Local null controllability of a fluid-rigid body interaction problem with Navier slip boundary conditions

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FSI problems

Fluid structure problems are coupling systems that involve generally a fluid and rigid or deformable structures.

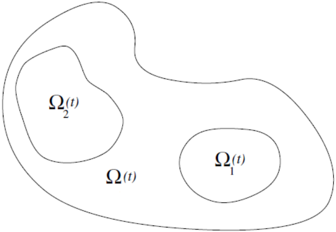


Figure – Fluid-solid interaction problem

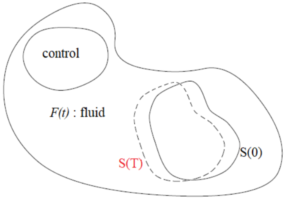
Objectives

The system writes :

$$\begin{cases} z' = Az + Bv + F(z), \\ a'(t) = Cz(t), \\ z(0) = z^0, \\ a(0) = a^0. \end{cases}$$

- z stands for the velocity of the fluid and the structure velocities and a stands for the position of the rigid body.
- v is the control.

Objectives : Find a distributed control v such that : $z(T) = 0$ and $a(T) = a_T$.



Specificities of this work

- FSI problems : free boundary problem.
- Navier boundary conditions.
- Non linear problem.
- Coupled system.

Boundary conditions

The non-slip (or Dirichlet) boundary conditions :

$$U_f = v_s \text{ on } \partial\Omega.$$

The slip (or Navier) boundary conditions :

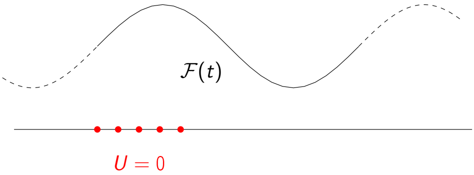
$$\begin{cases} (U_f - v_s)_n = 0 & \text{on } \partial\Omega, \quad \text{(Impermeability condition)} \\ [\mathbb{T}(U_f, P_f)n + \beta\,(U_f - v_s)]_\tau = 0 & \text{on } \partial\Omega, \quad \text{(Slip condition)} \end{cases}$$

- $\mathbb{T}(U_f, P_f)n$: The force exerted by the fluid on the surface.
- β : The friction coefficient.

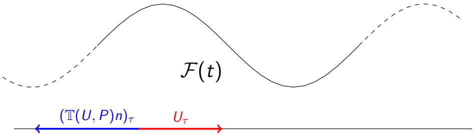
In the FSI problems, the vector U_f designates the fluid velocity and the vector v_s stands for the structure velocity.

Boundary conditions

Dirichlet boundary conditions



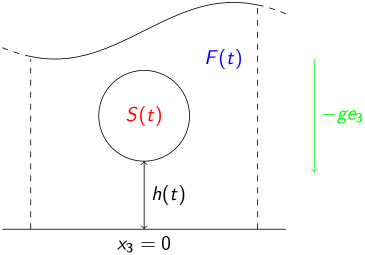
Navier boundary conditions



$$U_n = 0, \quad U_\tau = -\frac{1}{\beta}(\mathbb{T}(U,P)n)_\tau$$

The influence of the boundary conditions

Rigid structure



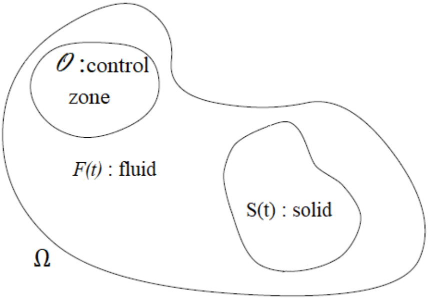
- M. Hillairet, D. Gérard-Varet, C. Wang. 2014 :
  - Dirichlet boundary conditions : No contact at $x = 0$.
  - Navier boundary conditions : Collision can occur at $x = 0$.

Previous works with Navier boundary conditions

Existence results :

- Existence of weak solutions : D. Gérard-Varet, M. Hillairet. 2014
- Uniqueness of the weak solution : N V. Chemetov, Š. Nečasová, B. Muha. 2017.
- Existence and uniqueness of strong solution : C. Wang. 2014

Problem setting



- $\mathcal{F}(t)$: the fluid domain.
- $S(t)$: the rigid body.
- $\Omega = \mathcal{F}(t) \cup S(t)$.

$$\partial\mathcal{F}(t) = \partial S(t) \cup \partial\Omega$$

Fluid equations

The incompressible Navier-Stokes system

$$\begin{cases} \partial_t U + (U\cdot\nabla)U - \nabla\cdot\mathbb{T}(U,P) = v1_{\mathcal{O}}, & \text{in } (0,T)\times\mathcal{F}(t), \\ \nabla\cdot U = 0, & \text{in } (0,T)\times\mathcal{F}(t), \end{cases}$$

- $\mathcal{F}(t)\subset\mathbb{R}^3$: Fluid domain that depends on time.
- $U(t,x)\in\mathbb{R}^3$: The fluid velocity.
- $P(t,x)$: The fluid pressure.

The non linear term :

$$(U\cdot\nabla)U = \sum_{j=1}^{3} U_j \frac{\partial U}{\partial x_j}.$$

The Cauchy stress tensor

$$\mathbb{T}(U,P) = -PI_2 + 2\nu D(U),$$

ν : The viscosity of the fluid.

$$[D(U)]_{i,j} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right).$$

Structure equations

The structure position :

$$X_S(t,y) = h(t) + R_{\theta(t)}y, \quad y \in S.$$

$h(t)$: The solid position of the solid.
θ : The rotation of the solid.
The structure velocity

$$U_S(t,x) = h'(t) + \omega(t)(x - h(t))^{\perp}, \quad x \in \partial S(t),$$

$\omega(t)$: The angular velocity of the solid.
Newton's laws :

$$\begin{cases} mh''(t) = -\int_{\partial S(t)} \mathbb{T}(U,P)n \, d\Gamma & t \in (0,T), \\ J\omega'(t) = -\int_{\partial S(t)} (x - h(t))^{\perp} \cdot \mathbb{T}(U,P)n \, d\Gamma & t \in (0,T). \end{cases}$$

$m > 0$ is the mass of the rigid structure and $J > 0$ its moment of inertia

Navier boundary conditions

On the fixed boundary part $\partial\Omega$, the boundary conditions write

$$\left\{ \begin{aligned} U_n &= 0 \quad \text{on } (0,T)\times\partial\Omega, , \\ [2\nu D(U)n + \beta_\Omega U]_\tau &= 0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \right.$$

Navier boundary conditions

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On the moving boundary $\partial S(t)$, the boundary conditions write

$$\begin{cases} (U - U_S)_n = 0 & \text{on } (0,T)\times\partial S(t), \\ [2\nu D(U)n + \beta_S (U - U_S)]_\tau = 0 & \text{on } (0,T)\times\partial S(t). \end{cases}$$

where β_Ω and β_S are the friction coefficients.

Navier boundary conditions

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where β_Ω and β_S are the friction coefficients.
Initial conditions

$$U(0,x) = u^0(x),\ x \in \mathcal{F}(0), \quad h(0) = h^0, \quad h'(0) = h^1, \quad \omega(0) = \omega^0,$$

Problem setting

$$\begin{cases} \partial_t U + (U\cdot\nabla)U - \nabla\cdot\mathbb{T}(U,P) = v1_{\mathcal{O}} & t\in(0,T),\ x\in\mathcal{F}(t), \\ \nabla\cdot U = 0 & t\in(0,T),\ x\in\mathcal{F}(t), \end{cases}$$

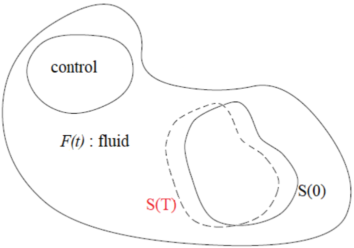
$$\begin{cases} mh''(t) = -\int_{\partial S(t)} \mathbb{T}(U,P)n\,d\Gamma & t\in(0,T), \\ J\omega'(t) = -\int_{\partial S(t)} (x-h(t))^{\perp}\cdot\mathbb{T}(U,P)n\,d\Gamma & t\in(0,T). \end{cases}$$

$$\begin{cases} U_n = 0 & t\in(0,T),\ x\in\partial\Omega, \\ \left[2\nu D(U)n + \beta_\Omega U\right]_\tau = 0 & t\in(0,T),\ x\in\partial\Omega, \\ (U-U_S)_n = 0 & t\in(0,T),\ x\in\partial S(t), \\ \left[2\nu D(U)n + \beta_S (U-U_S)\right]_\tau = 0 & t\in(0,T),\ x\in\partial S(t), \end{cases}$$

$$U(0,x) = u^0(x),\ x\in\mathcal{F}(0), \quad h(0) = h^0, \quad h'(0) = h^1, \quad \omega(0) = \omega^0,$$

Problem setting

Objective : Find a control v such that the velocities are equal to zero at final time T.



Main result

Theorem

Assume that $\beta_S > 0$. There exists $v \in L^2(0, T; [L^2(\mathcal{O})]^3)$ such that

$$U(T, .) = 0 \text{ in } \mathcal{F}(T), \quad h(T) = 0, \quad h'(T) = 0$$

$$\omega(T) = 0, \quad \theta(T) = 0,$$

provided that the velocities are small enough and the final state is close enough to the initial state.

Note : If $h(T) = h_T$ and $\theta(T) = \theta_T$, we can reduce the problem to $h(T) = 0$, $\theta(T) = 0$.

Previous null controllability results

Dirichlet boundary conditions :

- O. Imanuvilov, T. Takahashi. 2007.
- M. Boulakia and A. Osses. 2007.
- M. Boulakia and S. Guerrero. 2011.

Outline

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Change of variables

T. Takahashi. 2003.

- Change of variables : $X(t,\cdot) : \Omega \longrightarrow \Omega$ such that

$$X(t,\cdot) = X_S(t,\cdot) : S = S(T) \longrightarrow \mathcal{S}(t)$$

$$Y(t,\cdot) = X^{-1}(t,\cdot)$$

$$u(t,\cdot) = \nabla Y(t, X(t,\cdot))U(t, X(t,\cdot)), \quad p(t,\cdot) = P(t, X(t,\cdot)).$$

Change of variables

C. Wang. 2014.

Fluid equations :

$$u(t,y) = (\nabla Y)(t, X(t,y))U(t, X(t,y)), \quad p(y,t) = P(t, X(t,y)),$$

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u,p) = v1_{\mathcal{O}} + F(u,p) & t \in (0,T),\ y \in \mathcal{F}, \\ \nabla \cdot u = 0 & t \in (0,T),\ y \in \mathcal{F}, \end{cases}$$

Structure equations

$$u_S = h'(t) + \omega(t) y^\perp, \quad y \in \partial S,$$

$$\begin{cases} mh''(t) = -\int_{\partial S} \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T), \\ J\omega'(t) = -\int_{\partial S} y^\perp \cdot \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T), \end{cases}$$

Boundary conditions

$$\begin{cases} u_n = 0 & t \in (0,T),\ y \in \partial\Omega, \\ [2\nu D(u)n + \beta_\Omega u]_\tau = 0 & t \in (0,T),\ y \in \partial\Omega, \\ (u - u_S)_n = 0 & t \in (0,T),\ y \in \partial S, \\ [2\nu D(u)n + \beta_S (u - u_S)]_\tau = 0 & t \in (0,T),\ y \in \partial S. \end{cases}$$

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Linearized system

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u,p) = f + v 1_{\mathcal{O}} & t \in (0,T),\ y \in \mathcal{F}, \\ \nabla \cdot u = 0 & t \in (0,T),\ y \in \mathcal{F}. \end{cases}$$

$$\begin{cases} m h''(t) = -\int_{\partial S} \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T), \\ J\omega'(t) = -\int_{\partial S} y^{\perp} \cdot \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T). \end{cases}$$

$$u_S = h'(t) + \omega(t) y^{\perp}, \quad y \in \partial S.$$

Boundary conditions

$$\begin{cases} u_n = 0 & t \in (0,T),\ y \in \partial\Omega, \\ [2\nu D(u)n + \beta_\Omega u]_\tau = 0 & t \in (0,T),\ y \in \partial\Omega, \\ (u - u_S)_n = 0 & t \in (0,T),\ y \in \partial S, \\ [2\nu D(u)n + \beta_S (u - u_S)]_\tau = 0 & t \in (0,T),\ y \in \partial S. \end{cases}$$

$$u(0,y) = u^0(y),\ y \in \mathcal{F}, \quad h(0) = h^0, \quad h'(0) = h^1, \quad \omega(0) = \omega^0,$$

Linearized system

The linearized system writes :

$$\begin{cases} u' = Au + Bv + F, \\ a'(t) = Cu(t), \\ u(0) = u^0, \\ a(0) = a^0. \end{cases}$$

A is introduced in C. Wang. 2014,

$$a = (h, \theta), \quad Cu = (h', \omega).$$

Theorem (Null controllability of the linearized system)

For all F, u^0, a^0 there exists a control $v \in L^2(0, T; [L^2(\mathcal{O})]^2)$ such that $u(T) = 0$ and $a(T) = 0$.

Main steps of the proof

We use the theory of O. Imanuvilov, T. Takahashi. 2007.

- **Main ingredient :** Carleman estimate for the adjoint system of the linearized problem.
- **Difficulties** :
  - Control the velocity of the rigid body with the fluid velocity :

$$\|u_S \cdot n\|_{L^2(\partial S)} \leq C \, \|u\|_{L^2(\mathcal{F})} \, ,$$

$$\beta_S \, \|u_S \cdot \tau\|_{L^2(\partial S)} \leq C \Big(\|u\|_{L^2(\partial \mathcal{F})} + \|u\|_{L^2(\mathcal{F})} + \|(\nabla \times u)_\tau\|_{L^2(\partial \mathcal{F})} \Big).$$

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Fixed-point

$$\begin{cases} \partial_t u - \nabla \cdot \mathbb{T}(u,p) = v 1_{\mathcal{O}} + F(u,p) & t \in (0,T),\ y \in \mathcal{F}, \\ \nabla \cdot u = 0 & t \in (0,T),\ y \in \mathcal{F}, \end{cases}$$

$$u_S = h'(t) + \omega(t) y^{\perp}, \quad y \in \partial S,$$

$$\begin{cases} m h''(t) = -\int_{\partial S} \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T), \\ J\omega'(t) = -\int_{\partial S} y^{\perp} \mathbb{T}(u,p) n \, d\Gamma & t \in (0,T), \end{cases}$$

$$\begin{cases} u_n = 0 & t \in (0,T),\ y \in \partial\Omega, \\ [2\nu D(u) n + \beta_\Omega u]_\tau = 0 & t \in (0,T),\ y \in \partial\Omega, \\ (u - u_S)_n = 0 & t \in (0,T),\ y \in \partial S, \\ [2\nu D(u) n + \beta_S (u - u_S)]_\tau = 0 & t \in (0,T),\ y \in \partial S. \end{cases}$$

There exists a function $\rho \in C([0,T])$ such that the application

$$\Phi : f \longrightarrow F(u,p)$$

defines a contraction on

$$K = \left\{ f \in L^2(0,T;[L^2(\mathcal{F})]^2), \quad \left\| \frac{f}{\rho} \right\|_{L^2(0,T;[L^2(\mathcal{F})]^2)} \leq R \right\} \text{ and } \Phi(K) \subset K$$

Conclusion and perspectives

- Global controllability.
- Reduce the controls.

  Navier Stokes system : S.Guerrero and C.Montoyay. 2017
- Deformable structure : Dirichlet boundary condition : J.Lequeurre 2013

Thank you for your attention