Controllability of the Wave Equation on a Rough Compact Manifold [1]

Belhassen DEHMAN[2]

Webinar "Control in Time of Crisis"

November 2020

[1] Joint work with N. Burq (Univ. Paris Sud) and J. Le Rousseau (Univ. Paris 13)

[2] Faculté des Sciences de Tunis & Enit-Lamsin

Setting

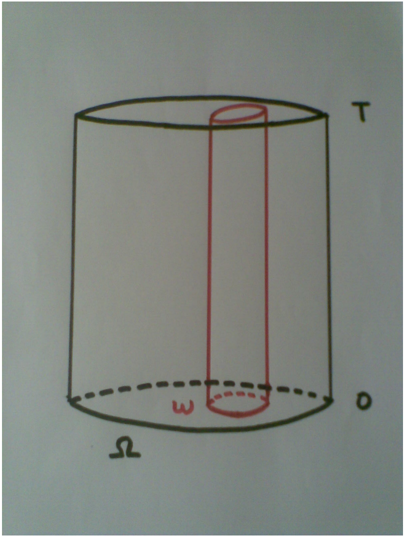
$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta_x u = 0 \qquad \text{in} \quad]0,+\infty[\times M \\ (u(0),\partial_t u(0)) = (u_0,u_1) \in H^1 \times L^2 \end{cases}$$

- M Riemannian manifold, connected, compact, without boundary, with dimension d.
- $M = \Omega$ open subset of $\mathbb{R}^d$, connected, bounded, with smooth enough boundary (homogeneous Dirichlet condition).

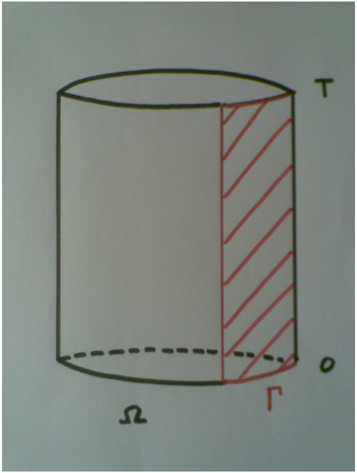
$$H = C([0,+\infty[, H^1) \cap C^1([0,+\infty[, L^2)$$

$$Eu(t) = ||\nabla_x u(t,.)||^2_{L^2(\Omega)} + ||\partial_t u(t,.)||^2_{L^2(\Omega)} = Eu(0)$$

ω open subset of Ω and $T > 0$ (suitable)



Γ open subset of $\partial\Omega$ and $T > 0$ (suitable)



The Goal

Provide internal or boundary exact control results in the case of **non smooth metrics**.
Given (u_0, u_1), find a control vector f (resp. g) s.t the solution of

$$\begin{cases} \partial_t^2 u - \Delta_x u = \chi_\omega f \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

resp.

$$\begin{cases} \partial_t^2 u - \Delta_x u = 0 \\ u = \chi_\Gamma g \qquad \text{on} \quad \partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

satisfies $u(T) = \partial_t u(T) = 0$.

Tool : By HUM , we need an observability estimate for the wave equation (W).

Observability estimates

Boundary observation

$$Eu(0) \leq c \int_0^T \int_\Gamma \left|\frac{\partial u}{\partial n}(t,x)\right|^2 d\sigma dt$$

Remark: The converse is "always" true :

$$\int_0^T \int_{\partial\Omega} \left|\frac{\partial u}{\partial n}(t,x)\right|^2 d\sigma dt \leq c\, Eu(0)$$

$\rightarrow$ Hidden regularity.

Internal observation

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt \qquad (O)$$

Or at least

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + c||(u_0, u_1)||^2_{L^2(M)\times H^{-1}(M)} \qquad (RO)$$

$\rightarrow$ unique continuation property....

Or either observation with loss

$$Eu(0) \leq c||u||^2_{H^m((0,T)\times\omega)}, \qquad m > 1 \qquad (OL)$$

$\rightarrow$ (RO) can be interpreted as propagation of the H^1 wave front of u.

Other Applications

$\rightarrow$ **Stabilization**

$$Eu(t) \leq C\, exp^{-\gamma t} Eu(0)$$

for solutions of the damped equation

$$\partial_t^2 u - \Delta_x u + a(x)\partial_t u = 0$$

$\rightarrow$ **Inverse problems**

Stability results, ...

State of the art

- 80' : Observability estimates under the Γ-condition of J.L. Lions.
  $\rightarrow$ Metric of class C^1.
  $\rightarrow$ Multiplier techniques.

- 90' : Microlocal conditions and microlocal tools (Rauch and Taylor (75'), Bardos, Lebeau and Rauch (92'), Burq and Gérard (97')).
  The geometric control condition **(G.C.C)** : a microlocal condition, stated in the compressed cotangent bundle of Melrose-Sjöstrand (82').

  $\rightarrow$ Microlocal and pseudo-differential techniques : propagation of wave front sets and supports of microlocal defect measures.
  $\rightarrow$ The condition is optimal but....... a priori needs smooth metric and smooth boundary.

- 97' N. Burq : Boundary observability: C^2-metric and C^3-boundary.
- The OL theorem of Fanelli-Zuazua (2014)

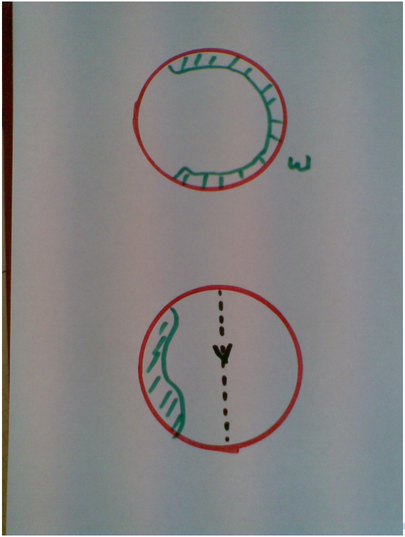
1-D setting, $\Omega =]0,1[$, $a(x)\partial_t^2 u - \partial_x^2 u = 0.$

$\rightarrow$ Classical boundary observation for $a(x) \in Z$ (Zygmund class) and boundary observation with **loss** for $a(x) \in LZ$ (Log-Zygmund class).
$\rightarrow$ For $a(x)$ worse that LZ, infinite loss of derivatives :
No Observability ! (See also the counter-example of Castro - Zuazua (03')).
$\rightarrow$ Proof: 1-dimensional technique: the sidewise energy estimates, i.e hyperbolic energy estimates by interchanging time $\longleftrightarrow$ space.
(Colombini, Spagnolo, Lerner, Métivier, Fanelli....)
$\rightarrow$ 1-D geometry: all characteristic rays reach the boundary in uniform time.

Question: What about dimensions higher than 1, where geometry is more evolved ???

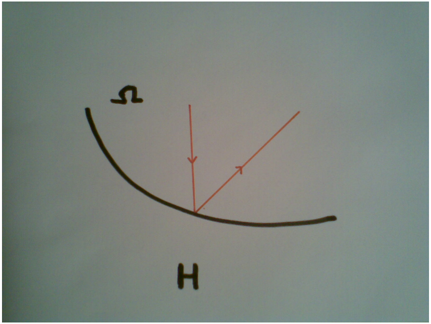
Geometric Control Condition I

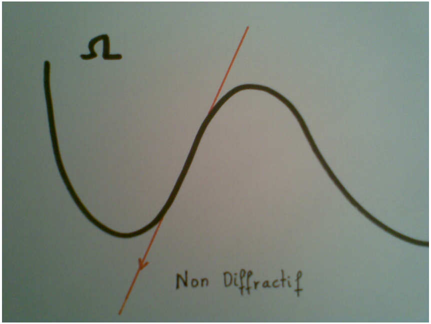
The couple (ω, T) satisfies the geometric control condition (G.C.C), if every geodesic of Ω issued at $t = 0$ and travelling with speed 1, enters in ω before the time T.



Geometric Control Condition II

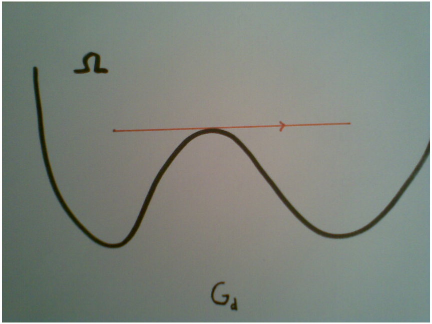
The couple (Γ, T) satisfies the geometric control condition (G.C.C), if every **generalized bicharacteristic** of the wave symbol, issued at $t = 0$ and travelling with speed 1, intersects the boundary subset Γ at a **nondiffractive** point, before the time T.



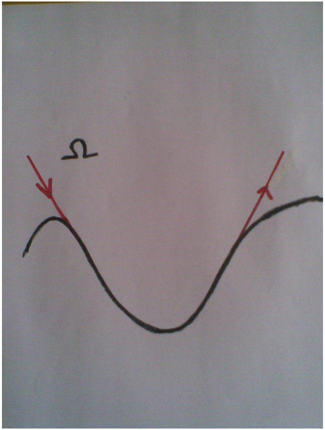


Hyperbolic

Non Diffractive



Diffractive



Glancing

Under GCC, observability/controllability is stable with respect to small Lipschitz perturbations.

Denote $A(x) = (a_{ij}(x))$, $d \times d$ symmetric definite positive matrix, and κ a real valued function, $\kappa(x) > 0$ (a density).

Denote $\mathcal{A} = (A, \kappa)$,

$$\Delta_{\mathcal{A}} = \frac{1}{\kappa(x)} \sum_{ij} \partial_j a_{ij}(x) \kappa(x) \partial_i$$

and consider the wave operator

$$P_{\mathcal{A}} = \partial_t^2 - \Delta_{\mathcal{A}}$$

$\rightarrow$ With $\kappa = (det\, A)^{-1/2}$, $\Delta_{\mathcal{A}}$ is the Laplace-Beltrami operator attached to the metric A.

Theorem (Burq-D-Le Rousseau 19')

Assume that $\mathcal{A} = (A, \kappa)$ *is* **smooth** *and that* (ω, T) *(resp.* (Γ, T)*) satisfies (GCC) for* $P_{\mathcal{A}}$*.*
Consider $\mathcal{U}_\varepsilon$ *an* ε*-neighborhood of* $\mathcal{A}$ *in* $W^{1,\infty}$*, and* $\mathcal{B} \in \mathcal{U}_\varepsilon$ *.*
Then for ε **small enough**, *the (classical) observability estimate holds true*

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 \kappa dx dt \quad \left(resp. \int_0^T \int_\Gamma \left|\frac{\partial u}{\partial n}(t,x)\right|^2 \kappa d\sigma dt \right)$$

for every solution of

$$P_{\mathcal{B}} u = \partial_t^2 u - \Delta_{\mathcal{B}} u = 0, \qquad u_{|\partial\Omega} = 0$$

Corollary

Under conditions above, we get exact controllability for $P_{\mathcal{B}} = \partial_t^2 - \Delta_{\mathcal{B}}$*, in time* T*.*

Comments

$\rightarrow$ The wave equation is well posed for $P_{\mathcal{B}}$ (Colombini, Spagnolo ...)

$\rightarrow$ No geometry setting for the metric B !!!

$\rightarrow$ One can take $\partial\Omega$ of class C^2.

$\rightarrow$ Under GCC, observability/controllability is stable with respect to small Lipschitz perturbations .

$\rightarrow$ For a given metric $B \in W^{1,\infty}$, we cannot decide if $P_{\mathcal{B}}$ is observable or not.

$\rightarrow$ In particular, what happens for $B \in C^1$???

Remark

Exact controlability property itself is *stable* by lower (first) order perturbations of the Laplace operator, but *not* by perturbations of the geometry/metric.
Consider the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$ endowed with its standard metric and with control domain $\omega = \{x \in \mathbb{S}^d; x_1 > 0\}$.

$\rightarrow$ Exact controlability holds true for this geometry (Lebeau 92')

$\rightarrow$ It doesn't hold for $\omega_\varepsilon = \{x \in \mathbb{S}^d; x_1 > \varepsilon\}$.
(which is ε-close to Lebeau's example in C^∞ topology).

1) The assumption that the asymptotic geometry satisfies GCC *cannot* be replaced by the assumption that is satisfies the exact controlability property.
2) Our perturbation argument will have to be performed on the proof of the fact that GCC implies exact controllability and *not* on the final property itself.

Contradiction argument

$$\|\mathcal{A}-\mathcal{A}_k\|_{W^{1,\infty}} \to 0, \quad \text{and for each } k, \quad \|(u_0^{k,p}, u_1^{k,p})\|_{H^1\times L^2} = 1$$

$$P_k u_k^p = \partial_t^2 u_k^p - \Delta_{\mathcal{A}_k} u_k^p = 0 \qquad \text{on} \quad (0,T)\times\Omega$$

and

$$\int_0^T \int_\omega |\partial_t u_k^p|^2 dxdt \le 1/p, \qquad \forall p \ge 1$$

Denoting $u_k^k \rightsquigarrow u_k$, $\|(u_0^k, u_1^k)\|_{H^1\times L^2} = 1$ and $\|u_k\|_{H^1((0,T)\times\omega)} \to 0$.
Moreover assume

$$u_k \rightharpoonup 0 \quad \text{weakly in} \quad H^1((0,T)\times\Omega)$$

Let μ be a microlocal defect measure attached to the sequence u_k.

$$P_0 u_k = (P_0 - P_k)u_k = (\Delta_{\mathcal{A}_k} - \Delta_{\mathcal{A}})u_k \longrightarrow 0 \quad \text{in} \quad H^{-1}$$

Consider $Q = q(t, x; D_t, D_x) \in Op(S^1_{cl})$.
And calculate the bracket

$$\Big([P_0, Q]u_k, u_k\Big)_{L^2} = \Big([\Delta_{\mathcal{A}_k} - \Delta_{\mathcal{A}}, Q]u_k, u_k\Big)_{L^2} + o(1)_{k\to\infty}$$

$$\rightsquigarrow \quad \Big([\partial_j(a^k_{ij} - a^0_{ij})\partial_i, Q]u_k, u_k\Big)_{L^2} \approx -\Big([(a^k_{ij} - a^0_{ij}), Q]\partial_i u_k, \partial_j u_k\Big)_{L^2}$$

Theorem (Calderon, Coifman-Meyer 85')

For any function $m \in W^{1,\infty}(\mathbb{R}^{d+1})$, the bracket $[m, Q]$ continuously maps $L^2(\mathbb{R}^{d+1})$ in itself and

$$\|[m, Q]\|_{L^2\to L^2} \leq C\|m\|_{W^{1,\infty}}$$

Thus $\Big([P_0, Q]u_k, u_k\Big)_{L^2} \longrightarrow 0$ and ${}^t H_{p_0}\mu = 0$.
μ is invariant along the hamiltonian flow of P_0 (propagation).

Behavior of the HUM control process

$$
(W) \quad \begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}
$$

We look for $f \in L^2(]0,T[\times\Omega)$, s.t

$$
(u(T), \partial_t u(T)) = (0,0)
$$

$$(W) \quad \begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

We look for $f \in L^2(]0,T[\times\Omega)$, s.t

$$(u(T), \partial_t u(T)) = (0,0)$$

By HUM and **under** (**G.C.C**), we can take f solution of

$$(W') \quad \begin{cases} \partial_t^2 f - \Delta_{\mathcal{A}} f = 0 & \text{in} \quad]0,T[\times\Omega \\ f = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (f(0), \partial_t f(0)) = (f_0, f_1) \in L^2 \times H^{-1} \end{cases}$$

The map

$$\begin{cases} \Lambda : H_0^1 \times L^2 \to L^2 \times H^{-1} \\ \\ \quad (u_0, u_1) \to (f_0, f_1) \end{cases}$$

is an isomorphism; this is HUM optimal control operator.

Denote by $f_{\mathcal{A}}$ the HUM control attached to $P_{\mathcal{A}}$, i.e the solution of (W').

Theorem (D-Lebeau, 2009)

In the setting above and under (G.C.C),
a) For all $s \geq 0$,

$$\Lambda : H^{s+1} \times H^s \to H^s \times H^{s-1}$$

is an isomorphism.

b)

$$\left\| \Lambda \psi(2^{-k} D) - \psi(2^{-k} D) \Lambda \right\| \leq C 2^{-k/2}$$

c) If M is a Riemannian manifold without boundary, Λ *is a pseudo differential operator.*

Here

$$\sum_{k \geq 0} \psi(2^{-k} D) = Id$$

is the Littlewood-Paley decomposition.

Behavior of the HUM control process

Let $\mathcal{A} = (A, \kappa)$ be smooth and such that (ω, T) satisfies (GCC).

Theorem

For any C^∞- neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exist $\mathcal{A}' \in \mathcal{U}$ and an initial data (u_0, u_1) , $||(\nabla_A u_0, u_1)||_{L^2 \times L^2} = 1$, s.t the respective solutions u and v of

$$\begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f_{\mathcal{A}} \\ \partial_t^2 v - \Delta_{\mathcal{A}'} v = \chi_\omega^2(x) f_{\mathcal{A}} \\ (u(0), \partial_t u(0)) = (v(0), \partial_t v(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

satisfy

$$E_A(u - v)(T) = E_A(v)(T) \geq 1/2$$

Moreover,

$$||f_{\mathcal{A}} - f_{\mathcal{A}'}||_{L^2((0,T)\times\omega)} \geq c > 0$$

Remarks

- (GCC) also satisfied by (ω, T) for the metric $\mathcal{A}'$.

  $\rightarrow \quad f_{\mathcal{A}'}$ is well defined.

- For fixed initial data, the map

$$(\mathcal{C}^\infty(\Omega))^{d^2+1} \longrightarrow L^2((0,T)\times\omega)$$

$$\mathcal{A} = (A,\kappa) \longrightarrow f_{\mathcal{A}}$$

  is not continuous.

- This result is not valid for control of smooth data.

Proof : Choosing $A' = (1+\varepsilon)A$, one has $CharP_{\mathcal{A}} \cap CharP_{\mathcal{A}'} = \{0\}$.....

Proof of the observability estimate

Remind the goal : Prove observability estimate for the waves with rough metric (of class C^1).

Strategy

→ First prove

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + ||(u_0, u_1)||^2_{L^2 \times H^{-1}}$$

→ In the smooth case :

Contradiction argument and propagation of micro local defect measures.

Back to non smooth metric

To be achieved : Prove a propagation result for μ, in a low regularity setting.

Key point : C^0 - ODE's

$\longrightarrow$ Study the propagation properties of a nonnegative Radon measure μ on $\mathbb{R}^d$, subject to

$$ {}^tX\mu = 0 \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^d), $$

i.e

$$ \langle \mu, X\chi \rangle = 0 \qquad \forall \chi \in C_0^1(\mathbb{R}^d), $$

where the vector field

$$ X = \sum_1^d a_j(x)\partial_{x_j} $$

has continuous coefficients.

Warm-up : The smooth case

If X has smooth (C^1 or Lipschitz) coefficients, it admits a flow given by

$$\Phi_s(x) = (x_1(s), ...x_d(s))$$

$$\frac{d\Phi_s}{ds} = X\Phi_s \,, \qquad\qquad \Phi_0(x) = x.$$

For $\chi \in C_0^1(\mathbb{R}^d)$,

$$\langle {}^t X\mu, \chi(\Phi_s)\rangle = \langle \mu, X\chi(\Phi_s)\rangle = \langle \mu, \frac{d}{ds}\chi(\Phi_s)\rangle = \frac{d}{ds}\langle \mu, \chi(\Phi_s)\rangle = 0$$

$$\langle \mu, \chi(\Phi_s)\rangle = \langle \mu, \chi\rangle, \qquad \forall s.$$

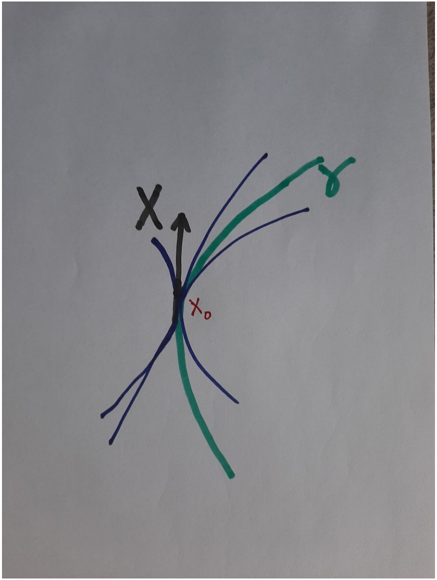
$\longrightarrow$ Propagation for the measure μ

Theorem (Ambrosio-Crippa 13' , Burq-D-Le Rousseau 19')

Consider X a continuous vector field on $\mathbb{R}^d$ and μ a nonnegative Radon measure on $\mathbb{R}^d$ solution to ${}^tX\mu = 0$ in the sense of distributions. Then, the support of the measure μ is a union of maximally extended integral curves of X.

1. In other words, if $x_0 \in \operatorname{supp}(\mu)$, then there exists a neighborhood of x_0 and an integral curve γ of X, $x_0 \in \gamma$, and $\gamma \subset \operatorname{supp}(\mu)$ locally.
2. In particular if the trajectory γ is unique, then $\gamma \subset \operatorname{supp}(\mu)$.
3. Propagation of the support of the measure μ.
4. Ambrosio-Crippa result is only valid in the free space (away from boundary).
5. The positivity of μ is crucial !

$\longrightarrow$ Proof : Ascoli theorem + positive commutators....



$x_0 \in \mathrm{supp}(\mu)$ and $\gamma \subset \mathrm{supp}(\mu)$

The main result

Theorem (Burq-D-Le Rousseau 19')

Assume that $\mathcal{A} = (A, \kappa)$ is of class C^1 and the domain Ω of class C^2, and that (ω, T) (resp. (Γ, T)) satisfies (GGCC).
Then the observability estimate

$$Eu(0) \le c \int_0^T \int_\omega |\partial_t u(t,x)|^2 \kappa dxdt \quad \left(resp. \int_0^T \int_\Gamma \left| \frac{\partial u}{\partial n}(t,x) \right|^2 \kappa d\sigma dt \right)$$

holds true for every solution of

$$P_{\mathcal{A}} u = \partial_t^2 u - \Delta_{\mathcal{A}} u = 0, \qquad u_{|\partial\Omega} = 0$$

Corollary

Under conditions above, we get exact controllability for $P_{\mathcal{A}} = \partial_t^2 - \Delta_{\mathcal{A}}$, in time T.

Denote by $h \in (0, h_0]$ a small parameter.

For $a \in L^1_{loc}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ such that

$$M(a) = \max_{|\alpha| \leq d+1} \operatorname{ess\,sup}_{(x,\xi)} \left|\partial_\xi^\alpha a(x,\xi)\right| (1 + |\xi|)^{d+1} < +\infty,$$

we set for $u \in L^2(\mathbb{R}^d)$

$$Op^h(a)u(x) = a(x, hD_x)u(x) = (2\pi)^{-d} \int e^{ix\cdot\xi} a(x, h\xi)\widehat{u}(\xi)d\xi.$$

Theorem (L^2-Continuity)

In the setting above, $a(x, hD_x)$ is uniformly bounded on $L^2(\mathbb{R}^d)$ and

$$\|a(x, hD_x)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_d M(a).$$

Theorem (Bracket Lemma)

Assume that $a(x,\xi) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. Then

$$\forall \theta = \theta(x) \in C(\mathbb{R}^d), \quad \|\theta\|_{L^\infty} \leq C, \quad \lim_{h \to 0} \|[a(x, hD_x), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = 0$$

Moreover if $\theta \in W^{1,\infty}(\mathbb{R}^d)$, then

$$\|[a(x, hD_x), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\, h \|\theta\|_{W^{1,\infty}}$$

$\longrightarrow$ Key tool : Schur Lemma.

Semi-calssical measures

We denote by $\Sigma(\mathbb{R}^{2d})$, the space of functions $a \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ defined previously and have compact supports (in the x variable)

Definition

Let $h_k \to 0$ for $k \to +\infty$, be a sequence of scales, and (u_k) a sequence bounded in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. A measure μ on $\mathbb{R}^{2d}$ is a semi-classical measure (s.c.m) for the sequence (u_k) at scale (h_k) iif for any $a(x,\xi) \in \Sigma(\mathbb{R}^{2d})$

$$\lim_{k \to +\infty} \Big(a(x, h_k D_x) u_k, u_k \Big)_{L^2(\mathbb{R}^d)} = \langle \mu, a \rangle.$$

Properties

- Support localization .
- Propagation up to the boundary (smooth metric).

The propagation equation

Theorem (Gérard-Leichtnam 93', Burq 97', Burq-D-Le Rousseau 19')

The two measures μ and ν satisfy in the distributions sense

$$ {}^tH_{p_{\mathcal{A}}}(\mu) = \int_{\rho\in\mathcal{H}\cup\mathcal{G}} \frac{\delta(\xi-\xi_+(\rho))-\delta(\xi-\xi_-(\rho))}{<\xi_+-\xi_-, n(x(\rho))>}\nu(d\rho). $$

Measures

Here μ is a s.c.m attached to (u_k) in $L^2(\mathbb{R}\times\mathbb{R}^d)$ and ν is a s.c.m attached to $(v_k = \frac{\partial u_k}{\partial \nu_k})$ in $L^2(\mathbb{R}\times\partial\Omega)$ (at some scale h_k).

Tool : Careful examination of the bracket $[P_{\mathcal{A}}, Q]$ + Weierstrass preparation theorem....

Consequence : The support of μ is a union of closed generalized bicharacteristic rays of $H_{p_{\mathcal{A}}}$.

$\longrightarrow$ Melrose - Sjöstrand theory for continuous hamiltonian field $H_{p_{\mathcal{A}}}$.

Proof of the propagation result (away from the boundary)

Proposition 1 Let X be a C^0-vector field on Ω an open set of $\mathbb{R}^d$. For a closed set F of Ω, the following two properties are equivalent :

1. The set F is a union of maximally extended integral curves of the vector field X.
2. For any compact $K \subset \Omega$ where the vector field X does not vanish,

$$\begin{cases} \forall \varepsilon > 0, \quad \exists \delta_0 > 0, \quad \forall x \in K \cap F, \quad \forall \delta \in]0, \delta_0], \\ \qquad\qquad B\big(x + \delta X(x), \delta\varepsilon\big) \cap F \neq \emptyset. \end{cases}$$

Proposition 2 If μ is a nonnegative measure on Ω solution to ${}^tX\mu = 0$ in the sense of distributions , then the closed set $F = supp(\mu)$ satisfies the second property in Proposition 1.

Proof of 2) $\Longrightarrow$ 1).
Let $n \geq 1$ and $x \in F$. Set $x_{n,0} = x$ and $\varepsilon = 1/n$ and apply 2).

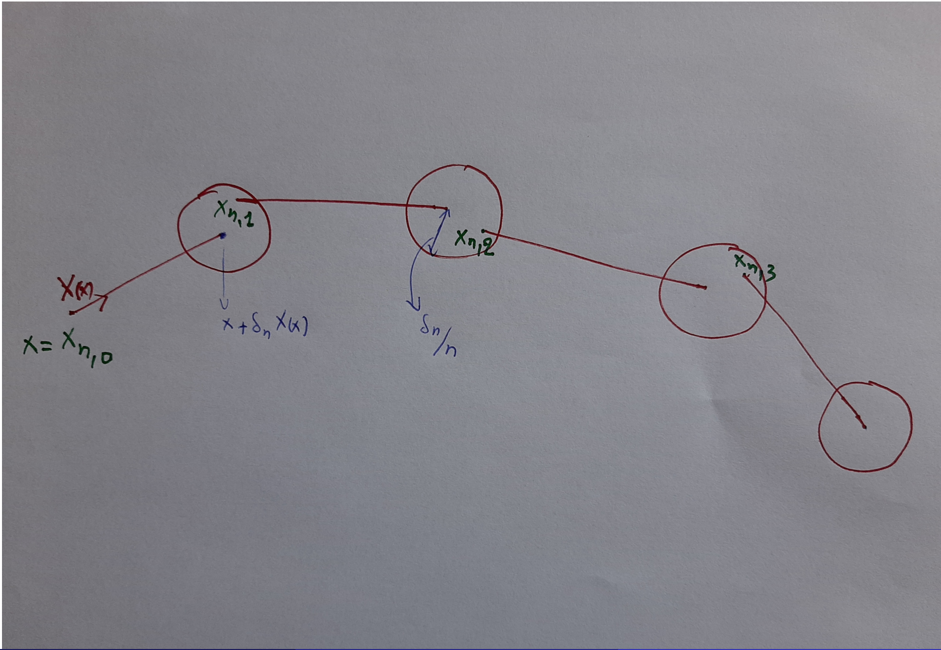
There exists $0 < \delta_n \leq 1/n$ and a point

$$x_{n,1} \in F \cap B\big(x_{n,0} + \delta_n X(x_{n,0}), \delta_n/n\big).$$

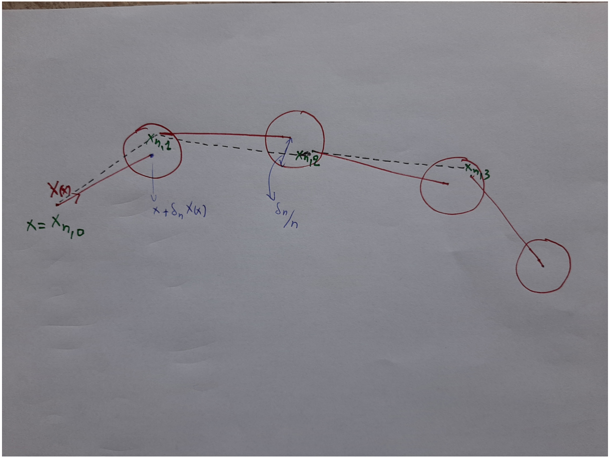
Perform this construction again, yet starting from $x_{n,1}$ instead of $x_{n,0}$.

If a sequence of points $x_{n,0}, x_{n,1}, \ldots, x_{n,L_n}$ is obtained in this manner one has

$$x_{n,\ell+1} \in F \cap B\big(x_{n,\ell} + \delta_n X(x_{n,\ell}), \delta_n/n\big), \qquad \ell = 0, \ldots, L_n - 1. \tag{1}$$



$x_{n,1}$
$x_{n,2}$
$x_{n,3}$
$X(x)$
$x = x_{n,0}$
$x + \delta_n X(x)$
δ_n/n



Define the affine curve (the dotted line)

$$\gamma_n(s) = x_{n,\ell} + (s - \ell\delta_n)\frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n}$$

for $s \in [\ell\delta_n, (\ell+1)\delta_n)$ and $\ell \leq L_n - 1$.

Note that $\gamma_n(s)$ remains in a compact set, uniformly with respect to n. In this compact set X is uniformly continuous.

Note also that since $x_{n,\ell} \in F$ for $\ell \leq L_n$ then one has

$$dist\big(\gamma_n(s), F\big) \leq \delta_n(C_K + 1/n), \qquad |s| \leq S. \tag{2}$$

From (1), for $\ell \geq 0$ and $s \in (\ell\delta_n, (\ell+1)\delta_n)$, we have

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} = \frac{\Big(x_{n,\ell} + \delta_n X(x_{n,\ell}) + \mathcal{O}(1/n)\Big) - x_{n,\ell}}{\delta_n}$$

$$= X(x_{n,\ell}) + \mathcal{O}(1/n).$$

Using the uniform continuity of the vector field X, we find

$$\dot{\gamma}_n(s) = X(\gamma_n(s)) + e_n(s),$$

where the error $|e_n|$ goes to zero *uniformly* with respect to $|s| \leq S$ as $n \to +\infty$.

Since the curve γ_n is continuous, we find

$$\gamma_n(s) = \gamma_n(0) + \int_0^s \dot{\gamma}_n(\sigma)d\sigma = x + \int_0^s X(\gamma_n(\sigma))d\sigma + \int_0^s e_n(\sigma)\, d\sigma. \quad (3)$$

We now let n grow to infinity. The family of curves $(s \mapsto \gamma_n(s), |s| \leq S)_{n\in\mathbb{N}^*}$ is equicontinuous and pointwise bounded; by the Arzelà-Ascoli theorem we can extract a subsequence $(s \mapsto \gamma_{n_p})_{p\in\mathbb{N}}$ that converges uniformly to a curve $\gamma(s), |s| \leq S$. Passing to the limit $n_p \to +\infty$, we find that $\gamma(s)$ is solution to

$$\gamma(s) = x + \int_0^s X(\gamma(\sigma))d\sigma.$$

Finally, for any $|s| \leq S$, there exists $(y_p)_p \subset F$ such that $\lim_{p\to+\infty} y_p = \gamma(s)$. Since F is closed we conclude that $\gamma(s) \in F$.