Inverse Problems and homogenization

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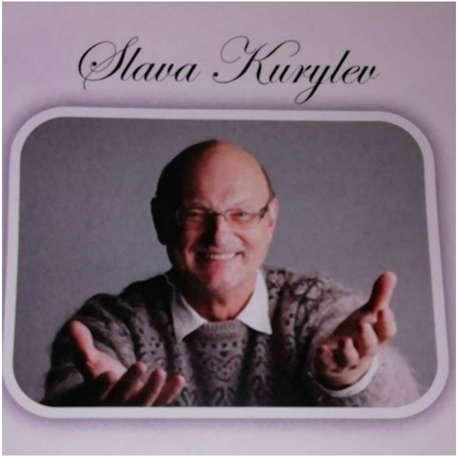
Dedicated to the memory of my friend and mentor

Slava Kurylev (1952-2019)



Slava was a leading figure in the field of inverse problems.
His enormous enthusiasm, vision and speed of mind lead him develop various ambitious programs (Inverse Problems in Relativity, Geometric Whitney theory)
With the spanish group, we were investigating the interplay between
homogenization,inverse problems and geometry

Menu.

- 1st Course. Calderon Problem
  - Intro and Forward Map.
  - Counterexamples. Homogenization.
  - If and only if condition for stability.
  - Quasiconformal maps.
- 2nd Course. Scattering.
  - Buckgheim approach.
  - Connection with $i\partial_t u = \Box u$
  - Taking averages improves the convergence.
- Dessert.
  - Homogenization in paralelizable manifolds.
  - 2-scale convergence.

Calderón meets QC maps and Non linear Fourier Transform

Uniformly strongly elliptic boundary value problems

Let $K \geqslant 1$, $\Omega \subset \mathbb{C}$ bounded domain. We say $\gamma \in \mathcal{G}(K, \Omega)$ when

- Compactly supported: $\mathrm{supp}(\gamma - 1) \subset \overline{\Omega}$.
- Strongly elliptic: $\|\gamma\|_\infty \leqslant K$, $\|\gamma^{-1}\|_\infty \leqslant K$.
- Isotropic conductivity: $\gamma : \mathbb{C} \to \mathbb{R}_+$.

Dirichlet BVP: prescribed electric voltage in the boundary, find voltage

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, \\ u_{|\partial\Omega} = f \in H^{1/2}(\partial\Omega). \end{cases}$$

$$\text{DtN map:} \qquad \Lambda_\gamma : f \mapsto (\gamma \partial_\nu u_{\gamma,f})|_{\partial\Omega}.$$

By integration by part we arrive to the weak formulation

$$\langle \Lambda_\sigma(\varphi), \psi \rangle = \int_\Omega \langle \sigma \nabla u_\varphi, \nabla \tilde{\psi} \rangle dx \wedge dy$$

Calderón's problem

The "forward map"

$$\begin{aligned} \Lambda : \quad \mathcal{G}(K,\Omega) &\to \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right), \\ \gamma &\mapsto \Lambda_\gamma, \end{aligned}$$

is continuous for the distance $\|\gamma_1 - \gamma_2\|_\infty$. Given boundary measurements can we recover the conductivity? That is, find the inverse map

$$\begin{aligned} \Lambda^{-1} : \quad \mathcal{L}\left(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)\right) &\to \mathcal{G}(K,\Omega), \\ \Lambda_\gamma &\mapsto \gamma. \end{aligned}$$

Difficulties

A problem is well-posed if the following conditions hold:

1. A solution exists (if we have perfect, complete data),
2. The solution is unique (planar case, see [Astala, Päivärinta Annnlas '06]), higher-D world record Caro Rogers Lpschitz conductivities (Pi 2018)
3. The solution depends continuously on the input (a priori conditions needed).

Unfortunately Calderón's CIP is severely "ill-posed". We will see that in general there is no hope for stability

Forward map for compactly supported

There is L^p continuity of the forward map for "compactly supported" conductivities:
Let $\{\gamma_j\}_{j=0}^{\infty} \subset \mathcal{G}(K, \tilde{\Omega})$ with $\gamma_j \to \gamma_0$ in L^p, $\tilde{\Omega} \subset\subset \Omega$.
Take u_0, u_j solution to Dirichlet BVP's with data φ. Let $\frac{1}{\tilde{p}} + \frac{1}{q} = \frac{1}{2}$. For $\tilde{p}$ big enough

$$\begin{aligned} \left|\langle(\Lambda_{\gamma_0} - \Lambda_{\gamma_j})\varphi, \varphi\rangle\right| &= \left|\int_\Omega (\gamma_0 - \gamma_j)\nabla u_0 \cdot \nabla u_j\right| \\ &\leqslant \|\gamma_j - \gamma_0\|_{L^{\tilde{p}}} \|\nabla u_0\|_{L^q(\tilde{\Omega})} \|\nabla u_j\|_{L^2} \\ &\leqslant \|\gamma_j - \gamma_0\|_{L^{\tilde{p}}} \|\varphi\|^2_{H^{1/2}} \end{aligned}$$

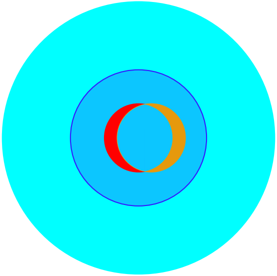
We have continuity of the forward map.
Tools The compact support was important higher integrability [Meyers'63]-[Astala'00].

Counterexamples. Homogenization

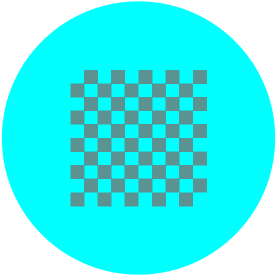
3 Mechanisms for inestability

- Counterexample to Stability in L^{∞} can be produced by of coefficients. Lack of regularity (Alessandrini)
  Solution: Measure stability in different norm.
- Compactness of the set of D-N maps. Mandache 2001. largely improved by Koch-Rulland-Salo 2020. The set of Dirichlet to Neuman is rather compact (quantified in terms of the entropy). The set of conductivities is rather sparse. Qualitatively: The set Y of conductivities has to be compact. Quantitatively: Logarithmic modulus of continuity.
- Oscillating sequences.

Stability counterexamples



Take a constant conductivity in $\mathbb{C}$.
Add the characteristic of $1/4\mathbb{D}$, i.e.
$\gamma_0 := 1 + \chi_{1/4\mathbb{D}}$.
Translate it ε to define $\gamma_\epsilon := 1 + \chi_{\varepsilon+1/4\mathbb{D}}$.
Clearly $\|\gamma_0 - \gamma_\varepsilon\|_\infty = 1$.
But $\|\gamma_0 - \gamma_\varepsilon\|_{\tilde{p}} \to 0$.
Thus, $\Lambda_\varepsilon \to \Lambda_0$, and L^∞ stability fails.



But take $\gamma_j \in \mathcal{G}(2, \mathbb{D})$ defined by
$\gamma_j(z) = 1 + \frac{1}{2}\chi_{\mathbb{Q}}(z)\chi_{chessboard}(jz)$.
The DtN maps converge as well [Alessandrini, Cabib], [Faraco, Kurylev, Ruiz].
But $\{\gamma_j\}$ has no L^p-convergent partial!! L^p stability fails in general! Thus, we seek a priori conditions.

Homogenization enters the game

Folklore: Counterexamples, one needs to look at rapidly oscillating sequences of conductivities.
A (The?) mechanism to have a grip on rapidly oscillating PDEs is the theory of homogenization. Homogenization can be described in several ways, G-convergence (H-converge), correctors, multiscale convergence.

Lemma

If σ_n G-converges to σ then $\Lambda_{\sigma_n} \to \Lambda_\sigma$ weakly.

The proof follows from the weak formulation,

$$\langle \Lambda_\sigma(\varphi), \psi \rangle = \int_\Omega \langle \sigma \nabla u_\varphi, \nabla \tilde{\psi} \rangle dx \wedge dy$$

and G convergence impying convergence of energies.
Therefore to have a richer sets of counterexamples we need to understand the following question.
Suppose that we have G-convergent sequence. When can we update the weak convergence to operator norm convergence?

F-Kurylev-Ruiz

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a domain. Assume that

$$\lim_{\delta \to 0} \delta^{-1} \left(\limsup_{n \to \infty} \|\sigma_n - \sigma\|_{L^\infty(\Omega_\delta)} \right) = 0 \tag{1}$$

and that $\sigma_n \in M_K$ converges to σ in the sense of the $G-$convergence. Then

$$\lim_{n \to \infty} \|\Lambda_{\sigma_n} - \Lambda_\sigma\|_{H(\partial\Omega) \to H^{-1/2}(\partial\Omega)} = 0.$$

(1), is a weak version of

$$\lim_{n \to \infty} \left(\|\nabla_\nu(\sigma_n - \sigma)\|_{L^\infty(\partial\Omega)} + \|\sigma_n - \sigma\|_{L^\infty(\partial\Omega)} \right) = 0. \tag{2}$$

Which is implied by

$$\lim_{n \to \infty} \|\Lambda_{\sigma_n} - \Lambda_\sigma\|_{H(\partial\Omega) \to H^{-1/2}(\partial\Omega)} = 0.$$

- Valid for arbitrary domains (No Lipchitz or even limitations on the dimension of the boundary)
- No regularity assumptions on the cofficients.
- General operators. Magnetic Laplacian, first order resolvent estimates. We call them L^{Λ}.
- Ellipticity is only needed near the boundary. Therefor the result has applications to imp obtaining isotropic almost cloackers

The proof in the case of Alessandrini and Cabib is related to decay of spherical harmonics in the unit ball. Under various assumptions the proof can be extended for some domains.
Here we argue in a complete different manner. More general, less quantitative.
Two main ideas:

- Cacciopoli inequality can be interpreted as improved regularity. Improved regularity means compactness.
- $\Lambda_n - \Lambda = \mathcal{T} \circ \mathcal{A}_n. = \mathcal{A}_n^* \circ \mathcal{T}^*$ where the sequence $\mathcal{A}_n$ converges strongly and $\mathcal{T}$ is compact. Thus by the finite range approximation of compact operators strong convergence follows.

Lemma

Let $w \in \mathring{H}^1(\Omega\backslash\Omega')$ *be a weak solution of*

$$L^\lambda w = f + divF \ \ on\ \Omega\backslash\Omega', \tag{3}$$

for $f \in L^2(\Omega)$ *and* F *being a vector field in* $L^2(\Omega\backslash\Omega')$*. Then, for any* Ω''*,* $\overline{\Omega}' \Subset \Omega''$*,* $\overline{\Omega}'' \Subset \Omega$*, there exists a* $C = C(\Omega, \Omega', \Omega'', K, \lambda)$ *such that*

$$\int_{\Omega\backslash\Omega''} |\nabla w|^2 \leqslant C\left(\int_{\Omega\backslash\Omega'} |w|^2 + \int_{\Omega\backslash\Omega'} |F|^2 + \int_{\Omega\backslash\Omega'} |f|^2\right). \tag{4}$$

Moreover if we choose $\Omega' = \Omega_{2\delta}$ *and* $\Omega'' = \Omega_\delta$ *the estimate is*

$$\int_{\Omega_\delta} |\nabla w|^2 \leqslant C\left(\delta^{-2}\int_{\Omega_{2\delta}} |w|^2 + \int_{\Omega_{2\delta}} |F|^2 + \int_{\Omega_{2\delta}} |f|^2\right). \tag{5}$$

Abstract Compactness arguments

Theorem (Mandache'01)

$\Lambda(\mathcal{G}(K, r_0\mathbb{D}))$ is a pre-compact subset of $\mathcal{L}(H^{1/2}(\partial\mathbb{D}), H^{-1/2}(\partial\mathbb{D}))$.

Let $\mathcal{F} \subset\subset \mathcal{G}(K, \widetilde{\Omega})$ in the L^p distance, with $\widetilde{\Omega} \subset\subset \Omega$. Then, $\mathcal{F}$ is L^p-stable for Ω.

Abstract argument, But no control on its modulus of continuity.

Theorem

Let $K \geqslant 1$, let $r_0 < 1$ and let $\mathcal{F} \subset \mathcal{G}(K, r_0\mathbb{D})$. The family $\mathcal{F}$ is L^2-stable for $\mathbb{D}$ if and only if it is pre-compact.

Alessandrini conjecture

Let $\tau_y f(x) = f(x-y)$. Integral modulus of continuity of f:

$$\omega_p f(t) := \sup_{|y|\leqslant t} \|f - \tau_y f\|_{L^p} \qquad \text{for } 0 \leqslant t \leqslant \infty,$$

Theorem (Kolmogorov-Riesz)

$\mathcal{F} \subset \mathcal{G}(K,\Omega)$ is L^p-precompact if and only if it has a uniform p-integral modulus of continuity $\omega_p f \leqslant \omega_{\mathcal{F}}$: $\mathcal{F} \subset \mathcal{G}(K,\Omega,p,\omega_{\mathcal{F}})$.

Conjecture: Quantify continuity of inverse mapping for any ω.

Solution to Alessandrini conjecture

Theorem (F-Prats)

Let $K \geqslant 1$, let $0 < p < \infty$, let Ω be a bounded domain and let ω be a modulus of continuity. Then the family $\mathcal{G}(K, \Omega, p, \omega)$ is L^2-stable for Ω. In particular

$$\|\gamma_1 - \gamma_2\|_{2s} \leqslant C\ {}_s\eta\left(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{\mathcal{L}(\partial\Omega)}\right)^{\frac{1}{2}\frac{1}{s}}$$

e for every $0 < s < \infty$. Moreover, if ω is continuous,

$$\eta(\rho) \lesssim_{K,p} (Id + \omega)\left(C_{K,p}\,\omega\left(\frac{C_K}{|\log(\rho)|^{\frac{1}{K}}}\right)^{b_{K,p}} + \frac{C_K}{|\log(\rho)|^{\alpha_K}}\right).$$

- Thus there is stability for every bounded domain and every modulus of continuity.
- No "compactly supported" condition!! Every conductivity has an integral modulus of continuity.
- This is a final step of a long program developed in collaboration with A.Ruiz. Based Various papers, Barcelo, Barceló, Clop, Ruiz and Rogers.
- With Barceló and Ruiz we dealt with L^∞ stability for Hölder coefficients. Improved to Dini continuous by MacOwen and Veteel (2020).

CGO Solutions. Non Linear Fourier transform

The CGOS move boundary conditions to infinity: family of solutions parameterized by $k \in \mathbb{C}$, which behave asymptotically as e^{ikz}:

$$\begin{cases} \nabla \cdot (\gamma \nabla u_\gamma(\cdot, k)) \equiv 0, \\ u_\gamma(z,k) = e^{ikz}\left(1 + R(z,k)\right), \text{ with } R(\cdot, k) \in W^{1,p} \end{cases}$$

Interesting behavior in k: for every z

$$\frac{\partial_{\overline{k}} u_\gamma(z,k)}{-i\overline{u_\gamma(z,k)}} = ct(k) =: \tau_\gamma(k).$$

(scattering transform), nonlinear Fourier Transform
Integral formula of the scattering transform and elliptic estimates yield stability from D-N map to scattering transform

$$\boxed{|\tau_1(k) - \tau_2(k)| \lesssim e^{C|k|}\rho}$$

Λ_γ

$R_\gamma(\cdot, k)$

τ_γ

$\log u_\gamma$

u_γ

γ

Pseudoanalytic equations

Next we need to understand how the pseudoanalitic equation (in the k variable) depends on τ.

$$\partial_{\overline{k}} u_\gamma(z,k) = -i\tau_\mu(k)\overline{u_\gamma(z,k)}.$$

• If τ_μ decays (depends on the regularity of μ as the classical Fourier transform) there is uniqueness and stability. Classical PDE

• In absence of decay, we get uniqueness and (bold) stability by using both variables at the same time:

$$\|u_1 - u_2\|_\infty \leqslant \iota(\|\Lambda_1 - \Lambda_2\|_{\mathcal{L}}).$$

$$\rho := \|\Delta\Lambda_\gamma\|_{\mathcal{L}}$$

$$\log u_\gamma - izk = o(k)$$

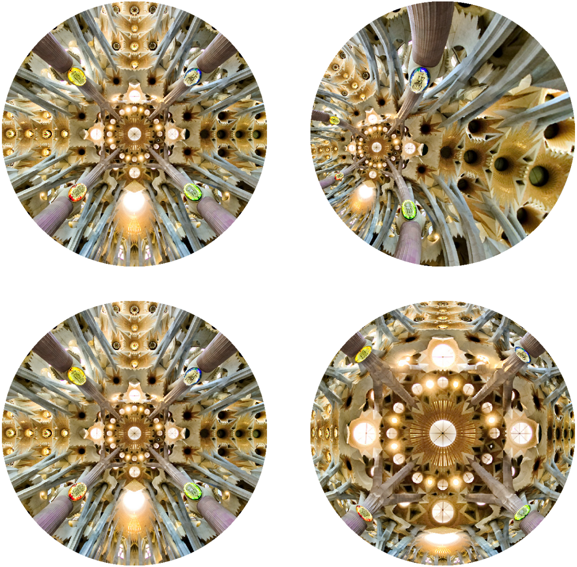
$$\|\Delta\nabla u_\gamma\|_2 \leqslant \iota(\rho)$$

$$\|\Delta\gamma\|_2 \leqslant \eta(\rho)$$

This is a topological argument in both variables: Very difficult.

• Then we use Cacciopoli typer arguments in terms of the modulus of continuity and interpolation to estimate $\|\nabla u_1 - \nabla_2\|_{L^2}$. From that, suitable pointwise estimates allow to estimate $\|\gamma_1 - \gamma_2\|$.

Quasiconformal mappings (Barcelona team: Clop, Prats)



Conformal mappings
Preserves angles
"Circles to circles"
Cauchy-Riemann:
$\frac{1}{2}(\partial_x f + i\partial_y f) = 0$
$\bar{\partial} f = 0$

Quasiconformal mappings
Angle distortion bounded.
"Circles to ellipses".
$|\bar{\partial} f| \leqslant k|\partial f|$

Hodge-* conjugation

Dictionary of divergence equation and Beltrami equation:
Let $\mu := \frac{1-\gamma}{1+\gamma}$. Let $f_\mu := \mathrm{Re}\, u_\gamma + i\, \mathrm{Im} u_{\gamma^{-1}}$. Then

$$\begin{cases} \bar\partial f_\mu = \mu\, \overline{\partial f_\mu} \\ f_\mu(z,k) = e^{ikz}\left(1 + M_\mu\right), \text{ with } M_\mu(\cdot,k) \in W^{1,p}(\mathbb{C}) \end{cases}$$

The logarithm $\varphi_\mu := \frac{\log(f_\mu)}{ik}$ is a quasiconformal principal mapping of $\mathbb{C}$.
Its inverse $\psi_k := \varphi_\mu(\cdot,k)^{-1}$ satisfies the linear Beltrami equation

$$\bar\partial \varphi_\mu(\cdot,k) = -\frac{\bar k}{k}\mu(\cdot)\, e_{-k}(\varphi_\mu(\cdot,k))\, \overline{\partial \varphi_\mu(\cdot,k)}.$$

$$\bar\partial \psi_k(\cdot) = -\frac{\bar k}{k}\mu \circ \psi_k(\cdot)\, e_{-k}(\cdot)\, \partial \psi_k(\cdot).$$

$\Delta\Lambda_\gamma$

$\Delta\tau_\mu$

$\log f_\mu$

$\log u_\gamma$

Δu_γ

$\Delta\gamma$

$\rho := \|\Delta\Lambda_\gamma\|_{\mathcal{L}}$

$|\Delta\tau_\mu(k)| \lesssim \rho e^{C|k|}$

$\log f_\mu$

$\log u_\gamma$

Δu_γ

$\Delta\gamma$

Subexponential behavior in k

We show that $\|\varphi_\mu(\cdot,k) - Id\|_{L^\infty} \leqslant \upsilon(|k|^{-1})$.
Tools:

- interaction of modulus of continuity with translation invariant operators and Fourier transform
- control of every term in the Neumann series respect to k
- Quantify how the composition with qc-maps affects the modulus of continuity.

$$\rho := \|\Delta\Lambda_\gamma\|_{\mathcal{L}}$$

$$\|\Delta M_\mu(\cdot,k)\|_{W\mathbb{D}^c}$$

$$|\Delta\tau_\mu(k)| \lesssim \rho e^{C|k|}$$

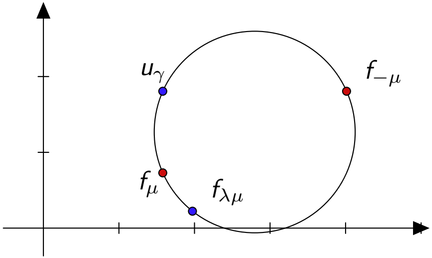
$$\log f_\mu - izk = o(k)$$

$$\log u_\gamma$$

$$\Delta u_\gamma$$

$$\Delta\gamma$$

Back to the conductivity



We see that $\log(u_\gamma) = \log(f_{\lambda\mu})$ for a $\lambda : \mathbb{C} \times \mathbb{C} \to \partial\mathbb{D}$ depending on the point. We infer the same asymptotic behavior

$$|\log(u_\gamma)(z,k) - izk| \leqslant |k| v(|k|^{-1}).$$

This is enough decay to use topological arguments

$$\rho := \|\Delta\Lambda_\gamma\|_{\mathcal{L}}$$

$$\|\Delta M_\mu(\cdot,k)\|_{W\mathbb{D}^c}$$

$$|\Delta\tau_\mu(k)| \lesssim \rho e^{C|k|}$$

$$\log f_\mu - izk = o(k)$$

$$\log u_\gamma - izk = o(k)$$

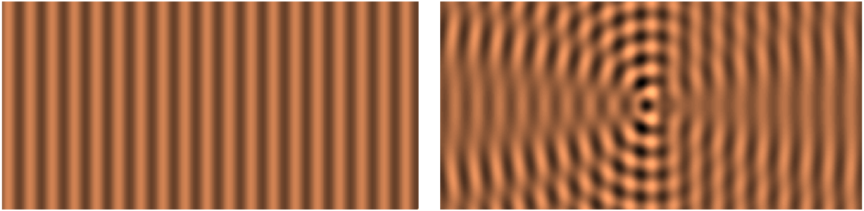
$$\Delta u_\gamma$$

$$\Delta\gamma$$

Calderón meets Carlsesson

Inverse scattering at a fixed energy k^2

- For all $\theta \in \mathbb{S}^{d-1}$ we send plane waves $e^{ik\theta\cdot x}$ toward an unknown object.
- For all $\vartheta \in \mathbb{S}^{d-1}$ we measure the scattered waves.



- If we consider $e^{ik\theta\cdot x}$ to be a sound wave (in air or in water for example), the task is to recover the speed of sound $c(x)$ at each x.
- If we consider $e^{ik\theta\cdot x}$ to be the wavefunction of a beam of neutrons fired at a nucleus, the task is to recover the nuclear potential $V(x)$ at each x.

The PDEs

The plane waves satisfy the Helmholtz equation

$$-\Delta u = k^2 u$$

and so we expect our scattered waves to satisfy distorted versions of this.

Sonar: The scattered waves are supposed to satisfy the accoustic equation

$$-\Delta u = \frac{k^2}{c^2} u.$$

We normalise so that the speed of sound is 1 away from the object.

Nuclear: The scattered waves are supposed to satisfy the time independent Schrödinger equation

$$-\Delta u = k^2 u - Vu.$$

- Writing $V = k^2(1 - \frac{1}{c^2})$, the models are equivalent.
- So from now on we consider only the quantum problem.

So our scattered solutions are supposed to satisfy

$$(-\Delta - k^2)u = -Vu,$$

and we want the solution $u = u_\theta$ that looks most like $e^{ik\theta\cdot x}$.

- That is, u_θ solves the Lippmann–Schwinger equation

$$u_\theta = e^{ik\theta\cdot x} - (-\Delta - k^2)^{-1}[Vu_\theta]$$

which can be written as

$$u_\theta(x) = e^{ik\theta\cdot x} - \int G_0(x-y)V(y)u_\theta(y)\,dy,$$

where in 2D

$$G_0(x-y) = e^{-ik\frac{x}{|x|}\cdot y}\frac{e^{ik|x|}}{\sqrt{k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big).$$

- Thus

$$u_\theta(x) = e^{ik\theta\cdot x} - A(\theta, \tfrac{x}{|x|})\frac{e^{ik|x|}}{\sqrt{k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big).$$

- The challenge is then to recover V from A

Reduction to the DN map

- Let u solve $\Delta u = (V - k^2)u$ with $u|_{\partial\Omega} = f$. Then

$$\Lambda_V : f \mapsto \nabla u \cdot n|_{\partial\Omega},$$

- The first step is to recover this map from the scattering amplitude A.
- First by Nachman's formula (1988),

$$\Lambda_V - \Lambda_0 = S_V^{-1} - S_0^{-1},$$

$$S_V[f] := \int_{\partial\Omega} G_V(x,y) f(y)\, dy, \qquad (-\Delta + V - k^2) G_V(x,y) = \delta(x-y).$$

- Then adapting the 3D work of Stefanov (1991), we obtain

$$G_V - G_0 = Formula(A).$$

- The challenge is then to recover V from $\Lambda_V - \Lambda_0$.

Alessandrini's identity

$$\left\langle (\Lambda_V - \Lambda_0)[u|_{\partial\Omega}], v|_{\partial\Omega} \right\rangle = \int_\Omega Vuv.$$

Proof :
As $\Delta u = Vu$ and $\Delta v = 0$, by integration by parts,

$$\begin{aligned}
\int_\Omega Vuv &= \int_\Omega \Delta uv \\
&= \int_{\partial\Omega} \nabla u \cdot n\, v - \int_\Omega \nabla u \cdot \nabla v \\
&= \left\langle \Lambda_V[u], v \right\rangle - \int_\Omega \nabla u \cdot \nabla v \\
&= \left\langle \Lambda_V[u], v \right\rangle - \int_{\partial\Omega} u \nabla v \cdot n + \int_\Omega u\Delta v \\
&= \left\langle \Lambda_V[u], v \right\rangle - \left\langle u, \Lambda_0[v] \right\rangle \\
&= \left\langle (\Lambda_V - \Lambda_0)[u], v \right\rangle
\end{aligned}$$

Strategy: The strategy of complex geometric optic solutions (Faddeev solutions) consists in chosing product $uv = e^{i\xi x}(1+r)$.
In $3D$ it works for bounded potencials (Sylvester-Uhlman Annals 1987).
Since for smooth conductivities it extends to conductivities in C^1.
There are analogous Alessandrini identities for less regular coefficients but the problem is to show that the remainder tends to zero.
The conductivity equation can be reduced to the Schröndinger at least formally $\gamma^{\frac{1}{2}} V = \Delta \gamma^{\frac{1}{2}}$
The current world record is by Caro-Rogers (Pi 2016) which prove the result for Lipschitz conductivites. (Haberman-Tataru C^1, Haberman better results for $n = 3, 4$)

[1] Prove that only one potential V can give rise to a given $\Lambda_V - \Lambda_0$.

- Bukhgeim (2008), $V \in C^1$
- Blåsten smanuvilov–Yamamoto (2015), $V \in L^p,\ p > 2$

[2] Give a formula which gives V in terms of $\Lambda_V - \Lambda_0$.

- Bukhgeim (2008) + Novikov–Santacesaria (2011), $V \in C^1$
- Astala–F–Rogers. , $V \in H^{1/2}$

[3] Give an algorithm which can compute V given $\Lambda_V - \Lambda_0$.

- Tejero thesis

Bukhgeim's solutions to Laplace's equation

- Quadratic phases!

$$\psi_{n,x}(z) = \tfrac{n}{8}\Big((z_1 - x_1)^2 - (z_2 - x_2)^2 + 2(z_1 - x_1)(z_2 - x_2)i\Big).$$

- Identifying (z_1, z_2) with $z_1 + iz_2$, we have $\psi_{n,x}(z) = \frac{n}{8}(z - x)^2$.

Thus $e^{i\psi_{n,x}}$ and $e^{i\overline{\psi}_{n,x}}$ are holomorphic and antiholomorphic, respectively.

- Writing

$$\Delta = (\partial_{z_1} + i\partial_{z_2})(\partial_{z_1} - i\partial_{z_2}),$$

we see that $e^{i\psi_{n,x}}$, $e^{i\overline{\psi}_{n,x}}$ are solutions to $\Delta v = 0$.

- The solutions grow exponentially at infinity, but $|e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}}| = 1$.

Bukhgeim's heuristic

Suppose that the potential V is smooth and that $e^{i\psi_{n,x}}$ were a solution to

$$\Delta u = Vu.$$

Then by Alessandrini's identity,

$$\begin{aligned}\left\langle (\Lambda_V - \Lambda_0)[e^{i\psi_{n,x}}], e^{i\overline{\psi}_{n,x}} \right\rangle &= \int_\Omega V\, e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \\ &= \int V(z)\, e^{i\frac{n}{4}\left((z_1-x_1)^2-(z_2-x_2)^2\right)}\, dz.\end{aligned}$$

Thus by the method of stationary phase,

$$\begin{aligned}\frac{n}{4\pi}\left\langle (\Lambda_V - \Lambda_0)[e^{i\psi_{n,x}}], e^{i\overline{\psi}_{n,x}} \right\rangle &= \int V(z)\, \frac{n}{4\pi}\, e^{i\frac{n}{4}\left((z_1-x_1)^2-(z_2-x_2)^2\right)}\, dz \\ &= V * K_n(x) \\ &\to V(x) \qquad \text{as} \quad n \to \infty.\end{aligned}$$

Making the heuristic precise

As before, but with $e^{i\psi_{n,x}}$ replaced by $u = e^{i\psi_{n,x}} + e^{i\psi_{n,x}} w$,

$$\frac{n}{4\pi}\left\langle (\Lambda_V - \Lambda_0)[u], e^{i\overline{\psi}_{n,x}} \right\rangle = \frac{n}{4\pi}\int_\Omega V\, u\, e^{i\overline{\psi}_{n,x}}$$
$$= \int V \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} + \int V\, w \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}}.$$

We expect

$$\int V\, \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \to V(x) \quad \text{as} \quad n \to \infty, \qquad \text{(conv)}$$

so we also need to prove

$$\int V\, w\, \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \to 0 \quad \text{as} \quad n \to \infty. \qquad \text{(remainder)}$$

Remainder is dealt with via Van der Corput Lemma and Cauchy transform estimates. By now it is well understood, so we focuse on the main term.

Main term: Carlesson meets Calderón

Writing $K_n(y) = \frac{n}{4\pi} e^{i\frac{n}{4}(y_1^2 - y_2^2)}$, it remains to prove

$$V * K_n(x) \to V(x) \quad \text{as} \quad n \to \infty. \qquad \text{(conv)}$$

As $V * K_n = \left(\widehat{V}\widehat{K}_n\right)^\vee$, we see that

$$V * K_n = \left(\widehat{V}(\xi)\, e^{-i\frac{1}{n}(\xi_1^2 - \xi_2^2)}\right)^\vee =: e^{i\frac{1}{n}\Box} V,$$

which, at time $t = 1/n$, solves

$$i\partial_t u + \Box u = 0, \quad u(\cdot, 0) = V,$$

where $\Box = \partial_{x_1x_1} - \partial_{x_2x_2}$.

Thus (conv) can be interpreted as the convergence of the solution to a time dependent equation to its initial data as time tends to zero.

Theorem (Astala-F-Rogers 16)

If $V \in H^{1/2}$ then (conv) holds for all $x \in \Omega \backslash E$ with $\dim_H(E) \leqslant 3/2$.

What happens below H^s

Inspired by Carlesson problem. First explicit example that can not be recovered.

$$V(x) = \sum_{j\geq 2} V_j(x) = \sum_{j\geq 2} 2^{(1-\beta)j+1}\cos(2^j x_2)\phi(2^j x_1)\phi(x_2)$$

Jorgue Tejero thesis was devoted to understand this issue. Three messages.

•If the potencials are piecewise smooth. The algoritm still converges.

• Buckgheim algorithm is very suitable for taking various averages, which gives recovery in H^s.

• The average algorithms seem to have better convergence properties.

Piecewise smooth reconstruction

Theorem

Let q be a piecewise-$W^{2,1}$ complex-valued potential, then

$$\lim_{\lambda\to\infty} \frac{\lambda}{\pi} \int_{\partial\Omega} e^{i\lambda\overline{\psi_x}} (\Lambda_q - \Lambda_0)[u_{\lambda,x}] = q(x), \quad a.e.\ x \in \Omega.$$

Moreover, if q is piecewise-$W^{s,1}$ with $2 < s < 3$ then

$$\left| q(x) - \frac{\lambda}{\pi} \int_{\partial\Omega} e^{i\lambda\overline{\psi_x}} (\Lambda_q - \Lambda_0)[u_{\lambda,x}] \right| \leq C_x \lambda^{1-\frac{s}{2}} \left(\|q\|_{D^{s,r}} + \|q\|^2_{D^{s,r}} \right)$$

for almost every $x \in \Omega$.

- $\|\cdot\|_{D^{s,r}}$ is a suitable norm for piecewise-$W^{s,1}$ potentials.

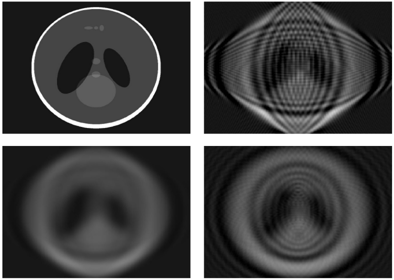
$B_{I_\lambda}(x) = \frac{\lambda}{4\pi}\left\langle (\Lambda_V - \Lambda_0)[u], e^{i\overline{\psi}_{\lambda,x}} \right\rangle$ When the Buckheim algorithm does not work we can take suitable averages.

Theorem (Tejero)

Let $s > 0$, let $q \in H^s(\mathbb{R}^2)$ be a complex-valued potential supported in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and let $\sigma = \lambda^{-1/4}$. Then

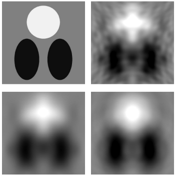
$$\lim_{\lambda \to \infty} \varphi_\sigma * BI_\lambda(x) = q(x) \quad \text{a.e. } x \in \Omega.$$

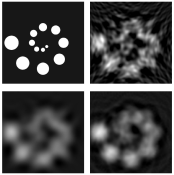
The main term improves



Taking suitale averages
Improve the pictures
,

Performance (on the main term) I





Different examples-
No simetries.
Still averaging does
much better

Dessert on manifolds

Back to Slava and Homogenization

Question: How to obtain explicit formula for the homogenization in arbitrary manifolds?

• Parallelizable manifold.

Set (M, g) a Riemanian manifold. We say that is is parallelizable if there exist global smooth vector fields $\Gamma(p) = (X_1, X_2, X_n)$.

•The thorus bundle. $\mathbb{T}M = \frac{TM}{\sim}$

$(p, v_1) \sim (q, v_2) \iff p = q and v_1 - v_2 = \sum k_i X_i$

• $M = \cup D_i$, D_i ϵVoronoi domain with center p_i and approximately ϵ.

• The partition of unity ψ_i related to D_i.

$$\mathrm{vol}\{\,\nabla\psi_i \neq 0\,\} \leqslant C'\epsilon^{\beta(n-1)+\alpha}, \mathrm{vol}(\cup_i \mathrm{supp}\nabla\psi_i) \leqslant C''\epsilon^{\alpha-\beta}.$$

It is not so easy how to speak of explicitly oscillating sequences in an arbitrary manifolds. A natural way, is to start with periodic functions in the tangent bundle and bring them back to the manifold via the exponential map. Given $f \in C^\infty(\mathbb{T}M))$ for every ϵ.

We define $H_{p_j,\epsilon}(q) = \frac{\exp_{p_j}^{-1}(q)}{\epsilon}$
Then,

$$\bullet f^\epsilon(q) = \sum \psi_j f(p_j, \frac{exp_{p_j}^{-1}(q)}{\epsilon}) = \sum \psi_j H^*_{\epsilon,p_j}(f)$$

An analogous definition can be stated for k, m tensors. Particularly if we start with 2 tensors.

$$\bullet, (A)^\epsilon(q) = \sum \psi_j(q) H^*_\epsilon(A).$$

The homogenized problem We define $A^* \in T^{0,2}(M)$ by stating that for any $e_1, e_2 \in T_p^*(M)$

$$A^*(p)[e_i, e_j] = \int_{\Gamma_p} A(p,v)[d_v w_i(p,v) + e_i, d_v w_j(p,v) + e_j] dv$$

Here $w_i(p,v)$ is the $\Gamma(p)$ periodic solution to the cell problem

$$-div_v(A(p,v)[\nabla_v w_i(p,v) + e_i]) = 0$$

Our main theorem is the following homogenization result.

Theorem (Guijarro-F-Kurylev-Ruiz)

Given $f \in H^{-1}(M)$ the unique solution $u_\epsilon \in H_0^1(\Omega)$ of the problem

$$div(A^\epsilon[du_\epsilon]) = f$$

converge weakly to $u^* \in H_0^1(\Omega)$ the unique solution to

$$div(A^*[du^*]) = f$$

•. Main tool. Studying two scale convergence on manifolds

• Natural definitions for functions. A sequence u^ϵ two scale converges to $u \in L^2(M)$ two scale converges to $u \in L^2(\mathbb{T}(M))$

$$\int_M f^\epsilon u_\epsilon \to \int_{\mathcal{T}M} f(p,v)u$$

• Various extension for forms or vector fields in $\mathbb{T}M$ but it gets technical to deal with the horizontal and vertical parts of the vector field.

Let X^ϵ a suitably defined oscillating vector field and $h = div_v(X)$.

The most difficult part is the following lemma.

Theorem 9.2. *Let $u \in H^1(M)$. Then, as $\epsilon \to 0$,*

$$\left| -\int_M \langle du, X^\epsilon \rangle \, dq - \frac{1}{\epsilon}\int_M u \cdot h^\epsilon \, dq - \int_M u \cdot \operatorname{div} \widetilde{X} \, dq \right| \to 0, \qquad (9.3)$$

Moreover, this convergence is uniform when u lies in a bounded set in $H^1(M)$.

• Potential pplications. E.g Darcy law, theories 3D to 2D in elasticity.

The end

Muchas Gracias!!