Controllability and stabilization for a degenerate wave equation in non divergence form with drift

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Controllability problem: prototype

$$(P) \quad \begin{cases} u_{tt} - x^{\alpha} u_{xx} - b(x) u_x = 0, & (t,x) \in Q \\ u(t,0) = 0, \ u(t,1) = f(t), & t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

- ⋆ $Q := (0, +\infty) \times (0, 1)$
- ⋆ $\alpha \in (0, 2)$
- ⋆ b can degenerate

The motivation

$$(E) \quad \rho(x)u_{tt}(t,x) = \mathcal{T}_x(t,x)u_x(t,x) + \mathcal{T}(t,x)u_{xx}(t,x)$$

⋆ $u(t,x)$ the vertical displacement of the string from the x axis at position $x \in (0, L)$ and time $t > 0$

⋆ $\rho(x)$ the mass density of the string at position x

⋆ $\mathcal{T}(t,x)$ the tension in the string at position x and time t.

If $\mathcal{T}(t,x) \equiv \mathcal{T}(x)$, (E) can be rewritten as

$$u_{tt} = a(x)u_{xx} + b(x)u_x$$

where $a(x) = \mathcal{T}(x)\rho^{-1}(x)$, $b(x) = \mathcal{T}_x(x)\rho^{-1}(x)$

General problem

$$(P) \quad \begin{cases} u_{tt} - a(x)u_{xx} - b(x)u_x = 0, & (t,x) \in Q_T \\ u(t,0) = 0, \ u(t,1) = f(t), & t \in (0,T) \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

⋆ $a(0) = 0$

⋆ b can degenerate

Problem

Null controllability at time T (N.C.):

Given (u_0, u_1) in a suitable space, we look for a control f s.t. if $u(t,x)$ solves (P)

$$\Downarrow$$

$$u(T,\cdot) = u_t(T,\cdot) = 0 \text{ in } [0,1].$$

Nondegenerate Model

The one dimensional nondegenerate wave equation

$$(NDP) \quad \begin{cases} u_{tt} - u_{xx} = 0, & (t,x) \in Q_T, \\ u(t,0) = 0, \quad u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

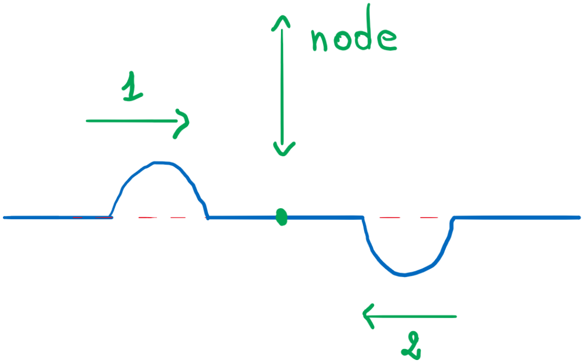
describes, e.g., the vibrations of a flexible string, with control on one end.

Here

- ⋆ $Q_T := (0,T) \times (0,1)$
- ⋆ $u = u(t,x)$ is the profile
- ⋆ $f = f(t)$ is the control

(N.C.) $\Leftrightarrow$ to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T.

Example



Parabolic problem

$$(BP)\quad \begin{cases} u_t - Au = 0, & (t,x) \in (0,T)\times\Omega, \\ u(t,0) = 0, & t \in (0,T), \\ u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

- ⋆ $Au = u''$ or $Au = (a(x)u')'$ or $Au = a(x)u''$
- ⋆ a can be strictly positive or can degenerate somewhere

(N.C.) $\Longleftrightarrow \exists\ f$ s.t. if $u(t,x)$ solves (BP) then

$$u(T,\cdot) = 0 \text{ in } \Omega.$$

Before going on with the problem, let us frame it

In Alabau, Cannarsa, F. 2006:

(N.C.) is studied if $Au = (au')'$ and $\Omega = (0,1)$ for

$$(BP) \quad \begin{cases} u_t - Au = 0, & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = 0, & t \in (0,T), \\ u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases}$$

In Biccari, Hernández-Santamaría, Vancostenoble 2020:

(N.C.) is studied if $Au = (x^\alpha u')'$, $\alpha < 1$, and $\Omega = (0,1)$ and $u(t,0) = f(t)$

Before going on with the problem, let us frame it

Parabolic problem

$$(PP) \quad \begin{cases} u_t - Au = f(t,x)\chi_\omega(x), & (t,x) \in (0,T) \times \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, & t \in (0,T), x \in \partial\Omega, \end{cases}$$

⋆ $Au = u''$ or $Au = (a(x)u')'$ or $Au = a(x)u''$

⋆ a can be strictly positive or can degenerate somewhere

(N.C.) $\iff \exists\ f$ s.t. if $u(t,x)$ solves (PP) then

$$u(T,\cdot) = 0 \text{ in } [0,1].$$

Parabolic problem. The non-degenerate case:

$a > 0$ on $\overline{\Omega}$ $\Rightarrow$ **(N.C.) HOLDS**

- ⋆ Fattorini, Russell 1971
- ⋆ Lebeau, Robbiano 1995
- ⋆ Fursikov, Imanuvilov 1996
- ⋆ D. Tataru 1997
- ⋆ Fernández-Cara, Zuazua 2000
- ⋆ ...

In all the cited papers a is at least of class C^1.

If a is not too regular:

- ⋆ Fernández-Cara, Zuazua 2002
- ⋆ Doubova, Osses, Puel 2002
- ⋆ Le Rousseau, Robbiano 2010
- ⋆ Benabdallah, Dermenjian, Le Rousseau 2007
- ⋆ Le Rousseau 2007
- ⋆ Koch, Tataru 2009
- ⋆ F., Mugnai 2016
- ⋆ ...

Parabolic problem. The degenerate case:

If $a(x_0) = 0$ with $x_0 \in [0, 1]$

First result: if, e.g., $a(x) = x^{K_1}(1-x)^{K_2}$ and K_1 or $K_2 \geq 2 \Rightarrow$ **Not null controllable!** (Cannarsa, Martinez, Vancostenoble 2004, Cannarsa, F., Rocchetti 2007)

Using a result of Micu and Zuazua 2001, Escauriaza, Seregin and Sverak 2003, 2004 for the heat equation on an unbounded domain.

Analogously if $a(x) = |x - x_0|^K$, $x_0 \in (0, 1)$ and $K \geq 2$ (F., Mugnai 2013 or 2016).

Null controllability if $K \geq 2$

If $a(x) \approx |x - x_0|^K$, with $x_0 \in [0, 1]$ and $K \geq 2$:

Regional null controllability:

- ⋆ Martinez, Raymond, Vancostenoble 2003
- ⋆ Cannarsa, Martinez, Vancostenoble 2004, 2008
- ⋆ Cannarsa, F., Vancostenoble 2006
- ⋆ Cannarsa, F. 2006
- ⋆ F., Mugnai 2013, 2016

Parabolic problem. The degenerate case:

$a(x) \approx |x - x_0|^K$ with $x_0 \in [0, 1]$ **(N.C.) holds** if $K \in (0, 2)$:

If $x_0 \in \{0, 1\}$

- ⋆ Cannarsa, Martinez, Vancostenoble 2005, 2008, 2009, 2016
- ⋆ Alabau-Boussaira, Cannarsa, F. 2006
- ⋆ Martinez, Vancostenoble 2006
- ⋆ Cannarsa, De Teresa 2009
- ⋆ Flores, De Teresa 2010
- ⋆ Ait Ben Hassi, Ammar Khodja, Hajjaj, Maniar 2011
- ⋆ Cannarsa, F., Rocchetti 2007, 2008
- ⋆ F. 2013

If $x_0 \in (0, 1)$

- ⋆ F., Mugnai 2013, 2016.
- ⋆ Beauchard, Cannarsa, Guglielmi 2014: **(N.C.)** for the **model of Grushin**

Coming back to the model problem...

$$(NDP) \quad \begin{cases} u_{tt} - u_{xx} = 0, & (t,x) \in Q_T, \\ u(t,0) = 0, \quad u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1). \end{cases}$$

It is well known that (N.C.) problem has a solution if $T > 2$ (Russell 1978).

Construction of the Control

Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$f(t) = y_x(t,1),$$

where y solves

$$(AP) \quad \begin{cases} y_{tt} - y_{xx} = 0, & (t,x) \in Q_T, \\ y(t,0) = y(t,1) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & x \in (0,1) \end{cases}$$

with $(y_0, y_1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(y_0,y_1) = \frac{1}{2}\int_0^T |y_x(t,1)|^2 dt + \int_0^1 u_0 y_1 dx - \langle u_1, y_0 \rangle_{H^{-1}(0,1),H_0^1(0,1)}$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Remark: the equation is nondegenerate.

Idea:

(N.C.) $\iff$ observability inequality

$$(OI) \quad \|(y_0, y_1)\|^2_{H^1_0(0,1)\times L^2(0,1)} \le C \int_0^T |y_x(t,1)|^2\, dt,$$

where $C = C(T) > 0$.
(OI) guarantees the coercivity of

$$J(y_0, y_1) = \frac{1}{2}\int_0^T |y_x(t,1)|^2 dt + \int_0^1 u_0 y_1\, dx - \langle u_1, y_0\rangle_{H^{-1}(0,1),H^1_0(0,1)}$$

thus J has a unique minimizer (y_0, y_1) and f, which drives the solution of (NDP) to 0 at time T, is given by

$$f(t) = y_x(t,1),$$

where y solves (AP) with initial data (y_0, y_1).

Coming back to the model problem...

If the control is distributed in ω

$$\begin{cases} u_{tt} - u_{xx} = f(t,x)\chi_\omega, & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

where χ_ω = characteristic function of $\omega \subset\subset (0,1)$

- ⋆ Privat, Trélat, Zuazua 2013
- ⋆ Ouzahra 2019
- ⋆ Sengouga 2020
- ⋆ Chaves-Silva, Rosier, Zuazua 2014: $-\Delta y_t + b(x)y_t$ and $\omega \leftrightarrow \omega(t)$

Remark: the equation is always nondegenerate.

Coming back to the original problem

If the equation is degenerate and in divergence form

$$(DDF) \quad \begin{cases} u_{tt} - (a(x)u_x)_x = 0, & (t,x) \in Q_T, \\ u(t,0) = 0, \quad u(t,1) = f(t), & t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

(N.C.) is proved in

- ⋆ Zhang, Gao 2016, 2018: $a(x) = x^K$ and $K \in (0,2)$ or $K \geq 2$
- ⋆ Bai, Chai 2020: $+\, c(t,x)u$
- ⋆ Alabau, Cannarsa, Leugering 2017: $a(x)$ s.t. $a(0) = 0$
- ⋆ Gueye 2014: $a(x) = x^K$ and $K \in (0,1)$ control at 0
- ⋆ Kogut, Kupenko, Leugering 2021 ...
- ⋆ Moumni, Salhi 2022 ...

In Kogut, Kupenko, Leugering 2021:

(N.C.) is proved for

$$(DDF) \quad \begin{cases} u_{tt} - (a(x)u_x)_x = 0, \\ u(t,c) = f_c(t), \quad u(t,d) = f_d(t), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases}$$

where $(t,x) \in (0,T) \times (c,d)$, $c < 1 < d$ and $a(x) > 0$ if $x \neq 1$ and $a(1) = 0$

In Moumni, Salhi 2022:

(N.C.) is proved for

$$
(DDF) \quad \begin{cases} u_{tt} - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, \\ \begin{cases} u(t,0) = 0, & \alpha \in (0,1), \\ \lim_{x\to 0}(x^\alpha u_x)(t,x) = 0, & \alpha \in (1,2) \end{cases} \\ u(t, 1+kt) = f(t), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases},
$$

where $(t,x) \in (0,T) \times (0, 1+kt)$ and $k \in \left(0, \frac{2-\alpha}{2}\right)$

Remark: $\alpha \neq 1$.

In Alabau, Cannarsa, Leugering 2017:

Assumptions on a:

$a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in (0,2)$ s.t.

$$a \in C^{[K_a]}(0,1],$$

$$\sup_{x \in (0,1]} \frac{x|a'(x)|}{a(x)} \le K_a$$

If $K_a \ge 2$ they prove that **(OI)** and hence **(N.C.) does not hold**.

General problem

$$(P) \quad \begin{cases} u_{tt} - a(x)u_{xx} - b(x)u_x = 0, & (t,x) \in Q_T \\ u(t,0) = 0, \;\; u(t,1) = f(t), & t \in (0,T) \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

- ⋆ $a(0) = 0$
- ⋆ b can degenerate

The (N.C.) problem for (P) is related to the observability problem for

$$(AP) \quad \begin{cases} y_{tt} - ay_{xx} - by_x = 0, & (t,x) \in Q := (0,+\infty) \times (0,1), \\ y(t,1) = y(t,0) = 0, & t > 0, \\ y(0,x) = y_T^0(x), & x \in (0,1), \\ y_t(0,x) = y_T^1(x), & x \in (0,1). \end{cases}$$

In particular (AP) is observable if and only if

$$C_T = \inf_{(y_T^0, y_T^1) \neq (0,0)} \frac{\int_0^T y_x^2(t,1)dt}{E_y(0)} > 0.$$

Definition

$c_t := \dfrac{1}{C_T}$ is the cost for observability (or the cost for controlling) at time T

Assumptions on a and b

Hypothesis 1 (H1)

a is:

⋆ **Weakly degenerate (WD)** at 0 : $a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in (0,1)$ s.t.

$$\sup_{x \in (0,1]} \frac{x|a'(x)|}{a(x)} \le K_a \quad (*)$$

or

⋆ **Strongly degenerate (SD)** at 0 : $a \in C^1[0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\ K_a \in [1,2)$ s.t. $(*)$ holds.

b is:

⋆ $b \in C[0,1]$

⋆ $\dfrac{b}{a} \in L^1(0,1)$

Remark on the condition $\frac{b}{a} \in L^1(0,1)$:

- ⋆ a is (WD) $\Rightarrow$ b can degenerate or not.
- ⋆ a is (SD) $\Rightarrow$ b has to degenerate at 0 since $\frac{1}{a} \notin L^1(0,1)$
- ⋆ $a(x) = x^K$, $K > 0$, and $b(x) = x^h$, $h \geq 0 \Rightarrow h > K - 1$.

Remark on K_a:

If $K_a \geq 2$, the boundary observability fails

Example 1: Given $T > 0$, consider the problem

$$(1) \quad \begin{cases} y_{tt} - a y_{xx} = 0, & (t,x) \in Q_T, \\ y(t,1) = y(t,0) = 0, & t \in (0,T), \\ y(0,x) = y_T^0(x), y_t(0,x) = y_T^1(x) & x \in (0,1) \end{cases}$$

with $a(x) = x^{K_a}$ and $K_a > 2$. Using the standard change of variables

$$X := \int_x^1 \frac{dy}{\sqrt{a(y)}}, \quad Y(t,X) := \frac{1}{\sqrt[4]{a(x)}} y(t,x),$$

Remark on K_a:

Problem (1) can be rewritten as

$$\begin{cases} Y_{tt} - Y_{xx} + c(X)Y = 0, \\ Y(t,0) = 0, \\ Y(0,X) = Y_T^0(X) := \frac{1}{(1+\frac{K_a-2}{2}X)^{\frac{K_a}{2(2-K_a)}}} y_T^0\left(\left(1+\frac{K_a-2}{2}X\right)^{\frac{2}{2-K_a}}\right), \\ Y_t(0,X) = Y_T^1(X) := \frac{1}{(1+\frac{K_a-2}{2}X)^{\frac{K_a}{2(2-K_a)}}} y_T^1\left(\left(1+\frac{K_a-2}{2}X\right)^{\frac{2}{2-K_a}}\right), \end{cases}$$

where

- ⋆ $(t, X) \in (0, T) \times (0, +\infty)$
- ⋆ $c(X) = \frac{K_a(K_a-4)}{16}\left(\frac{K_a-2}{2}X + 1\right)^{-2}$
- ⋆ $x = \left(1 + \frac{K_a-2}{2}X\right)^{\frac{2}{2-K_a}}$.

Remark: $Y_X(\cdot, 0) \equiv 0$ in $[0, T]$ when the support of Y_T^0 and Y_T^1 is sufficiently far from $X = 0$. Thus, (1) is not observable on $[0, T]$.

Remark on K_a:

Example 2: Given $T > 0$, consider the problem (1) with $K_a = 2$.
Using the standard change of variables

$$X := -\log x, \quad Y(t,X) := \frac{1}{\sqrt[4]{a(x)}} y(t,x),$$

with $a(x) = x^{K_a}$, (1) can be rewritten as

$$\begin{cases} Y_{tt} - Y_{xx} + \frac{1}{4} Y = 0, & (t,x) \in (0,T) \times (0,+\infty), \\ Y(t,0) = 0, & t \in (0,T), \\ Y(0,X) = Y_T^0(X) := e^X y_T^0(e^{-X}), & X \in (0,+\infty), \\ Y_t(0,X) = Y_T^1(X) := e^X y_T^1(e^{-X}), & X \in (0,+\infty). \end{cases}$$

As before, one can conclude that (1) is not observable on $[0,T]$.

Remark: For this reason we assume $K_a \in [0,2)$.

The (N.C.) problem for (P) is related to the observability problem for

$$(AP)\quad \begin{cases} y_{tt} - a y_{xx} - b y_x = 0, & (t,x) \in Q := (0,+\infty) \times (0,1), \\ y(t,1) = y(t,0) = 0, & t > 0, \\ y(0,x) = y_T^0(x), & x \in (0,1), \\ y_t(0,x) = y_T^1(x), & x \in (0,1). \end{cases}$$

In particular (AP) is observable if and only if

$$C_T = \inf_{(y_T^0, y_T^1) \neq (0,0)} \frac{\int_0^T y_x^2(t,1)\,dt}{E_y(0)} > 0.$$

Definition

$c_t := \dfrac{1}{C_T}$ is the cost for observability (or the cost for controlling) in time T

Definition of the energy:

Setting

$$\eta(x) := e^{\int_{\frac{1}{2}}^{x} \frac{b(y)}{a(y)} dy}$$

and

$$\sigma(x) := \frac{a(x)}{\eta(x)}$$

with $x \in [0, 1]$ we define

Definition

$$E_y(t) = \frac{1}{2} \int_0^1 \left(\frac{1}{\sigma} y_t^2(t,x) + \eta y_x^2(t,x) \right) dx, \quad \forall\, t \geq 0.$$

Suitable Hilbert spaces

As in Cannarsa, F., Rocchetti 2007 we consider:

$$L^2_{\frac{1}{\sigma}}(0,1) := \left\{ u \in L^2(0,1) \;\middle|\; \|u\|_{\frac{1}{\sigma}} < \infty \right\},$$

$$\langle u, v \rangle_{\frac{1}{\sigma}} := \int_0^1 uv \frac{1}{\sigma} dx,$$

$$H^1_{\frac{1}{\sigma}}(0,1) := L^2_{\frac{1}{\sigma}}(0,1) \cap H^1_0(0,1),$$

$$\langle u, v \rangle_{1,\frac{1}{\sigma}} := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 u'v' dx,$$

$$H^2_{\frac{1}{\sigma}}(0,1) := \left\{ u \in H^1_{\frac{1}{\sigma}}(0,1) \;\middle|\; Au \in L^2_{\frac{1}{\sigma}}(0,1) \right\},$$

$$\langle u, v \rangle_{2,\frac{1}{\sigma}} := \langle u, v \rangle_{1,\frac{1}{\sigma}} + \langle Au, Av \rangle_{\frac{1}{\sigma}}.$$

Here

$$Au := au_{xx} + bu_x = \sigma(\eta u_x)_x, \quad D(A) := H^2_{\frac{1}{\sigma}}(0,1)$$

Existence Theorem

Theorem (Boutaayamou, F., Mugnai, 2022)

If $(y_T^0, y_T^1) \in H^1_{\frac{1}{\sigma}}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1)$, then $\exists!$ **mild solution** y of (AP) and

$$y \in C^1([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1)) \cap C([0,+\infty); H^1_{\frac{1}{\sigma}}(0,1)).$$

Moreover, if $(y_T^0, y_T^1) \in H^2_{\frac{1}{\sigma}}(0,1) \times H^1_{\frac{1}{\sigma}}(0,1)$, the solution y is **classical.**

Note: To prove the well posedness of (AP) it is sufficient to require

"**(H0):** a and $b \in C[0,1]$, $\dfrac{b}{a} \in L^1(0,1)$, $a(0) = 0$ and
$\exists\ K_a \in (0,2)$ s.t.

$$x \mapsto \frac{x^{K_a}}{a(x)}$$

is nondecreasing near $x = 0$".

Conservation of the energy

Theorem (Boutaayamou, F., Mugnai, 2022)

Let y be a mild solution of (AP). Then

$$E_y(t) = E_y(0), \quad \forall\, t \geq 0.$$

Also in this case it is sufficient to require (H0).

Problem:

$$(**) \quad CE_y(0) \leq \int_0^T y_x^2(t,1)dt$$

Problem:

$$(**) \quad CE_y(0) \leq \int_0^T y_x^2(t,1)dt$$

In the (SD) case:

Hypothesis 2 (H2):

⋆ a and b are as in (H1) and such that $\dfrac{xb}{a} \in L^\infty(0,1)$,

Define, for L sufficiently large,

$$g(x) = x^{K_a-1} e^{\int_1^x \frac{|b|}{a}dy} \int_1^x \frac{e^{\int_t^1 \frac{|b|}{a}dy}}{t^{K_a}} dt + L x^{K_a-1} e^{\int_1^x \frac{|b|}{a}dy}$$

Note :

⋆ g is continuous, increasing, non negative in $[0,1]$ with $g(1) = L$.

⋆ It is essential that

$$K_a > 1$$

The observability of (AP)

Theorem (Boutaayamou, F., Mugnai, 2022)

Assume (H2) and $K_a > 1$. If y solves (AP), then for any $T > 0$

$$(**) \quad CE_y(0) \leq \int_0^T y_x^2(t,1)dt,$$

where $C := \frac{2}{\eta(1)g(1)}\left(T - 2\max\left\{1, \frac{g^2(1)}{a(1)}\right\}\right)$.

The observability of (AP)

Corollary

Assume (H2). If

$$T > 2\max\left\{1, \frac{g^2(1)}{a(1)}\right\}$$

then (AP) is observable in time T. Moreover

$$\frac{2}{g(1)\eta(1)}\left(T - 2\max\left\{\frac{g^2(1)}{a(1)}, 1\right\}\right) \le C_T.$$

Problem:

$$(**) \quad CE_y(0) \le \int_0^T y_x^2(t,1)dt$$

In the (WD) case and if $K_a = 1$:

Hypothesis 3 (H3):

⋆ a and b are as in (H1) and such that $K_a < 2 - 2M$

where $M := \dfrac{\|b\|_\infty}{a(1)}$

Note:

⋆ we cannot consider the function g.

⋆ $\dfrac{xb}{a} \in L^\infty(0,1)$ is obvious in the case $K_a \le 1$ since $\dfrac{x}{a(x)} \le \dfrac{1}{a(1)}$

⋆ if $M < \dfrac{1}{2}$, then $K_a < 2 - 2M$ is clearly satisfied

The observability of (AP)

Theorem (Boutaayamou, F., Mugnai, 2022)

Assume (H3). If y solves (AP), then for any $T > 0$

$$(**) \quad CE_y(0) \leq \int_0^T y_x^2(t,1)dt,$$

where $C := \frac{1}{\eta(1)}\left(TC_{a,b} - \max\left\{4, \frac{4}{a(1)}, \frac{K_a \max \eta}{a(1) \min \eta}\right\}\right)$. Here

$$C_{a,b} := 2 - K_a - 2\frac{\|b\|_\infty}{a(1)} > 0$$

The observability of (AP)

Corollary

Assume (H3). If

$$T > \frac{1}{C_{a,b}} \max \left\{ 4, \frac{4}{a(1)}, \frac{K_a \max \eta}{a(1) \min \eta} \right\},$$

then (AP) is observable in time T. Moreover

$$\frac{1}{\eta(1)} \left(T C_{a,b} - \max \left\{ 4, \frac{4}{a(1)}, \frac{K_a \max \eta}{a(1) \min \eta} \right\} \right) \leq C_T.$$

Some Remarks

In the (WD) case the estimate (**) still holds under different assumptions. In particular, (H3) can be substituted by

Hypothesis 4 (H4):

a and b are as in (H1), $\left(\frac{xb}{a}\right)_x \in L^\infty(0,1)$, $\left(\frac{xb}{a}\right)_{xx}$ is a measurable function defined in $[0,1]$ so that

- $(H4_1)$: $\exists\ C_b \in \left(0, \frac{1}{2}\right)$ s.t. for a.e. $x \in [0,1]$

$$\left|\frac{b}{a}\left(\frac{xb}{a}\right)_x + \left(\frac{xb}{a}\right)_{xx}\right| \leq C_b \frac{1}{x^2}, \quad \text{or}$$

- $(H4_2)$: $b \geq 0$ and $M < 1$

where $M := \dfrac{\|b\|_\infty}{a(1)}$.

If $(H4_1)$

Theorem (Boutaayamou, F., Mugnai, 2022)

Assume (H4) with $(H4_1)$ and

$$(*\circ) \quad \frac{2C_b \max_{[0,1]} \eta}{\min_{[0,1]} \eta} < 1.$$

If y solves (AP), then $\exists\ C > 0$ s.t. for any $T > 0$

$$(**) \quad CE_y(0) \le \int_0^T y_x^2(t,1)dt,$$

where $C := \frac{2}{\eta(1)} \left(C_{a,b} T - 2 \max \left\{ M^2 \frac{\max_{[0,1]} \eta}{a(1) \min_{[0,1]} \eta}, 1, \frac{1}{a(1)} \right\} \right)$, with $C_{a,b} := 1 - \max \left\{ a, \frac{2C_b \max_{[0,1]} \eta}{\min_{[0,1]} \eta} \right\}$ and $M := \frac{\|b\|_\infty}{a(1)}$.

Corollary

Assume (H4) with $(H4_1)$. If

$$T > \frac{2}{C_{a,b}} \max\left\{ M^2 \frac{\max_{[0,1]} \eta}{a(1) \min_{[0,1]} \eta}, 1, \frac{1}{a(1)} \right\},$$

then (AP) is observable in time T. Moreover

$$\frac{2}{\eta(1)} \left(C_{a,b} T - 2 \max\left\{ M^2 \frac{\max_{[0,1]} \eta}{a(1) \min_{[0,1]} \eta}, 1, \frac{1}{a(1)} \right\} \right) \leq C_T,$$

where $M := \dfrac{\|b\|_\infty}{a(1)}$.

If $(H4_2)$

The constant in (**) is different and here we have

Corollary

Assume Assume (H4) with $(H4_2)$. If

$$T > \frac{2}{1 - \max\{M, K_a\}} \max\left\{1, \frac{1}{a(1)}\right\},$$

then (AP) is observable in time T. Moreover

$$\frac{2}{\eta(1)}\left((1 - \max\{M, K_a\})T - 2\max\left\{\frac{1}{a(1)}, 1\right\}\right) \leq C_T,$$

where $M := \dfrac{\|b\|_\infty}{a(1)}$.

Remarks on $(H4_1)$

$$(\circ\circ)\quad \left|\frac{b}{a}\left(\frac{xb}{a}\right)_x + \left(\frac{xb}{a}\right)_{xx}\right| \le C_b \frac{1}{x^2} \text{ with } C_b \in \left(0, \frac{1}{2}\right)$$

$\star$ The condition $\frac{2C_b \max_{[0,1]} \eta}{\min_{[0,1]} \eta} < 1$ implies that

$$2C_b \max_{[0,1]} \eta < \min_{[0,1]} \eta < \max_{[0,1]} \eta \Rightarrow C_b < \frac{1}{2}$$

$\star$ $(\circ\circ)$ is similar to the assumption made in Cannarsa, F., Rocchetti 2007 for the parabolic case.

$\star$ If $a(x) = \alpha x^K$ $(\alpha > 0)$ and $b(x) = \beta x^K$, $(\circ\circ)$ is satisfied if $|\beta| < \frac{\alpha}{2}$.

$\star$ If $a(x) = x^K$ and $b(x) = x^h$ with $K > 0$ and $h \ge 0$, then $(\circ\circ)$ is always satisfied since it requires $h > K - 1$.

Returning to null controllability:

Transposition method (as in Alabau, Cannarsa, Leugering 2017)

Definition

$f \in L^2_{\text{loc}}[0,+\infty)$, $(u_0,u_1) \in L^2_{\frac{1}{\sigma}}(0,1) \times H^{-1}_{\frac{1}{\sigma}}(0,1)$, u solves (P) by transposition if $u \in \mathcal{U}$ and $\forall\ T>0$ and $\forall\ (v^0_T, v^1_T) \in \mathcal{H}_0$

$$\langle u_t(T), v^0_T\rangle_{H^{-1}_{\frac{1}{\sigma}}(0,1),H^1_{\frac{1}{\sigma}}(0,1)} + \int_0^1 \frac{1}{\sigma} u(T) v^1_T\,dx + \int_0^1 \frac{1}{\sigma} u_0 v_t(0,x)\,dx$$

$$= \langle u_1, v(0)\rangle_{H^{-1}_{\frac{1}{\sigma}}(0,1),H^1_{\frac{1}{\sigma}}(0,1)} + \int_0^T \eta(1) f(t) v_x(t,1)\,dt,$$

where v solves the backward problem (AP) with data (v^0_T, v^1_T).

- $\mathcal{U} := C^1([0,+\infty); H^{-1}_{\frac{1}{\sigma}}(0,1)) \cap C([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1))$
- $\mathcal{H}_0 := H^1_{\frac{1}{\sigma}}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1)$

Remark

Note:

If $y(t,x) := v(T-t,x)$, then y satisfies (AP) with $y_T^0(x) = v_T^0(x)$ and $y_T^1(x) = -v_T^1(x)$. Since (AP) admits a unique solution y, then there exists a unique v solving the backward problem (AP) so that

$$v \in C^1([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1)) \cap C([0,+\infty); H^1_{\frac{1}{\sigma}}(0,1))$$

which depends continuously on the data $V_T := (v_T^0, v_T^1) \in \mathcal{H}_0$.

Returning to null controllability:

Theorem (Boutaayamou, F., Mugnai, 2022)

Assume one of the previous hypothesis. Then, given $(u_0, u_1) \in L^2_{\frac{1}{\sigma}}(0,1) \times H^{-1}_{\frac{1}{\sigma}}(0,1)$ and fix T as before, there exists a control $f \in L^2(0,T)$ such that the solution of (P) satisfies

$$u(T,\cdot) = u_t(T,\cdot) = 0 \text{ in } [0,1].$$

Idea of the proof:

Consider the bilinear form $\Lambda : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}$

$$\Lambda(V_T, W_T) := \eta(1) \int_0^T v_x(t,1) w_x(t,1) dt,$$

where v and w are the solutions of (AP) associated to the final data $V_T := (v_T^0, v_T^1)$ and $W_T := (w_T^0, w_T^1)$, respectively.

Properties of Λ

⋆ Λ is coercive thanks to (**)

⋆ Λ is continuous: we need to prove

$$\int_0^T y_x^2(t,1) dt \leq C E_y(0)$$

for a positive constant C.

Idea of the proof:

Theorem (Boutaayamou, F., Mugnai, 2022)

Assume (H1). If y solves (AP), then for any $T > 0$

$$(***) \quad \int_0^T y_x^2(t,1)dt \leq CE_y(0),$$

where $C := 2(2 + K_a + M)T + 4\max\left\{1, \frac{1}{a(1)}\right\}$

Idea of the proof:

Consider $\mathcal{L} : \mathcal{H}_0 \to \mathbb{R}$ defined as

$$\mathcal{L}(V_T) := \langle u_1, v(0) \rangle_{H^{-1}_{\frac{1}{\sigma}}(0,1), H^1_{\frac{1}{\sigma}}(0,1)} - \int_0^1 \frac{1}{\sigma} u_0 v_t(0,x) dx.$$

By the Lax-Milgram Theorem, $\exists!\ \bar{V}_T := (\bar{v}_T^0, \bar{v}_T^1) \in \mathcal{H}_0$ such that

$$(\circ) \quad \Lambda(\bar{V}_T, W_T) = -\mathcal{L}(W_T), \quad \forall W_T \in \mathcal{H}_0.$$

Setting $f(t) := \bar{v}_x(t,1)$, where $\bar{v}$ solves

$$\begin{cases} v_{tt} - a v_{xx} - b v_x = 0, & (t,x) \in Q := (0,+\infty) \times (0,1), \\ v(t,1) = v(t,0) = 0, & t > 0, \\ v(T,x) = \bar{v}_T^0(x), & x \in (0,1), \\ v_t(T,x) = \bar{v}_T^1(x), & x \in (0,1), \end{cases}$$

if u solves (P) (by transposition) with f as boundary control, then

$$u(T,\cdot) = u_t(T,\cdot) = 0 \text{ in } [0,1]$$

thanks to $(\circ)$.

Stabilization: prototype

$$(P_s) \quad \begin{cases} u_{tt} - x^{\alpha} u_{xx} - b(x)u_x = 0, & (t,x) \in Q \\ u(t,0) = 0, & t > 0 \\ u_t(t,1) + \eta u_x(t,1) + \beta u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

⋆ $Q := (0, +\infty) \times (0,1)$

⋆ $\alpha > 0$

⋆ b can degenerate

⋆ β is a non negative constant and $\eta(x) := e^{\int_{\frac{1}{2}}^{x} \frac{b(y)}{a(y)} dy}$

Problem

Asymptotic stability of solutions (AS):

Given (u_0, u_1) in a suitable space, we look for conditions s.t. if $u(t, x)$ solves (P_s) then

$$\text{“}\lim_{t\to\infty} E_u(t) = 0\text{”},$$

where E_u is the energy associated to the solution u of the problem.

Previous results

ODE's or Systems of ODE's:

$$u'' + h(t)u' = f(u).$$

$h \geq 0$

f **linear: Hatvani 1996**
Hatvani, Krisztin 1997
f **nonlinear: Pucci, Serrin 1993, 1994**

Previous results

Hyperbolic PDE's:

$$u_{tt} + h(t)u_t = \Delta u + f(u)$$

$h > 0$

$f \equiv 0$: Cox, Zuazua 1994
López, Gómez 1997
Martinez 2000
Alabau-Boussouira 2004
Haraux - Martinez - Vancostenoble 2005

f nonlinear with linear growth: Zhang 1994
f "generic": Messaoudi 2000
Pucci, Serrin 1996
Lasiecka, Toundykov 2006

Previous results

Hyperbolic PDE's:

$$u_{tt} + h(t)u_t = \Delta u + f(u)$$

- ⋆ h is integrally positive
- ⋆ h is on-off
- ⋆ h is positive-negative
- ⋆ distributed or localized h with delay

Haraux, Martinez, Vancostenoble 2005; Zhang 1994; F., Mugnai 2008, 2011; Chen 1981; Lagnese 1983, 1988; Lasieska, Triggiani 1987; Komornik, Zuazua 1990; Datko, Lagnese, Polis 1986; Nicaise, Pignotti 2006; Xu, Yung, Li 2006; F., Pignotti 2016, ...

Monographs:
Alabau-Boussouira 2012; Bastin, Coron 2016

Always non degenerate!

Coming back to the original problem

If the equation is degenerate and in divergence form

$$(DDP_s) \quad \begin{cases} u_{tt} - (a(x)u_x)_x = 0, & (t,x) \in Q_T, \\ u(t,0) = 0, \quad g(u(t,1)) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

(AS) is proved in

⋆ Alabau, Cannarsa, Leugering 2017: $a(x)$ s.t. $a(0) = 0$.
Here

$$g(u(t,1)) = u_t(t,1) + u_x(t,1) + \beta u(t,1)$$

and a is only (WD).

General problem

$$(P_s) \quad \begin{cases} u_{tt} - a(x)u_{xx} - b(x)u_x = 0, & (t,x) \in Q \\ u(t,0) = 0, & t > 0 \\ u_t(t,1) + \eta u_x(t,1) + \beta u(t,1) = 0, & t > 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), & x \in (0,1) \end{cases}$$

- ⋆ $Q := (0,+\infty) \times (0,1)$
- ⋆ $a(0) = 0$
- ⋆ b can degenerate
- ⋆ β is a non negative constant and $\eta(x) := e^{\int_{\frac{1}{2}}^{x} \frac{b(y)}{a(y)} dy}$

Suitable Hilbert spaces

$$K^1_{\frac{1}{\sigma},0}(0,1) := \left\{ u \in L^2_{\frac{1}{\sigma}}(0,1) \cap H^1(0,1) : u(0) = 0 \right\},$$

$$\langle u, v \rangle_{1,\frac{1}{\sigma}} := \langle u, v \rangle_{\frac{1}{\sigma}} + \int_0^1 u'v'dx,$$

$$K^2_{\frac{1}{\sigma},0}(0,1) := \left\{ u \in K^1_{\frac{1}{\sigma},0}(0,1) \;\middle|\; Au \in L^2_{\frac{1}{\sigma}}(0,1) \right\},$$

$$\langle u, v \rangle_{2,\frac{1}{\sigma}} := \langle u, v \rangle_{1,\frac{1}{\sigma}} + \langle Au, Av \rangle_{\frac{1}{\sigma}}.$$

Here

$$Au := au_{xx} + bu_x = \sigma(\eta u_x)_x, \quad D(A) := K^2_{\frac{1}{\sigma},0}(0,1).$$

Existence Theorem

Hypothesis (H0): a and $b \in C[0,1]$, $\frac{b}{a} \in L^1(0,1)$, $a(0)=0$ and $\exists\ K_a \in (0,2)$ s.t.

$$x \mapsto \frac{x^{K_a}}{a(x)}$$

is nondecreasing near $x=0$

Theorem (F., Mugnai, 2022)

If $(u_0,u_1) \in K^1_{\frac{1}{\sigma},0}(0,1) \times L^2_{\frac{1}{\sigma}}(0,1)$, then $\exists!$ mild solution u of (P_s) and $u \in C^1([0,+\infty); L^2_{\frac{1}{\sigma}}(0,1)) \cap C([0,+\infty); K^1_{\frac{1}{\sigma},0}(0,1))$.
Moreover, if $(u_0,u_1) \in \mathcal{K}$, the solution u is classical.

$$\mathcal{K} := \Big\{(u,v) \in K^2_{\frac{1}{\sigma},0}(0,1) \times K^1_{\frac{1}{\sigma},0}(0,1) :$$
$$(\eta u')(1) + v(1) + \beta u(1) = 0\Big\}$$

Definition of the energy

Definition

$$E_u(t) = \frac{1}{2}\int_0^1 \left(\frac{1}{\sigma}u_t^2(t,x) + \eta u_x^2(t,x)\right)dx + \beta u^2(t,1), \quad \forall\, t \geq 0.$$

Theorem (F., Mugnai, 2022)

Assume (H0) and let u be a mild solution of (P). Then

$$\frac{dE_u(t)}{dt} = -u_t^2(t,1), \quad \forall\, t \geq 0.$$

Hence E_u is nonincreasing.

Assumptions on a and b

Hypothesis 1 (H1)

a is:

⋆ **Weakly degenerate (WD)** at 0 : $a \in C[0,1] \cap C^1(0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\, K_a \in (0,1)$ s.t.

$$\sup_{x \in (0,1]} \frac{x|a'(x)|}{a(x)} \leq K_a \quad (*)$$

or

⋆ **Strongly degenerate (SD)** at 0 : $a \in C^1[0,1]$, $a(0) = 0$, $a > 0$ on $(0,1]$ and $\exists\, K_a \in [1,2)$ s.t. $(*)$ holds.

b is:

⋆ $b \in C[0,1]$

⋆ $\dfrac{b}{a} \in L^1(0,1)$

Stabilization

Theorem (F., Mugnai, 2022)

Assume (H1) and $K_a \leq 1$. Suppose that $\exists\ \epsilon_0 > 0$ s.t.

$$(2 - K_a) - 2x|b| \geq \epsilon_0 a.$$

Then

$$E_u(t) \leq E_u(0)e^{1-\frac{t}{C}}, \quad \forall\ t \in [C, +\infty)$$

where $C = C(a(1), \min_{[0,1]} \eta, \max_{[0,1]} \eta, \beta, K) > 0$ is independent of (u_0, u_1).

Idea of the proof

Lemma

Assume (H0). Then $\exists\, C_{HP} > 0$ s.t.

$$\int_0^1 \frac{u^2}{\sigma} dx \leq C_{HP} \int_0^1 (u')^2 dx \quad \forall\, u \in K^1_{\frac{1}{\sigma},0}.$$

Thus, for all $u \in K^1_{\frac{1}{\sigma},0}$,

$$\|u\|_1^2 := \int_0^1 (u')^2 dx, \quad |||u|||_1^2 := \int_0^1 \eta (u')^2 dx + \beta u^2(1)$$

are equivalent to

$$\|u\|^2_{1,\frac{1}{\sigma}} := \int_0^1 (u')^2 dx + \int_0^1 \frac{u^2}{\sigma}.$$

Idea of the proof

Lemma

Assume (H1). Then, for all $\lambda \in \mathbb{R}$,

$$\int_0^1 z'\phi' dx + \beta z(1)\phi(1) = \lambda \phi(1)$$

for all $\phi \in K^1_{\frac{1}{\sigma},0}$ admits a unique solution $z \in K^1_{\frac{1}{\sigma},0}$ s.t.

$$|||z|||_1^2 \leq C\lambda^2 \quad \|z\|^2_{L^2_{\frac{1}{\sigma}}(0,1)} \leq \frac{C\lambda^2}{C_{HP}}$$

for $C > 0$. Moreover, $z \in K^2_{\frac{1}{\sigma},0}$ and $\begin{cases} \sigma(\eta z')' = 0, \\ \eta z'(1) + \beta z(1) = \lambda. \end{cases}$

We will apply this result with $\lambda = u(t, 1)$

Idea of the proof

Lemma (see Komornik, 1994)

Assume that $E : [0, +\infty) \to [0, +\infty)$ is a nonincreasing function and that $\exists C > 0$ such that

$$\int_t^{\infty} E(s)ds \leq CE(t) \quad \forall\, t \in [0, +\infty).$$

Then $E(t) \leq E(0)e^{1-\frac{t}{C}}$ for all $t \in [C, +\infty)$.

We can apply the previous Lemma with E_u, obtaining the thesis.

Work in progress and open problems:

- **Stabilization for the previous problem with**
  - ⋆ $K_a > 1$
  - ⋆ **a memory or a damping term**
  - ⋆ **a nonlinear damping at** $x = 1$
- **Controllability for degenerate wave equations with**
  - ⋆ **Control localized in the interior of the domain**
  - ⋆ **Control on the degeneracy point**
  - ⋆ **Bilinear Control**
  - ⋆ **Moving Control**
  - ⋆ **Memory term**
  - ⋆ **Damping term**
- **Controllability and Stabilization for the previous problem with a nonlinear term**

Thank you for your attention