary actuation - unbounded operators.

Dynamic extension [Curtain & Zwart, 95], [Prieur & Trélat, Aut '18], [Katz & EF, Aut '21]:

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) := 1 - x$$

Results in better behaved $\{b_n\}_{n=1}^{\infty} \Rightarrow$ convergence in stronger norms.

Point measurement & actuation - dynamic extension

Existing results on KSE:

- Distributed state-feedback/observer-based control via modal decomposition [Christofides & Armaou. SCL '00]
- Boundary control, small anti-diffusion [Liu & Krstić. Nonlin Analysis. '01]
- State-feedback stabilization of KSE under boundary/non-local actuation [Cerpa. Commun. Pure Appl. Anal, '10], [Cerpa, Guzman & Mercado. ESAIM, '17], [Guzman, Marx & Cerpa. CPDE '19]

  $\rightarrow$ Different boundary conditions $\Rightarrow$ no explicit estimates on eigenvalues and eigenfunctions

  $\rightarrow$ Theoretically possible but computationally expensive

Point measurement & actuation - dynamic extension

Equivalent ODE-PDE system:

$$\dot{u}(t) = v(t), \quad w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) - r(x)v(t)$$

with

$$\begin{aligned} &u(0) = 0, \\ &w(0,t) = 0, \quad w(1,t) = 0, \\ &w_{xx}(0,t) = 0, \quad w_{xx}(1,t) = 0. \end{aligned}$$

- New measurement: $y(t) = w(x_*, t) + r(x_*)u(t)$.
- $u(t)$ - additional state, $v(t)$ - control input
- Given $v(t)$, $u(t)$ is computed by

$$\dot{u}(t) = v(t), \quad u(0) = 0$$

Modal decomposition using Sturm-Liouville operator for KSE:

$$\lambda_n = \pi^2 n^2, \; \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x), \quad n \geq 1$$

Point measurement & actuation - modal decomposition

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$$

$$\Downarrow$$

$$\dot{w}_n(t) = (-\lambda_n^2 + \nu\lambda_n)w_n(t) + b_n v(t),\ \ w_n(0) = \langle z_0, \phi_n \rangle\,,$$

$b_n = -\sqrt{\frac{2}{\lambda_n}}$ $\qquad$ $\ell^2(\mathbb{N})$ sequence, nonzero elements.

Point measurement & actuation - modal decomposition

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$$

$$\Downarrow$$

$$\dot{w}_n(t) = (-\lambda_n^2 + \nu\lambda_n)w_n(t) + b_n v(t), \; w_n(0) = \langle z_0, \phi_n \rangle \,,$$

$$b_n = -\sqrt{\tfrac{2}{\lambda_n}} \qquad \ell^2(\mathbb{N}) \text{ sequence, nonzero elements.}$$

Let $\delta > 0$ be a desired decay rate. Let $N_0 \in \mathbb{N}$ satisfy:

$$-\lambda_n^2 + \nu\lambda_n < -\delta, \quad n > N_0.$$

$N \geq N_0$-dimensional observer: $\hat{w}(x,t) = \sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x)$.

$$\dot{\hat{w}}_n(t) \quad = (-\lambda_n^2 + \nu\lambda_n)\hat{w}_n(t) + b_n v(t) - \ell_n \left[\hat{w}(x_*,t) + r(x_*)u(t) - y(t)\right], \quad t \geq 0$$

$N_0 + 1$-dimensional observer-based controller:

$$v(t) = K_0 \hat{w}^{N_0}(t), \quad \hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T.$$

Point measurement & actuation - closed-loop system

Closed-loop system for $t \geq 0$:

$$\dot{X}(t) = FX(t) + \mathcal{L}\zeta(t),$$
$$\dot{w}_n(t) = (-\lambda_n^2 + \nu\lambda_n)w_n(t) + b_n\tilde{K}X(t), \quad n > N,$$

where

$$X(t) = \text{col}\left\{\hat{w}^{N_0}(t), e^{N_0}(t), \hat{w}^{N-N_0}(t), e^{N-N_0}(t)\right\},$$

$$F = \begin{bmatrix} \tilde{A}_0 + \tilde{B}_0K_0 & \tilde{L}_0C_0 & 0 & \tilde{L}_0C_1 \\ 0 & A_0 - L_0C_0 & 0 & -L_0C_1 \\ B_1K_0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix}.$$

Furthermore,

$$\zeta^2(t) \leq \left\|w_x(\cdot,t) - \sum_{n=1}^{N} w_n(t)\phi_n'(\cdot)\right\|^2 \leq \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t)$$

Point measurement & actuation - stability analysis

For H^1-stability we use

$$V(t) = X^T(t)PX(t) + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t),$$

where $P > 0$, leading to LMIs:

$$\begin{bmatrix} PF + F^T P + 2\delta P + \frac{2\alpha}{\pi^2 N}\tilde{K}^T\tilde{K} & P\mathcal{L} \\ * & -\beta \end{bmatrix} < 0,$$

$$\begin{bmatrix} -\lambda_{N+1} + \nu + \frac{2\delta+\beta}{2\lambda_{N+1}} & \frac{1}{\sqrt{2}} \\ * & -\alpha \end{bmatrix} < 0.$$

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If there exists $0 < P \in \mathbb{R}^{(2N+1)\times(2N+1)}$ and scalars $\alpha, \beta > 0$ s.t. the LMIs hold, then:

$$\|w(\cdot,t)\|_{H^1} + |u(t)| + \|w(\cdot,t) - \hat{w}(\cdot,t)\|_{H^1} \leq Me^{-\delta t}\|w(\cdot,0)\|_{H^1}.$$

with some constant $M > 0$.

Point measurement & actuation - Example

Consider

$$
\begin{gathered}
z_t(x,t) = -z_{xxxx}(x,t) - 10 z_{xx}(x,t), \\
z(0,t) = u(t), \quad z(1,t) = 0, \\
z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0.
\end{gathered}
$$

with $y(t) = z(\pi^{-1}, t)$. The open-loop is unstable.

Point measurement & actuation - Example

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with $y(t) = z(\pi^{-1},t)$. The open-loop is unstable.

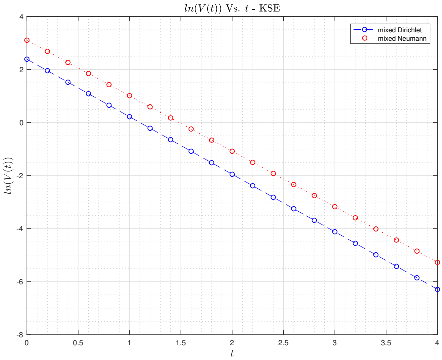
- Let $\delta = 1 \to N_0 = 1$.
- The observer and controller gains:

$$K_0 = [7.1415, 26.0901]\,, \; L_0 = 2.3419.$$

- LMIs are feasible for $N_{min} = 3$.

  Simulations $\to$ the same $N_{min} = 3$ that preserves stability!

Point measurement & actuation - Example

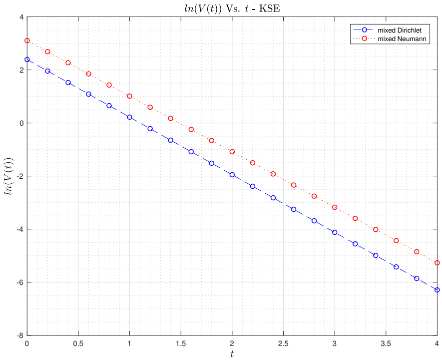


Initial condition: $u(0) = 0,\ z(x,0) = w(x,0) = (x - x^2)^3,\ x \in [0,1]$.
Computed linear fits (log-linear scales):

$$p_V(t) \approx -2.0898t + 2.2112.$$

Close to theoretical $2\delta = 2$.

Point measurement & actuation - Example



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Close to theoretical $2\delta = 2$.

- Point measurement & actuation for heat eq in [Katz & EF, EJC '21]

L^2-gain and ISS analysis

In [Katz & EF,TAC'22] we consider

$$\begin{aligned} &z_t(x,t) = -z_{xxxx}(x,t) - \nu z_{xx}(x,t) + d(x,t), \\ &z(0,t) = u(t), \quad z(1,t) = 0, \quad z_{xx}(0,t) = z_{xx}(1,t) = 0 \end{aligned}$$

with in-domain point measurement

$$y(t) = z(x_*,t) + \sigma(t), \; x_* \in (0,1).$$

The disturbances satisfy

$$\begin{aligned} &d \in L^2((0,\infty); L^2(0,1)) \cap H^1_{\mathsf{loc}}((0,\infty); L^2(0,1)), \\ &\sigma \in L^2(0,\infty) \cap H^1_{\mathsf{loc}}(0,\infty). \end{aligned}$$

Dynamic extension:

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) := 1 - x$$

L^2-gain and ISS analysis

Let $\gamma > 0$ and $\rho_w, \rho_u \geq 0$ be scalars. We introduce the performance index

$$J = \int_0^\infty \left[\rho_w^2 \left\| w(\cdot,t) \right\|_{L^2}^2 + \rho_u^2 u^2(t) - \gamma^2 \left(\left\| d(\cdot,t) \right\|_{L^2}^2 + \sigma^2(t) \right) \right] dt.$$

L^2-gain and ISS analysis

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We find conditions that guarantee along the closed-loop

$$\begin{array}{l} \dot{V} + 2\delta V + W \leq 0, \\ W = \rho_w^2 \left\|w(\cdot,t)\right\|_{L^2}^2 + \rho_u^2 u^2(t) - \gamma^2 \left(\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right), \\ V(t) = |X_N(t)|_P^2 + \sum_{n=N+1}^\infty \lambda_n w_n^2(t) \end{array}$$

$$\Downarrow$$

- $\delta = 0 \Rightarrow J \leq 0$
- $\delta > 0$ and $\rho_w = \rho_u = 0 \Rightarrow$ ISS, i.e. for some $\overline{M} > \underline{M} > 0$:

$$\begin{aligned} \underline{M} \quad \left[|u(t)|^2 + \left\|w(\cdot,t)\right\|_{H^1}^2\right] \leq & \overline{M} e^{-2\delta T} \left\|w(\cdot,0)\right\|_{H^1}^2 \\ & + \frac{\gamma^2}{2\delta} \sup_{0 \leq t \leq T} \left[\left\|d(\cdot,t)\right\|_{L^2}^2 + \sigma^2(t)\right] \quad \forall T > 0, \end{aligned}$$

L^2-gain and ISS analysis

Our L^2-gain analysis results in the following LMI:

$$\Psi_N^{(1)} = \left[\begin{array}{cc|cc} \Phi_N^{(1)} + \Xi & P\mathcal{L} & P & P\mathcal{L} \\ * & -2\left(\theta_{N+1}^{(1)} - \frac{\lambda_{N+1}}{2\alpha} - \frac{\lambda_{N+1}}{2\gamma^2}\right) & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{-\gamma^2 I} \end{array}\right] < 0,$$

$$\Phi_N^{(1)} = PF + F^T P + 2\delta P + \frac{2\alpha}{\pi^2 N}\tilde{K}_0^T \tilde{K}_0,$$

$$\Xi = \Xi_1^T \Xi_1, \ \Xi_1 = \begin{bmatrix} \rho_u & 0 & 0 & 0 & 0 \\ 0 & \rho_w I_{N0} & \rho_w I_{N0} & 0 & 0 \\ 0 & 0 & 0 & \rho_w I_{N-N_0} & \rho_w I_{N-N_0} \end{bmatrix}.$$

Novelty: proof of the LMI feasibility for large enough γ and N

- Ξ: positive term, which is not multiplied by a decision variable and does not decay with N (compare with $\frac{2\alpha}{\pi^2 N}\tilde{K}^T\tilde{K}$)
- For ISS with $d(x,t) \equiv 0$, the LMI feasibility for N implies its feasibility for $N+1$. Thus, increasing N does not deteriorate the performance.

Plan

1. Introduction: delays, spatial & modal decomposition
2. Constructive finite-dimensional observer-based control
3. Delayed and sampled-data implementation
4. Predictors and subpredictors
5. Semilinear PDEs

Delayed implementation - problem formulation

[Katz & EF, Aut '21]

$$z_t(x,t) = z_{xx}(x,t) + qz(x,t) + b(x)u(t-\tau_u(t)),$$
$$z_x(0,t) = 0, \; z(1,t) = 0,$$

$$y(t) = z(0, t-\tau_y(t))$$

Consider $b \in H^1(0,1), \; b(1) = 0.$

- $\tau_y(t)$ - known measurement delay, $\tau_y(t) \leq \tau_M$
- $\tau_u(t)$ - unknown input delay, $\tau_u(t) \leq \tau_M$
- C^1 delays or sawtooth delays (correspond to sampled-data or networked control)

Delayed implementation - problem formulation

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- $\tau_u(t)$ - unknown input delay, $\tau_u(t) \le \tau_M$
- C^1 delays or sawtooth delays (correspond to sampled-data or networked control)

$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t-\tau_u(t)),$$
$$z_n(0) = \langle z_0, \phi_n \rangle =: z_{0,n}, \quad b_n = \langle b, \phi_n \rangle.$$

Let $N_0 \in \mathbb{N}$ satisfy

$$-\lambda_n + q < -\delta_\tau, \quad n > N_0.$$

N_0 - the controller dimension. $N \ge N_0$ - the observer dimension.

Delayed implementation - observer design

- Finite-dimensional observer: $\hat{z}(x,t) := \sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x)$.

$$\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_n u(t) - \ell_n \left[\sum_{n=1}^{N} c_n \hat{z}_n(t-\tau_y(t)) - y(t)\right],$$
$$\hat{z}_n(t) = 0, \quad t \leq 0, \quad c_n = \phi_n(0) = \sqrt{2}, \quad 1 \leq n \leq N.$$

$\{\ell_n\}_{n=1}^{N}$ - scalar observer gains.

- Controller: $u(t) = K_0 \hat{z}^{N_0}(t)$.

- Closed-loop system for $t \geq 0$:

$$\dot{X}(t) = FX(t) + F_1 X(t-\tau_y(t)) + F_2 \tilde{K} X(t-\tau_u(t)) + \mathcal{L}\zeta(t-\tau_y(t)),$$
$$\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n \tilde{K} X(t-\tau_u(t)), \ n > N.$$
$$\zeta^2(t-\tau_y(t)) \leq \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t-\tau_y(t))$$

Delayed implementation - closed-loop system

We use Lyapunov functional for H^1-stability

$$V(t) = V_{\text{nom}}(t) + \sum_{i=1}^{2} V_{S_i}(t) + \sum_{i=1}^{2} V_{R_i}(t),$$
$$V_{\text{nom}}(t) = X^T(t)PX(t) + \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t),$$

- $V_{S_i}(t)$ and $V_{R_i}(t)$ compensate delays in $X(t)$
- Halanay inequality to compensate $\zeta(t - \tau_y(t))$:

Theorem (Halanay's inequality)

Let $0 < \delta_1 < \delta_0$ *and* $V : [-\tau, \infty) \longrightarrow [0, \infty)$ *be an absolutely continuous s.t.*

$$\dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-\tau \le \theta \le 0} V(t+\theta) \le 0, \quad t \ge 0.$$

Then $V(t) \le e^{-2\delta_\tau t} \sup_{-\tau \le \theta \le 0} V(\theta),\ t \ge 0$ *where* $\delta_\tau = \delta_0 - \delta_1 e^{2\delta_\tau \tau}$.

$$-2\delta_1 \sup_{-\tau_M \le \theta \le 0} V(t+\theta) \le -2\delta_1 \left|X(t-\tau_y(t))\right|_P^2 - 2\delta_1 \zeta^2(t - \tau_y(t))$$

- We prove: the resulting LMIs are feasible for large N and small τ_M.

Reduced-order LMIs

[Katz et al, ECC'21 & Aut under review]

Consider heat equation with Neumann actuation

$$\begin{array}{l} z_t(x,t) = z_{xx}(x,t) + qz(x,t), \\ z_x(0,t) = 0, \quad z_x(1,t) = u(t). \end{array}$$

Non-local measurement

$$y(t) = \langle c, z(\cdot,t) \rangle \,, \quad c \in L^2(0,1).$$

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Non-local measurement

$$y(t) = \langle c, z(\cdot,t)\rangle, \quad c \in L^2(0,1).$$

- No dynamic extension for L^2-stability:

$\rightarrow$ $\lambda_n = \pi^2 n^2,\ n \geq 0\ ;\ \phi_0(x) = 1,\ \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x), \quad n \geq 1$

$$\begin{aligned} &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) + b_n u(t), t \geq 0, \\ &b_0 = 1, \quad b_n = (-1)^n\sqrt{2}, \quad - \ \ell^\infty(\mathbb{N}) \end{aligned}$$

$\rightarrow$ The estimation error tail $\zeta(t)$ satisfies

$$\zeta^2(t) \leq \underbrace{\|c\|_N^2}_{\sum_{n=N+1} c_n^2 \overset{N\to\infty}{\to} 0} \sum_{n=N+1}^{\infty} z_n^2(t),$$

Reduced-order closed-loop system

The reduced-order closed-loop system is given by

$$\dot{X}_0(t) = F_0 X_0(t) + \mathcal{L}_0 C_1 e^{N-N_0}(t) + \mathcal{L}_0 \zeta(t),$$
$$\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + b_n \mathcal{K}_0 X_0(t), \; n > N.$$

where

$$F_0 = \begin{bmatrix} A_0 + B_0 K_0 & L_0 C_0 \\ 0 & A_0 - L_0 C_0 \end{bmatrix},$$
$$X_0(t) = \mathrm{col}\left\{ \hat{z}^{N_0}(t), e^{N_0}(t) \right\}.$$

What about $\hat{z}^{N-N_0}(t)$ and $e^{N-N_0}(t)$?

$$\begin{aligned} \dot{\hat{z}}^{N-N_0}(t) &= A_1 \hat{z}^{N-N_0}(t) + B_1 \mathcal{K}_0 X_0(t) \Rightarrow \text{exp. decaying provided } X_0(t) \text{ is} \\ \dot{e}^{N-N_0}(t) &= A_1 e^{N-N_0}(t) \qquad \Rightarrow \text{exp. decaying} \end{aligned}$$

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- Advantages of the reduced-order closed-loop:
  - → Takes into account the fast-slow structure of the dynamics
  - → Reduced-order LMIs, which are more computationally efficient
  - → Trivializes proofs of LMIs feasibility for large N, and of feasibility for N implies $N+1$

Stability analysis

For L^2-stability we use

$$V(t) = V_0(t) + p_e \left| e^{N-N_0}(t) \right|^2, \; V_0(t) = |X_0(t)|^2_{P_0} + \sum_{n=N+1}^{\infty} z_n^2(t)$$

where $0 < P \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$, $p_e \to \infty$ leading to the reduced-order LMI:

$$\begin{bmatrix} \Phi_0 & P_0\mathcal{L}_0 & 0 \\ * & -2\left(\lambda_{N+1} - q - \delta\right)\|c\|_N^{-2} & 1 \\ * & * & -\frac{\alpha\|c\|_N^2}{\lambda_{N+1}} \end{bmatrix} < 0,$$

$$\Phi_0 = P_0F_0 + F_0^T P_0 + 2\delta P_0 + \frac{2\alpha}{\pi^2 N}\mathcal{K}_0^T\mathcal{K}_0.$$

$\rightarrow$ The LMI dimension does not grow with N

Stability analysis

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where $0 < P \in \mathbb{R}^{(2N_0+1)\times(2N_0+1)}$, $p_e \to \infty$ leading to the reduced-order LMI:

$$\begin{bmatrix} \Phi_0 & P_0\mathcal{L}_0 & 0 \\ * & -2\left(\lambda_{N+1} - q - \delta\right) \|c\|_N^{-2} & 1 \\ * & * & -\frac{\alpha\|c\|_N^2}{\lambda_{N+1}} \end{bmatrix} < 0,$$

$$\Phi_0 = P_0F_0 + F_0^TP_0 + 2\delta P_0 + \tfrac{2\alpha}{\pi^2 N}\mathcal{K}_0^T\mathcal{K}_0.$$

$\rightarrow$ The LMI dimension does not grow with N

- Since we don't use dynamic extension, we can treat general time-varying delays & sampled-data control via a ZOH
- In the numerical example we easily verify LMIs for $N = 30$, whereas feasibility of the full-order LMIs could be verified for $N \leq 9$.
- To enlarge delays, in [Katz & EF, Aut under review] we compensate constant part of the input delay via classical predictor.

Sampled-data implementation via dynamic extension

In [Katz & EF, Aut '21], we consider:

$$z_t(x,t) = z_{xx}(x,t) + az(x,t),\ t \geq 0,$$
$$z_x(0,t) = 0, \quad z(1,t) = u(t)$$

Sampled-data in measurements:

- Sampling instances $0 = s_0 < s_1 < \cdots < s_k < \ldots, \lim_{k\to\infty} s_k = \infty$

$$s_{k+1} - s_k \leq \tau_{M,y},\ \forall k \in \mathbb{Z}_+, \quad \tau_{M,y} > 0.$$

- Quantizer $q : \mathbb{R} \to \mathbb{R}$

$$|q[r] - r| \leq \Delta, \quad \text{for all } r \in \mathbb{R} \tag{1}$$

  where $\Delta > 0$ is the quantization error bound

Discrete-time in-domain point measurement:

$$y(t) = q\left[z(x_*, s_k)\right],\ x_* \in [0,1),\ t \in [s_k, s_{k+1}).$$

Dynamic extension:

$$w(x,t) = z(x,t) - u(t)$$

Sampled-data implementation via dynamic extension

Sampled-data in actuation:

Controller holding times $0 = t_0 < t_1 < \cdots < t_j < \ldots, \lim_{j\to\infty} t_j = \infty$

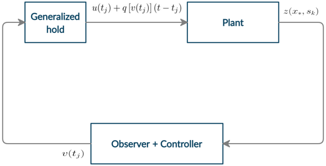
$$t_{j+1} - t_j \leq \tau_{M,u}, \ \forall j \in \mathbb{Z}_+, \quad \tau_{M,u} > 0.$$

$u(t)$ is generated by a generalized hold device:

$$\dot{u}(t) = q[v(t_j)], \ t \in [t_j, t_{j+1}), \quad u(0) = 0.$$

Generalized hold - given $v(t_j)$, the control signal is computed as:

$$u(t) = u(t_j) + q[v(t_j)](t - t_j), \ t \in [t_j, t_{j+1}), \ j = 0, 1, \ldots$$



Sampled-data implementation via dynamic extension

Dynamic extension:

$$w(x,t) = z(x,t) - u(t)$$

leads to the equivalent ODE-PDE system

$$\begin{aligned}
&\dot{u}(t) = q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}), \\
&w_t(x,t) = w_{xx}(x,t) + aw(x,t) + au(t) - q\left[v(t_j)\right],
\end{aligned}$$

with homogeneous boundary conditions and

$$y(t) = q\left[w(x_*, s_k) + u(s_k)\right], \; t \in [s_k, s_{k+1})$$

$(N_0 + 1)$-dimensional observer-based controller

$$\begin{aligned}
&\dot{u}(t) = q\left[v(t_j)\right], \quad t \in [t_j, t_{j+1}), \\
&v(t_j) = -K_0 \hat{w}^{N_0}(t_j), \\
&\hat{w}^{N_0}(t) = \left[u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\right]^T
\end{aligned}$$

Sampled-data implementation via dynamic extension

Reduced-order closed-loop system for $t \geq 0$:

$$\begin{aligned}
\dot{X}_0(t) \quad &= F_0 X_0(t) + \mathcal{L}\mathcal{C}\Upsilon_y(t) - \mathcal{B}\tilde{K}_0\Upsilon_u(t) + \mathcal{B}\sigma_u(t) \\
&\quad + \mathcal{L}C_1 e^{-A_1\tau_y} e^{N-N_0}(t) + \mathcal{L}\zeta(t-\tau_y) + \mathcal{L}\sigma_y(t), \\
\dot{w}_n(t) \quad &= (-\lambda_n + a)w_n(t) + b_n\left[\tilde{K}_a X_0(t) + \tilde{K}_0\Upsilon_u(t)\right] \\
&\quad - b_n\sigma_u(t), \; n > N, \quad t \geq 0
\end{aligned}$$

Here

$$\begin{aligned}
\tau_y(t) = t - s_k, \quad t \in [s_k, s_{k+1}), \quad \tau_y(t) \leq \tau_{M,y}, \\
\tau_u(t) = t - t_j, \quad t \in [t_j, t_{j+1}), \quad \tau_u(t) \leq \tau_{M,u}
\end{aligned}$$

and the quantization errors σ_u, σ_y are treated as disturbances

$$\max\left(\|\sigma_u\|_\infty, \|\sigma_y\|_\infty\right) \leq \Delta.$$

Sampled-data implementation via dynamic extension

For H^1-ISS analysis, we use a Wirtinger-based Lyapunov functional
- efficient for sampled-data control

Challenge:
$V(t)$ may have jump discontinuities at $s_k,\ k \in \mathbb{Z}_+$ and inside the intervals $[s_k, s_{k+1})$, where we want to apply Halanay's inequality.

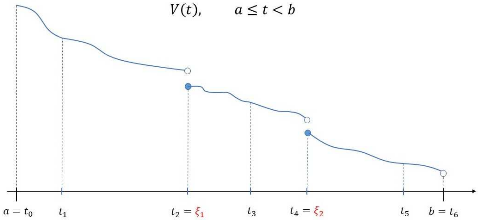


Figure 2: Possible behavior of $V(t)$

Sampled-data implementation via dynamic extension

Proposition (ISS Halanay's inequality)

Let $V:[a,b)\to[0,\infty)$ be a bounded function, where $b-a\le h$ for some $h>0$. Assume that $V(t)$ is continuous on $[t_i,t_{i+1})$, $i=0,\dots,N-1$, where $a=:t_0<t_1<\dots<t_{N-1}<t_N:=b$ and

$$\lim_{t\nearrow t_i} V(t)\ge V(t_i),\quad i=1,2,\dots,N-1.$$

Assume further that for some $d\ge 0$ and $\delta_0>\delta_1>0$

$$D^+V(t)\le -2\delta_0 V(t)+2\delta_1\sup\nolimits_{a\le\theta\le t}V(\theta)+d,\ t\in[a,b)$$

where $D^+V(t)$ is the right upper Dini derivative. Then

$$V(t)\le e^{-2\delta_\tau(t-a)}V(a)+d\int_a^t e^{-2\delta(t-s)}ds,\ t\in[a,b)$$

where $\delta=\delta_0-\delta_1$ and $\delta_\tau>0$ is the unique solution of the equation $\delta_\tau=\delta_0-\delta_1 e^{2\delta_\tau h}$.

- Reduced-order LMIs for ISS

$$\|w(\cdot,t)\|^2_{H^1}+\|\hat w(\cdot,t)\|^2_{H^1}+u^2(t)\le M_0e^{-2\delta_\tau t}\|w(\cdot,0)\|^2_{H^1}+M_1\Delta^2$$

- The LMIs are always feasible for large enough N and small enough samplings
- The LMIs feasibility for N implies feasibility for $N+1$.

Sampled-data implementation - Example

Consider

$$z_t(x,t) = z_{xx}(x,t) + 3z(x,t),\ t \geq 0,$$
$$z_x(0,t) = u(t), \quad z(1,t) = 0$$

with $y(t) = q\left[z(0,s_k)\right],\ t \in [s_k, s_{k+1})$. The open-loop is unstable.

Let $\delta = 10^{-2}$, resulting in $N_0 = 1$. The corresponding observer and controller gains are

$$L_0 = 1.0837, \quad K_0 = [12.6755, -12.7348].$$

The LMIs were found to be feasible for $N = 2$ with $\tau_{M,y} = 0.05$ and $\tau_{M,u} = 0.09$.

Sampled-data implementation - Example

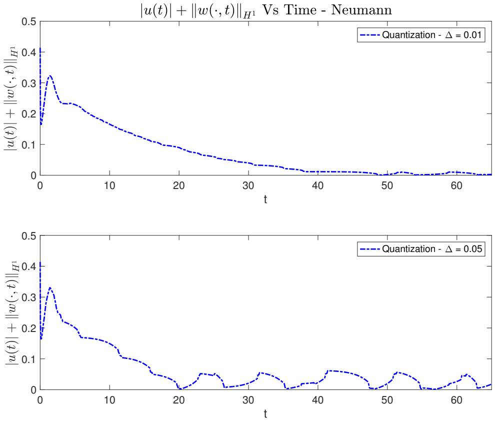
Given

$$z_0(x) = 3(x - x^2)^2, \quad x \in [0, 1].$$

the closed-loop system was simulated with varying sampling intervals

$$s_{k+1} = s_k + 0.5(1 + U_k)\tau_{M,y}, \quad U_k \sim Unif(0,1)\ random,$$
$$t_{j+1} = t_j + 0.5(1 + U_j)\tau_{M,u}, \quad U_j \sim Unif(0,1)\ random$$

and $\Delta \in \{0.01, 0.05\}$ - quantization error bound.



Plan

Predictor/Subpredictors

In [Katz & Fridman, L-CSS '21] we consider

$$z_t(x,t) = z_{xx}(x,t) + qz(x,t),\ x \in [0,1],\ t \geq 0,$$
$$z_x(0,t) = 0, \quad z_x(1,t) = u(t-r)$$

with known delay r and

$$y(t) = \langle c, z(\cdot,t) \rangle,\ t \geq 0, \quad c \in L^2(0,1)$$

Challenge:

Observer-based L^2-stabilization for arbitrarily large delay r
via efficient reduced-order LMIs.

Predictor/Subpredictors via reduced-order LMIs

To compensate r we employ a chain of M sub-predictors

$$\hat{z}_1^{N_0}(t-r) \mapsto \cdots \mapsto \hat{z}_i^{N_0}\left(t - \tfrac{M-i+1}{M} r\right) \mapsto \cdots \mapsto \hat{z}_M^{N_0}(t - \tfrac{r}{M}) \mapsto z^{N_0}(t)$$

Here $\hat{z}_M^{N_0}(t)$ predicts the value of $z^{N_0}(t + \frac{r}{M})$.

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Here $\hat{z}_M^{N_0}(t)$ predicts the value of $z^{N_0}(t + \frac{r}{M})$.

Intuition: $\hat{z}_1^{N_0}(t) \approx z^{N_0}(t+r) \Rightarrow u(t-r) \approx -K_0 z^{N_0}(t)$.
Novelty:

- Closed-loop system includes the state $z^{N_0}(t)$ (not $\hat{z}^{N_0}(t)$), subpredictor estimation errors $X_e(t)$ and tail $z_n(t), n > N$

  $\Rightarrow$ eliminates r from ODEs of $z^{N_0}(t)$ and $z_n(t),\ n > N$ and decreases it to $\frac{r}{M}$ in $X_e(t)$.
- L^2-stability in terms of reduced-order LMIs
- LMIs feasibility for arbitrary constant delays provided M and N are large.

Predictor/Subpredictors via reduced-order LMIs

We also consider compensation of r using a classical predictor:

$$\bar{z}(t) = e^{A_0 r}\hat{z}^{N_0}(t) + \int_{t-r}^{t} e^{A_0(t-s)}B_0 u(s)ds,\ u(t) = -K_0\bar{z}(t)$$

The resulting reduced-order closed-loop system consists of ODEs for $\bar{z}(t),\ e^{N_0}(t)$ and $z_n(t),\ n > N$.

Lyapunov L^2-stability analysis leads to reduced-order LMI.
Given any $r > 0$, the LMI is feasible provided N is large enough.

- For both sub-predictors and predictors, we prove LMIs feasibility for arbitrary constant delays provided observer dimension is large.

Plan

Semilinear PDEs

- [Karafyllis, IJC '21] - design of CLF for global boundary L^2-stabilization of semilinear 1D heat eq with linear growth bound
- In [Katz & EF, under review] - global distributed and boundary stabilization of a semilinear heat equation with unknown/known semilinearity exhibiting linear growth bound

In [Katz & EF, L-CSS '22] we consider regional stabilization of

$$\begin{aligned} &z_t(x,t) = -z_{xxxx}(x,t) - \nu z_{xx}(x,t) - \tfrac{1}{2}\left(z^2(x,t)\right)_x, \\ &z(0,t) = 0, \quad z(1,t) = u(t), \quad z_{xx}(0,t) = 0, \quad z_{xx}(1,t) = 0 \end{aligned}$$

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Dynamic extension:

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) = x$$

leads to

$$\begin{array}{l} \dot{u}(t) = -\kappa u(t) + v(t), \ u(0) = 0, \ \kappa > 0 \\ w_t(x,t) = -w_{xxxx}(x,t) - \nu w_{xx}(x,t) + \kappa r(x)u(t) \\ \qquad\qquad -r(x)v(t) - [w(x,t) + xu(t)]\,[w_x(x,t) + u(t)], \\ w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0. \end{array}$$

Semilinear PDEs

Modal decomposition: $w(x,t) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x)$

$$\begin{aligned}
\dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right) w_n(t) + \kappa b_n u(t) - b_n v(t) \\
&\quad -w_n^{(1)}(t) - w_n^{(2)}(t),\ t \geq 0, \\
w_n^{(1)}(t) &= \langle [w(\cdot,t) + \cdot u(t)]\, w_x(\cdot,t), \phi_n \rangle, \\
w_n^{(2)}(t) &= \langle w(\cdot,t) + \cdot u(t), \phi_n \rangle\, u(t)
\end{aligned}$$

Controller:

$$v(t) = -Kw^N(t),\ w^N(t) = \mathrm{col}\left\{u(t), w_n(t)\right\}_{n=1}^{N}.$$

Closed-loop system for $t \geq 0$:

$$\begin{aligned}
\dot{w}^N(t) &= (A - BK)w^N(t) - w^{N,(1)}(t) - w^{N,(2)}(t), \\
\dot{w}_n(t) &= \left(-\lambda_n^2 + \nu\lambda_n\right) w_n(t) + \kappa b_n u(t) - b_n v(t) \\
&\quad -w_n^{(1)}(t) - w_n^{(2)}(t).
\end{aligned}$$

Semilinear PDEs

For H^1-stability analysis of the closed-loop system, we consider

$$V(t) = \left|w^N(t)\right|_P^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t),$$

where $0 < P \in \mathbb{R}^{(N+1)\times(N+1)}$.

To manage with semilinearity, let $0 < \sigma \in \mathbb{R}$ and assume

$$\|w_x(\cdot,t)\|^2 + u^2(t) < \sigma^2, \quad t \in [0,\infty).$$

We use the Young/Sobolev inequalities and Parseval's equality in the cross terms and derive LMIs depending on tuning parameter σ.

- From LMIs we find K, δ and radius ρ of attractive ball $\|z_x(\cdot,0)\|^2 < \rho^2$, starting from which the solutions are exp decaying

$$\|w(\cdot,t)\|_{H^1}^2 + u^2(t) \leq Me^{-2\delta t} \|w(\cdot,0)\|_{H^1}^2\ , \ t \geq 0$$

Conclusions

A dream about efficient finite-dimensional observer-based control comes true:

a LMI-based method is introduced for parabolic PDEs via modal decomposition.

- $\rightarrow$ Observer dimension, ISS & L^2-gain, delay bounds are found from LMIs.
- $\rightarrow$ LMIs are proved to be asymptotically feasible and
  they are almost not conservative in examples.
- $\rightarrow$ LMIs may be verified by users without any background in PDEs!
- $\rightarrow$ Large input delays are compensated by predictors.
- $\rightarrow$ For point measurement and actuation via dynamic extension,
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- $\rightarrow$ Modal decomposition for *semilinear* parabolic PDEs
- $\rightarrow$ Delay compensation for semilinear heat

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Thank You!