# Rapid and finite-time stabilization

#### Jean-Michel Coron Sorbonne Université (Paris 6), Laboratoire J.-L. Lions





# Webinar "Control in Time of Crisis" September 8 2020

## Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws

#### 1 Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- 3 Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws

# Cart-inverted pendulum: the equilibrium



# Instability of the equilibrium



# Instability of the equilibrium



# Stabilization of the equilibrium



◆□ > ◆□ > ◆臣 > ◆臣 > 善臣 - のへで

# Double inverted pendulum (CAS, ENSMP/La Villette)







# Rapid stabilization



#### Motivation of the stabilization/rapid stabilization problems

#### 2 Some results in finite dimension

- Stabilizability problem
- Linear systems and applications to nonlinear systems
- $\bullet$  Necessity to remove the  $C^1\mbox{-smoothness}$  of the feedback laws
- Finite-time stabilization of linear controllable systems
- Obstruction to the stabilizability
- Finite-time stabilization and time-varying feedback laws

#### 3 Small-time stabilization of 1-D linear heat equations

- 4 Rapid exponential stabilization of Korteweg-de Vries equations
- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- Finite-time stabilization of 1-D linear balance laws

We consider the control system  $\dot{y} = f(y, u)$  where y in  $\mathbb{R}^n$  is the state and u in  $\mathbb{R}^m$  is the control. We assume that f(0, 0) = 0.

#### Problem

Does there exists  $u : \mathbb{R}^n \to \mathbb{R}^m$  vanishing at 0 such that  $0 \in \mathbb{R}^n$  is (locally) asymptotically stable for  $\dot{y} = f(y, u(y))$ ? (If the answer is yes, one says that the control system is locally asymptotically stabilizable.)

#### Remark

The map  $u : y \in \mathbb{R}^n \mapsto \mathbb{R}^m$  is called a feedback (or feedback law). The dynamical system  $\dot{y} = f(y, u(y))$  is called the closed loop system.

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

The regularity of  $y \mapsto u(y)$  is an important point. With u continuous, asymptotic stability implies the existence of a smooth strict Lyapunov function and one has robustness with respect to small actuator errors as well as small measurement errors.

If u is discontinuous, one needs to define the notion of solution of the closed loop system  $\dot{y}=f(y,u(y))$  and study carefully the robustness of the closed loop system.

Let T > 0. Given two states  $y^0$  and  $y^1$ , does there exist a control  $t \in [0,T] \mapsto u(t)$  which steers the control system from  $y^0$  to  $y^1$ , i.e. such that

(1) 
$$(\dot{y} = f(y, u(t)), y(0) = y^0) \Rightarrow (y(T) = y^1)?$$

If the answer is yes, the control system is said to be controllable on [0, T].

The control system is

(1) 
$$\dot{y} = Ay + Bu, \ y \in \mathbb{R}^n, \ u \in \mathbb{R}^m,$$

where  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times m}$ .

#### Theorem (Kalman's rank condition (1960))

The linear control system  $\dot{y} = Ay + Bu$  is controllable on [0,T] if and only if

▲日▶ ▲□▶ ▲ヨ▶ ▲ヨ▶ ヨー シタの

(2) Span 
$$\{A^i Bu; u \in \mathbb{R}^m, i \in \{0, 1, \dots, n-1\}\} = \mathbb{R}^n$$
.

We assume that  $(y_e, u_e)$  is an equilibrium, i.e.,  $f(y_e, u_e) = 0$ . Many possible choices for natural definitions of local controllability. The most popular one is Small-Time Local Controllability (STLC): the state remains close to  $y_e$ , the control remains close to  $u_e$  and the time is small.







▲ロト ▲園 ▶ ▲ 臣 ▶ ▲臣 ▶ ▲ 臣 → りへぐ



▲□▶ ▲圖▶ ▲≣▶ ▲≣▶ = = -の��



◆□ ▶ ◆昼 ▶ ◆ 重 ▶ ◆ 国 ▶ ◆ □ ▶



## The linear test

We consider the control system  $\dot{y} = f(y, u)$  where the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}^m$ . Let us assume that  $f(y_e, u_e) = 0$ . We are interested in the small-time local controllability of  $\dot{y} = f(y, u)$  around  $(y_e, u_e)$ . L. Nirenberg, besides to be a great mathematician, always gave great advices when you have no more idea to solve a given problem. I was told that one of his famous advices is

# The linear test

We consider the control system  $\dot{y} = f(y, u)$  where the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}^m$ . Let us assume that  $f(y_e, u_e) = 0$ . We are interested in the small-time local controllability of  $\dot{y} = f(y, u)$  around  $(y_e, u_e)$ . L. Nirenberg, besides to be a great mathematician, always gave great advices when you have no more idea to solve a given problem. I was told that one of his famous advices is

Have you tried to linearize?

# The linear test

We consider the control system  $\dot{y} = f(y, u)$  where the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}^m$ . Let us assume that  $f(y_e, u_e) = 0$ . We are interested in the small-time local controllability of  $\dot{y} = f(y, u)$  around  $(y_e, u_e)$ . L. Nirenberg, besides to be a great mathematician, always gave great advices when you have no more idea to solve a given problem. I was told that one of his famous advices is

#### Have you tried to linearize?

We follow Nirenberg's advice. The linearized control system at  $(y_e,u_e)$  is the linear control system  $\dot{y}=Ay+Bu$  with

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

(1) 
$$A := \frac{\partial f}{\partial y}(y_e, u_e), \ B := \frac{\partial f}{\partial u}(y_e, u_e).$$

If the linearized control system  $\dot{y} = Ay + Bu$  is controllable, then  $\dot{y} = f(y, u)$  is small-time locally controllable at  $(y_e, u_e)$ .

# Stabilizability of linear controllable systems

Notations. For a matrix  $M \in \mathbb{R}^{n \times n}$ ,  $P_M$  denotes the characteristic polynomial of M:  $P_M(z) := \det (zI - M)$ .

Let us denote by  $\mathcal{P}_n$  the set of polynomials of degree n in z such that the coefficients are all real numbers and such that the coefficient of  $z^n$  is 1. One has the following theorem.

Theorem (Pole shifting theorem, M. Wonham (1967))

Let us assume that the linear control system  $\dot{y} = Ay + Bu$  is controllable. Then

(1) 
$$\{P_{A+BK}; K \in \mathbb{R}^{m \times n}\} = \mathcal{P}_n$$

#### Corollary

If the linear control system  $\dot{y} = Ay + Bu$  is controllable, there exists a linear feedback  $y \mapsto u(y) = Ky$  such that  $0 \in \mathbb{R}^n$  is (globally) asymptotically stable for the closed loop system  $\dot{y} = Ay + Bu(y)$ .

# Application to nonlinear controllable systems

We assume that f(0,0) = 0. Let us consider the linearized control system  $\dot{y} = Ay + Bu$  of  $\dot{y} = f(y,u)$  at  $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ :

(1) 
$$A := \frac{\partial f}{\partial y}(0,0), \ B := \frac{\partial f}{\partial u}(0,0).$$

Let us assume that the linearized control system  $\dot{y} = Ay + Bu$  is controllable. Then, by the pole-shifting theorem, there exists  $K \in \mathbb{R}^{m \times n}$ such that  $\sigma(A + BK) = \{-1\}$ . Let us consider the feedback u(y) = Ky. Then, if X(y) := f(y, u(y)), X'(0) = A + BK. Hence, by Lyapunov's first theorem,  $0 \in \mathbb{R}^n$  is locally asymptotically stable for the closed loop system  $\dot{y} = f(y, u(y))$ .

In conclusion, if the linearized control system is controllable, then

- The control system  $\dot{y} = f(y, u)$  is small-time locally controllable at (0, 0).
- The control system  $\dot{y} = f(y, u)$  is locally asymptotically stabilizable (at the equilibrium (0, 0)).

# A first notion of rapid stabilization: Rapid exponential stabilization

We consider the control system

(1) 
$$\dot{y} = f(y, u),$$

where the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}^m$ . We assume that f(0,0) = 0. A first possible notion for rapid exponential is the rapid exponential stabilization. It is the following property: For every  $\nu > 0$ , there exist a feedback law  $y \in \mathbb{R}^n \mapsto u(y) \in \mathbb{R}^m$ , C > 0 and r > 0 such that, for every solution of the closed loop system  $\dot{y} = f(y, u(y))$  such that  $|y(0)| \leq r$ , one has

(2) 
$$|y(t)| \leqslant C e^{-\nu t} |y(0)|, \ \forall t \ge 0$$

One has the following theorem.

## Theorem (Corollary of the Pole shifting theorem)

If the linear control system  $\dot{y} = Ay + Bu$  is controllable, the rapid exponential stabilization property holds for this control system. If the linearized control control system at  $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$  of  $\dot{y} = f(y,u)$  is controllable, then  $\dot{y} = f(y,u)$  is rapidly exponentially stabilizable.

# An example: Cart-inverted pendulum



▲□▶▲圖▶▲≧▶▲≧▶ 差 のへぐ

Let

(1) 
$$y_1 := \xi, y_2 := \theta, y_3 := \dot{\xi}, y_4 := \dot{\theta}, u := F,$$

The dynamics of the cart-inverted pendulum system is  $\dot{y}=f(y,u),$  with  $y=(y_1,y_2,y_3,y_4)^{\rm tr}$  and

$$f := \begin{pmatrix} y_3 \\ y_4 \\ \frac{mly_4^2 \sin y_2 - mg \sin y_2 \cos y_2}{M + m \sin^2 y_2} + \frac{u}{M + m \sin^2 y_2} \\ \frac{-mly_4^2 \sin y_2 \cos y_2 + (M + m)g \sin y_2}{(M + m \sin^2 y_2)l} - \frac{u \cos y_2}{(M + m \sin^2 y_2)l} \end{pmatrix}.$$

▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●

For the cart-inverted pendulum, the linearized control system around  $(0,0)\in\mathbb{R}^4\times\mathbb{R}$  is  $\dot{y}=Ay+Bu$  with

(1) 
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{mg}{M} & 0 & 0 \\ 0 & \frac{(M+m)g}{Ml} & 0 & 0 \end{pmatrix}, B = \frac{1}{Ml} \begin{pmatrix} 0 \\ 0 \\ l \\ -1 \end{pmatrix}$$

One easily checks that this linearized control system satisfies the Kalman rank condition and therefore is controllable. Hence the cart-inverted pendulum is small-time locally controllable at  $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$  and is rapidly exponentially stabilizable (at the equilibrium (0,0)).

◆□ → ◆□ → ◆三 → ◆三 → ○ ● ● ● ●

Let us consider the control system

(1) 
$$\dot{y}_1 = y_1 - y_2^3, \, \dot{y}_2 = u,$$

where the state is  $(y_1, y_2)^{tr} \in \mathbb{R}^2$  and the control is  $u \in \mathbb{R}$ . The linearized control system of (1) at  $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}$  is

(2) 
$$\dot{y}_1 = y_1, \, \dot{y}_2 = u,$$

which is not controllable. However the nonlinear control system (1) is small-time locally controllable around  $(0,0) \in \mathbb{R}^2 \times \mathbb{R}$ . This can been seen by the return method, i.e. constructs small (but not 0) trajectories going from 0 to 0 and having a linearized control system which is controllable. This can been also checked by using criteria relying on iterated Lie brackets.

But there is no  $u \in C^1(\mathbb{R}^2, \mathbb{R})$  such that  $0 \in \mathbb{R}^2$  is asymptotically stable for the closed loop system  $\dot{y} = X(y)$ 

(1) 
$$X_1(y) = y_1 - y_2^3, X_2(y) = u(y).$$

Indeed, one has

(2) 
$$X'(0) = \begin{pmatrix} 1 & 0 \\ k_1 & k_2 \end{pmatrix}$$

and we cannot have both trace  $X'(0) \leq 0$  and det  $X'(0) \geq 0$ . However, as proved by Dayawansa and Martin (1989), there are continuous feedback laws u such that  $0 \in \mathbb{R}^2$  is asymptotically stable for  $\dot{y}_1 = y_1 - y_2^3$ ,  $\dot{y}_2 = u(y)$ . See also below.

Since we have to relax the regularity on the feedback laws, one can be more ambitious and look for finite-time stability instead of asymptotic stability as shows the following simple example. One considers the simplest control system

where the state is  $y \in \mathbb{R}$  and the control is  $u \in \mathbb{R}$ . We consider the feedback law  $u(y) := -(3/2)|y|^{1/3}\mathrm{sign}(y)$ . The solution to the Cauchy problem  $\dot{y} = -(3/2)|y|^{1/3}\mathrm{sign}(y)$ ,  $y(0) = y^0$  is

(2) 
$$y(t) = ||y^0|^{2/3} - t|^{3/2} \operatorname{sign}(|y^0|^{2/3} - t) \text{ if } t \in [0, |y^0|^{2/3}],$$
  
(3)  $y(t) = 0 \text{ if } t \ge |y^0|^{2/3}.$ 

This feedback law leads to finite-time stability.

Let  $X \in C^0(\mathbb{R}^n; \mathbb{R}^n)$  and  $y_e \in \mathbb{R}^n$  be such that  $X(y_e) = 0$ . One adopts the following definition

Definition (Finite-time stable)

One says that  $y_e$  is finite-time stable for  $\dot{y} = X(y)$  if it is stable and there exists  $\eta > 0$  and  $\tau > 0$  such that

(1) 
$$(\dot{y} = X(y) \text{ and } |y(0) - y_e| < \eta) \Rightarrow (y(t) = 0, \forall t \ge \tau)$$
Let us now consider the control system  $\dot{y} = f(y, u)$  with  $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$  of class  $C^1$  and let  $(y_e, u_e) \in \mathbb{R}^n \times \mathbb{R}^m$  be an equilibrium of  $\dot{y} = f(y, u)$ , i.e.  $f(y_e, u_e) = 0$ .

#### Definition (Finite-time stabilizable)

One says that  $(y_e, u_e)$  is finite-time stabilizable for  $\dot{y} = f(y, u)$  if there exists  $u \in C^0(\mathbb{R}^n, \mathbb{R}^m)$  such that  $u(y_e) = u_e$  and  $y_e$  is finite-time stable for  $\dot{y} = f(y, u(y))$ .

#### Let us consider finite dimensional control systems of the following form

$$(1) \qquad \qquad \dot{y} = Ay + Bu$$

where  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ , the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}^m$ .

#### Theorem (JMC-L. Praly (1991))

The control system (1) is finite-time stabilizable if and only if it is controllable.

(1) 
$$\dot{y}_1 = y_2, \, \dot{y}_2 = u,$$

where the state is  $(y_1, y_2)^{tr} \in \mathbb{R}^2$  and the control is  $u \in \mathbb{R}$ . We use another very important Nirenberg's advice

・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・
・

(1) 
$$\dot{y}_1 = y_2, \, \dot{y}_2 = u,$$

where the state is  $(y_1, y_2)^{tr} \in \mathbb{R}^2$  and the control is  $u \in \mathbb{R}$ . We use another very important Nirenberg's advice

Have you tried the dimension 2?

◆□▶ ◆□▶ ◆□▶ ◆□▶ □ ● ● ●

(1) 
$$\dot{y}_1 = y_2, \, \dot{y}_2 = u,$$

where the state is  $(y_1, y_2)^{tr} \in \mathbb{R}^2$  and the control is  $u \in \mathbb{R}$ . We use another very important Nirenberg's advice

#### Have you tried the dimension 2?

We go one step further and try the dimension 1. Then the 1-dimensional linear control system is

(2) 
$$\dot{y}_1 = y_2,$$

where the state is  $y_1 \in \mathbb{R}$  and the control is  $y_2 \in \mathbb{R}$ . One then notes that, for  $\alpha \in (0, 1)$ , the feedback law

(3) 
$$\bar{y}_2(y_1) := -\operatorname{sign}(y_1)|y_1|^{\alpha} := -\{y_1\}^{\alpha}$$

finite-time stabilizes the control system (2). See above for  $\alpha = 1/3$ .

The standard backstepping approach is a method to stabilize the control system

(1) 
$$\dot{y}_1 = f(y_1, y_2), \, \dot{y}_2 = u_2,$$

where the state is  $(y_1^{\text{tr}}, y_2^{\text{tr}})^{\text{tr}} \in \mathbb{R}^n = \mathbb{R}^{n_1+n_2}$ ,  $y_1 \in \mathbb{R}^{n_1}$ ,  $y_2 \in \mathbb{R}^{n_2}$ , and the control is  $u \in \mathbb{R}^{n_2}$  if one knows how to stabilize the control system

(2) 
$$\dot{y}_1 = f(y_1, y_2),$$

where the state is  $y_1 \in \mathbb{R}^{n_1}$  and the control is  $y_2 \in \mathbb{R}^{n_2}$  by means of a feedback law of class  $C^1$ . Let us recall the method. Just to simplify the notation we assume that  $n_2 = 1$  and that  $(y_e, u_e) = (0, 0)$ . Let  $\bar{y}_2 : \mathbb{R}^{n_2} \to \mathbb{R}, y_1 \mapsto \bar{y}_2$  be of class  $C^1$ , vanishing at  $0 \in \mathbb{R}^{n_1}$  and such that 0 is asymptotically for the closed-loop system  $\dot{y} = f(y_1, \bar{y}_2(y_1))$ . Let  $V \in C^{\infty}(\mathbb{R}^{n_1}), y_1 \mapsto L(y_1)$ , be a Lyapunov function of  $\dot{y}_1 = f(y_1, \bar{y}_2(y_1))$ . We consider the control Lyapunov function for the control system (1)

(3) 
$$V(y_1, y_2) := L(y_1) + \frac{1}{2}(y_2 - \bar{y}_2(y_1))^2$$

The natural idea behind this definition is to penalize the fact that  $y_2 \neq \bar{y}_2(y_1)$ .

Along the trajectories of  $\dot{y}_1 = f(y_1, y_2), \ \dot{y}_2 = u_2$ , one has

$$\dot{V} = (L'(y_1) - (y_2 - \bar{y}_2(y_1))\bar{y}_2'(y_1)) f(y_1, y_2) + (y_2 - \bar{y}_2(y_1))u = L'(y_1)f(y_1, \bar{y}_2(y_1)) + (y_2 - \bar{y}_2(y_1)) \left(L'(y_1)\frac{f(y_1, y_2) - f(y_1, \bar{y}_2(y_1))}{y_2 - \bar{y}_2(y_1)} - \bar{y}_2'(y_1)f(y_1, y_2) + u\right).$$

Hence the feedback law

$$u(y_1, y_2) := \bar{y}_2'(y_1)f(y_1, y_2) - L'(y_1)\frac{f(y_1, y_2) - f(y_1, \bar{y}_2(y_1))}{y_2 - \bar{y}_2(y_1)} - (y_2 - \bar{y}_2(y_1))$$

leads to

 $\dot{V} = L'(y_1)f(y_1, \bar{y}_2(y_1)) - (y_2 - \bar{y}_2(y_1))^2 < 0$  for  $|y_1| + |y_2|$  small but not 0.

Hence this feedback law asymptotically stabilizes  $\dot{y}_1 = f(y_1, y_2), \, \dot{y}_2 = u_2$ ,

Let us follow this method for  $\dot{y}_1=y_2$ ,  $\dot{y}_2=u_2$ ,  $\bar{y}_2(y_1)=-\{y_1\}^\alpha.$  One takes  $L(y_1)=y_1^2/2$  and

(1) 
$$V(y_1, y_2) = \frac{1}{2}y_1^2 + \varphi(y_1, y_2) := \frac{1}{2}y_1^2 + \frac{1}{2}(y_2 + \{y_1\}^{\alpha})^2,$$

the idea of  $\varphi$  being, again, to penalize the fact that  $y_2 \neq -\{y_1\}^{\alpha}$ . Unfortunately this V is not of class  $C^1$  on the full line  $y_1 = 0$ . There are other  $\varphi$  which are more regular and which also penalizes the fact that  $y_2 \neq -\{y_1\}^{\alpha}$ . For example,

(2) 
$$\varphi(y_1, y_2) = \int_{-\{y_1\}^{\alpha}}^{y_2} \left(\{s\}^{1/\alpha} - y_1\right) ds \\ = \frac{\alpha}{1+\alpha} |y_2|^{(1+\alpha)/\alpha} + y_1 y_2 + \frac{1}{1+\alpha} |y_1|^{(1+\alpha)},$$

which is of class  $C^1$  and satisfies  $\varphi(y_1, y_2) \ge 0$  with equality if and only if  $y_2 = -\{y_1\}^{\alpha}$ . For homogeneity issues, one then replaces (1) by

(3) 
$$V(y_1, y_2) = \frac{\alpha}{1+\alpha} |y_1|^{1+\alpha} + \varphi(y_1, y_2) \\ = \frac{\alpha}{1+\alpha} |y_2|^{(1+\alpha)/\alpha} + y_1 y_2 + |y_1|^{1+\alpha}.$$

With this new V one has, along the trajectories of  $\dot{y}_1 = y_2$ ,  $\dot{y}_2 = u$ ,

(1) 
$$\dot{V} = \left(\{y_2\}^{(1/\alpha)} + y_1\right)u + y_2^2 + (1+\alpha)\{y_1\}^{\alpha}y_2.$$

Note that, if  $y_2 + \{y_1\}^{\alpha} = 0$ , then

(2) 
$$\dot{V} = -\alpha |y_1|^{2\alpha} \leqslant 0.$$

Hence, by homogeneity argument, one sees that, if

(3) 
$$u := -k\{y_2 + \{y_1\}^{\alpha}\}^{2\alpha - 1},$$

then, if k > 0 is large enough, there exists  $\delta > 0$ ,

(4) 
$$\dot{V} \leqslant -\delta V^{2\alpha/(1+\alpha)}$$

Note that u defined by (3) is continuous and vanishes at 0 if  $2\alpha > 1$ . Hence, taking  $\alpha \in (1/2, 1)$ , the u defined by (3) leads to stabilization in finite time for  $\dot{y}_1 = y_2$ ,  $\dot{y}_2 = u$  provided that k > 0 is large enough.

## The general case $\dot{y}_1 = y_2$ , $\dot{y}_2 = y_3$ ... $\dot{y}_{n-1} = y_n$ , $\dot{y}_n = u$

We consider the control linear control system

(1) 
$$\dot{y}_1 = y_2, \, \dot{y}_2 = y_3, \, \dots, \, \dot{y}_{n-1} = y_n, \, \dot{y}_n = u,$$

where the state is  $(y_1, y_2, \ldots, y_{n-1}, y_n)^{tr} \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}$ . Adapting the above construction and taking  $\alpha \in ((n-1)/n, 1)$  one can get feedback laws leading to finite-time stabilization. See JMC and L. Praly (1992), P. Bhat and D. Bernstein (1998, 2002), Y. Hong (2002), Y. Hong, Y. Xu, and J. Huang (2002), Y. Hong and Z.-P. Jiang (2006), E. Moulay and W. Perruguetti (2006), E. Bernuau, W. Perruguetti, D. Efimov, and E. Moulay (2015), B. d'Andréa-Novel, JMC, and W. Perruguetti (2020). Note that it follows from this result that any linear controllable system in finite dimension are finite-time stabilizable by means of stationary feedback laws. The finite-time stabilizability of nonlinear systems having having a controllable linearized control system at  $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$  follows from homogeneity arguments (one uses here L. Rosier's result on the existence of homogeneous Lyapunov for homogeneous vector fields). 

#### Theorem (R. Brockett (1983))

If the control system  $\dot{y} = f(y, u)$  is locally asymptotically stabilizable then

▲□▶ ▲□▶ ▲三▶ ▲三▶ - 三 - のへで

(B) the image by f of every neighborhood of  $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$  is a neighborhood of  $0 \in \mathbb{R}^n$ .



# The slider



The slider is actuated by two propellers producing forces  $F_L$  and  $F_R$ . The sum of these two forces is directly linked to the acceleration of the vehicle, whereas the difference acts on the angular dynamics. Let us denote  $\tau_1 = F_L + F_R$  and  $\tau_2 = F_R - F_L$ , the dynamics can be written:

(1) 
$$\begin{cases} m\ddot{\xi}_1 = \cos(\psi)\tau_1, \\ m\ddot{\xi}_2 = \sin(\psi)\tau_1, \\ I\ddot{\psi} = \tau_2, \end{cases}$$

where m is the slider mass and I is the moment of inertia of the slider about its center of mass.

Let

(1) 
$$\begin{cases} y_1 = \xi_1, y_2 = \dot{\xi}_1, y_3 = \xi_2, y_4 = \dot{\xi}_2, \\ y_5 = \psi, y_6 = \dot{\psi}, u_1 = \frac{\tau_1}{m}, u_2 = \frac{\tau_2}{I}. \end{cases}$$

Then the dynamics of the slider can be written in the form  $\dot{y} = f(y, u)$  with

(2) 
$$f(y,u) := (y_2, u_1 \cos(y_5), y_4, u_1 \sin(y_5), y_6, u_2)^{\text{tr}}.$$

One has the following theorem.

#### Theorem

The slider control system is small-time locally controllable at the equilibrium  $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$  but does not satisfy the Brockett condition.

For the Brockett condition, consider the equation

(1) 
$$(y_2, u_1 \cos(y_5), y_4, u_1 \sin(y_5), y_6, u_2)^{\mathsf{tr}} = (0, 0, 0, \delta, 0, 0)^{\mathsf{tr}}.$$

Instead of u(y), use u(t, y). Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function). First use of time-varying feedback laws:

- n = 1: E. Sontag and H. Sussmann (1980).
- For a driftless control system with n = 3 and m = 2: C. Samson (1992).

In order to deal with systems for which the linearized system is not controllable, we use the following definition.

#### Definition

The origin (of  $\mathbb{R}^n$ ) is *locally continuously reachable in small time* for the control system  $\dot{y} = f(y, u)$  if, for every positive real number T, there exist a positive real number  $\varepsilon$  and  $u : \bar{B}_{\varepsilon} \to L^1((0, T); \mathbb{R}^m)$  such that

(1) 
$$u \in C^0\left(\bar{B}_{\varepsilon}; L^1\left((0,T); \mathbb{R}^m\right)\right)$$

(2) 
$$Sup\{|u(a)(t)|; t \in (0,T)\} \to 0 \text{ as } a \to 0,$$

(3) 
$$((\dot{y} = f(y, u(a)(t)), y(0) = a) \Rightarrow (y(T) = 0)), \forall a \in \bar{B}_{\varepsilon}.$$

In order to deal with systems for which the linearized system is not controllable, we use the following definition.

#### Definition

The origin (of  $\mathbb{R}^n$ ) is *locally continuously reachable in small time* for the control system  $\dot{y} = f(y, u)$  if, for every positive real number T, there exist a positive real number  $\varepsilon$  and  $u : \bar{B}_{\varepsilon} \to L^1((0, T); \mathbb{R}^m)$  such that

(1) 
$$u \in C^0\left(\bar{B}_{\varepsilon}; L^1\left((0,T); \mathbb{R}^m\right)\right)$$

(2) 
$$Sup\{|u(a)(t)|; t \in (0,T)\} \to 0 \text{ as } a \to 0,$$

(3) 
$$((\dot{y} = f(y, u(a)(t)), y(0) = a) \Rightarrow (y(T) = 0)), \forall a \in \bar{B}_{\varepsilon}.$$

#### Open problem: Small-time local controllability and continuous reachability

Assume that f is analytic and that  $\dot{y} = f(y, u)$  is small-time locally controllable at  $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$ . Is the origin (of  $\mathbb{R}^n$ ) locally continuously reachable in small time for the control system  $\dot{y} = f(y, u)$ ?

#### Theorem (JMC (1995))

Assume f is analytic, that  $0 \in \mathbb{R}^n$  is locally continuously reachable in small time for the control system  $\dot{y} = f(y, u)$ , and that  $n \notin \{2, 3\}$ . Then, for every positive real number T, there exist  $\varepsilon$  in  $(0, +\infty)$  and u in  $C^0(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ , of class  $C^\infty$  on  $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ , T-periodic with respect to time, vanishing on  $\mathbb{R} \times \{0\}$  and such that, for every  $s \in \mathbb{R}$ ,

(1) 
$$((\dot{y} = f(y, u(t, y)) \text{ and } y(s) = 0) \Rightarrow (y(\tau) = 0, \forall \tau \ge s)),$$

(2) 
$$(\dot{y} = f(y, u(t, y)) \text{ and } |y(s)| \leq \varepsilon) \Rightarrow (y(\tau) = 0, \forall \tau \geq s + T)).$$

In particular 0 is finite-time stable for the closed-loop system  $\dot{y} = f(y, u(t, y)).$ 

# An example: The slider



Let us recall that the dynamics of the slider can be written in the form  $\dot{y}=f(y,u)$  with

(1) 
$$f(y,u) := (y_2, u_1 \cos(y_5), y_4, u_1 \sin(y_5), y_6, u_2)^{\text{tr}}.$$

As mentioned above, the slider does not satisfies the Brockett condition at the equilibrium  $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$  and therefore is not locally asymptotically stabilizable by means of stationary feedback laws (i.e. feedback laws u(y)). However it follows from our result on the stabilizability in small time that the following proposition holds.

#### Proposition

The slider is asymptotically and even in small time stabilizable by means of periodic time-varying feedback laws (i.e. feedback laws u(t, y) which are periodic with respect to time).

Construction of a time varying feedback stabilizing the slider in small time (B. d'Andréa-Novel, JMC, and W. Perruquetti (2019)).

#### Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws

We consider the heat control system

(1) 
$$y_t - y_{xx} = 0, \ y(t,0) = 0, \ y(t,1) = u(t), \ t \in [0,+\infty), \ x \in [0,1],$$

where, at time  $t \in [0, +\infty)$ , the state is  $y(t) \in L^2(0, 1)$ ,  $x \in (0, 1) \mapsto y(t)(x) := y(t, x)$  and the control is  $u(t) \in \mathbb{R}$ . We are interested in

- The rapid exponential stabilization of (1),
- 2 The finite time stabilization of (1).

Note that the linear control system (1) is known to be null controllable.

Concerning the rapid stabilization we want to prove the following theorem

#### Theorem

Let  $\lambda > 0$ . There a feedback law  $u_{\lambda} : L^2(0,1) \to \mathbb{R}$  such that there exists  $C(\lambda) > 0$  such that, for every solution of

(2) 
$$y_t - y_{xx} = 0, \ y(t,0) = 0, \ y(t,1) = u_\lambda(y(t,\cdot)), \ t \in [0,+\infty), \ x \in [0,1],$$

#### one has

(3) 
$$|y(t,\cdot)|_{L^2} \leq C(\lambda)e^{-\lambda t}|y(0,\cdot)|_{L^2}, \, \forall t \ge 0.$$

This theorem is a simple corollary of a pole shifting theorem due to D. Russell (1976). Let us give a proof due to D. Bošković, M. Krstic and W. Liu (2001). It relies on backstepping.

1. Backstepping was initially a recursive method to stabilize finite dimensional control system of the form  $\dot{x} = f(x, y)$ ,  $\dot{y} = u$ . See above. 2. First application to PDE: JMC and B. d'Andréa-Novel (1998).



American Mathematical Society

JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.

▲ロ▶ ▲周▶ ▲ヨ▶ ▲ヨ▶ ヨー のへ⊙

1. Backstepping was initially a recursive method to stabilize finite dimensional control system of the form  $\dot{x} = f(x, y)$ ,  $\dot{y} = u$ . See above. 2. First application to PDE: JMC and B. d'Andréa-Novel (1998). 3. This method has been used on the discretization of partial differential equations by D. Bošković, A. Balogh and M. Krstic in 2003. 4. A key modification of the method is introduced by D. Bošković, M. Krstic and W. Liu in 2001: They saw that at the continuous level, the backstepping method corresponds to a Volterra transformation of the second kind for the transformation T and well chosen target systems. 5. For a survey on this method with Volterra transformations of the second kind, see the book by M. Krstic and A. Smyshlyaev in 2008.

## Backstepping and the 1D heat equation

We consider the heat control system

(1) 
$$y_t - y_{xx} = 0, \ y(t,0) = 0, \ y(t,1) = u(t), \ t \in [0,+\infty), \ x \in [0,1],$$

where, at time  $t \in [0, +\infty)$ , the state is  $y(t) \in L^2(0, 1)$ ,  $x \in (0, 1) \mapsto y(t)(x) := y(t, x)$  and the control is  $u(t) \in \mathbb{R}$ . We are interested in the rapid exponential stabilization of this linear (controllable) control system. Let  $\lambda \in \mathbb{R}$ . Consider the following controlled system (called the target system)

(2) 
$$z_t - z_{xx} = -\lambda z, \ z(t,0) = 0, \ z(t,1) = v(t), \ t \in [0,+\infty), \ x \in [0,1],$$

where, at time  $t \in [0, +\infty)$ , the state is  $z(t) \in L^2(0, 1)$ ,  $x \in (0, 1) \mapsto z(t)(x) := z(t, x)$  and the control is  $v(t) \in \mathbb{R}$ . Note that for (2) with z = 0, one has

(3) 
$$|z(t)|_L^2 \leqslant e^{-\lambda t} |z(0)|_L^2, \forall t \ge 0.$$

D. Bošković, M. Krstic and W. Liu in 2001 looks for maps  $T^{-1}: L^2(0,1) \to L^2(0,1) \ y \mapsto z$  and  $K: L^2(0,1) \to \mathbb{R}, z \mapsto Kz$  such that the target system (2) is transformed into the initial system (1) if u = Kz + v. They choose to look for  $T^{-1}$  in the class of Volterra transform of the second kind:

(1) 
$$z(x_1) := y(x_1) - \int_0^{x_1} k(x_1, x_2) y(x_2) dx_2.$$

One of the advantages of the Volterra transforms of the second kind is that there are invertible (if k is smooth enough, for example in  $L^2((0,1) \times (0,1))$ ). Note that, once T is defined, we must take

(2) 
$$Kz = \int_0^1 k(1, x_2) y(x_2) dx_2.$$

Moreover, the feedback law  $u(y):=\int_0^1 k(1,s)y(s)ds$  leads for z to the closed loop system

(3) 
$$z_t - z_{xx} = -\lambda z, \ z(t,0) = z(t,1) = 0,$$

which insures exponential stability for z with an exponential decay rate (in  $L^2(0,1)$ ) at least equal to  $\lambda$ .

Since  $y\in L^2(0,1)\to z\in L^2(0,1)$  is an isomorphism the same holds for the closed loop system

(1) 
$$y_t - y_{xx} = 0, \ y(t,0) = 0, \ y(t,1) = \int_0^1 k(1,s)y(s)ds,$$

which shows the rapid exponential stabilizability of the initial heat control system (with a method to compute a feedback law leading to an exponential stability with an exponential decay rate as large as we want).

Straightforward computations show that the y system is equivalent to the z system if and only if k satisfies the following equation, called the kernel equation,

(1) 
$$\begin{cases} k_{11} - k_{22} = \lambda k, & 0 < x_2 < x_1 < 1, \\ k(x_1, 0) = 0, & 0 < x_1 < 1, \\ k(x, x) = -\frac{\lambda}{2}x, & 0 < x < 1, \end{cases}$$

 $k_{ii} := \partial_{x_i x_i}^2 k, \ i \in \{1, 2\}.$ 

### A method to prove the existence of k

D. Bošković, M. Krstic and W. Liu in 2001 proposed the following iterative scheme. Let us make the following change of variables

 $t=x_1-x_2,\,s=x_1+x_2$  and define  $G(s,t):=k(x_1,x_2)$  on  $\mathcal{T}_0:=\{(s,t);\,t\in[0,1],s\in[t,2-t]\}.$  Then k satisfies the kernel equation if and only if

(1) 
$$\begin{cases} G_{st} = -\frac{\lambda}{4}G, & \text{in } \mathcal{T}_0, \\ G(s,s) = 0, & \text{in } [1,2], \\ G(s,0) = \frac{\lambda}{4}s, & \text{in } [0,2]. \end{cases}$$

One integrates the first equation of (1) with respect to t from 0 to t. One gets, using also the third equality of (1),

(2) 
$$G_s(s,t) = G_s(s,0) - \frac{\lambda}{4} \int_0^t G(s,t_1) dt_1 = \frac{\lambda}{4} - \frac{\lambda}{4} \int_0^t G(s,t_1) dt_1.$$

We integrate this equation with respect to s from t to s. Using also the second equation of (1), we get

(1) 
$$G(s,t) = \frac{\lambda}{4}(s-t) - \frac{\lambda}{4}\int_{t}^{s}\int_{0}^{t}G(s_{1},t_{1})dt_{1}ds_{1}$$

One defines inductively  $G^n: \mathcal{T}_0 \to \mathbb{R}, n \in \mathbb{N} \setminus \{0\}$ , by requiring

(2) 
$$G^1(s,t) = 0,$$

(3) 
$$G^{n+1}(s,t) = \frac{\lambda}{4}(s-t) - \frac{\lambda}{4}\int_t^s \int_0^t G^n(s_1,t_1)dt_1ds_1$$

One gets, by induction on n, that

(4) 
$$G^{n}(s,t) = -\sum_{k=1}^{n} \frac{(s-t)s^{k-1}t^{k-1}(-\lambda)^{k}}{(k-1)!k!4^{k}},$$

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 ● のへで

a sum which converges as  $n \to +\infty$ .

Let

(1) 
$$I(x) := \sum_{k=1}^{+\infty} \frac{(-x)^{2k-1}}{(k-1)!k!2^{2k-1}}.$$

#### Then

(2) 
$$G(s,t) = \frac{\lambda}{2}(s-t)\frac{I(\sqrt{\lambda st})}{\sqrt{\lambda st}},$$
  
(3) 
$$k(x_1,x_2) = \frac{\lambda}{2}x_2\frac{I(\sqrt{\lambda(x_1^2-x_2^2)})}{\sqrt{\lambda(x_1^2-x_2^2)}}.$$

◆□ → < 団 → < 三 → < 三 → 三 </p>

# How to recover the null controllability with the backstepping method (JMC and H.-M. Nguyen (2015))

From now on we assume that  $\lambda > 1$ . Looking at the explicit expression of the kernel k, one sees that

(1) 
$$|k|_{H^1(\Delta)} \leqslant C e^{C\sqrt{\lambda}},$$

where

(2) 
$$\Delta := \{ (x_1, x_2); \ 0 < x_2 < x_1 < 1 \}.$$

The inverse transform of

(3) 
$$z(x_1) := y(x_1) - \int_0^{x_1} k(x_1, x_2) y(x_2) dx_2,$$

has the form

(4) 
$$y(x_1) := z(x_1) - \int_0^{x_1} l(x_1, x_2) z(x_2) dx_2.$$

The exact expression of l shows that

$$(5) |l|_{H^1(\Delta)} \leqslant C\lambda.$$
So if we apply the backstepping for  $\lambda$  and during the interval of time  $[0,\tau],$  we have

(1) 
$$|y(\tau)|_{L^2} \leq C\lambda |z(\tau)|_{L^2} \leq C\lambda e^{-\lambda\tau} |z(0)|_{L^2} \leq C\lambda e^{-\lambda\tau} e^{C\sqrt{\lambda}} |y(0)|_{L^2}.$$

Similar estimates holds for the control y(t, 1). Let T > 0, and for  $n \in \mathbb{N} \setminus \{0, 1\}$ , let  $t_n = T(1 - 1/n^2)$  and  $\lambda_n = n^8$ . Let  $t_1 := 0$  and  $\lambda_1 := 1$ . During the interval  $[t_n, t_{n+1})$  we apply the feedback law coming from the backstepping with  $\lambda := \lambda_n$ 

Proposition (H.-M. Nguyen and JMC (2015))

(2) 
$$\lim_{t \to T^{-}} |y(t, \cdot)|_{L^{2}} = 0,$$

$$\lim_{t \to T^-} u(t) = 0.$$

Hence this is a new method to prove the null controllability of the heat equation in small time.

A D > 4 回 > 4 回 > 4 回 > 1 回

#### The estimates

(1) 
$$|k|_{H^{1}(\Delta)} \leq Ce^{C\sqrt{\lambda}},$$
  
(2)  $|l|_{H^{1}(\Delta)} \leq C\lambda.$ 

are crucial for this method. Note that one can find related estimates in G. Lebeau and L. Robbiano (1995) (in every space dimension). Let us now turn to the case of the following more general 1-D parabolic equations

(3) 
$$\begin{cases} y_t(t,x) = (a(x)y_x(t,x))_x + c(x)y(t,x) & \text{in } (\tau_1,\tau_2) \times [0,1], \\ y(t,0) = 0, \ y(t,1) = u(t) & \text{for } t \in (\tau_1,\tau_2). \end{cases}$$

The target system is then

(4) 
$$\begin{cases} z_t(t,x) = (a(x)z_x(t,x))_x + c(x)z(t,x) - \lambda z & \text{in } (\tau_1,\tau_2) \times [0,1], \\ z(t,0) = 0, \ z(t,1) = v(t) & \text{for } t \in (\tau_1,\tau_2). \end{cases}$$

We assume that  $a \in H^2(0,1)$ , a > 0 in [0,1], and that  $c \in H^1(0,1)$ .

## Proposition (H.-M. Nguyen and JMC (2015))

There exists a kernel k which allows to transform the initial y system into the z system and one has, for  $\lambda \in [1, +\infty)$ ,

(1) 
$$|k|_{H^1(\Delta)} \leq C e^{C\sqrt{\lambda}}$$
  
(2)  $|l|_{H^1(\Delta)} \leq C\lambda.$ 

(2)

#### Remark

Our proof is different from the iterative scheme mentioned above. We interpret the kernel equation on k (and l) as a wave equation defined in  $[0,1]^2$ . Estimates (1) and (2) follow from an energy type estimate for the wave equation which is somehow nonstandard in the sense that the energy not only contains the gradient of the solutions but also the solutions: the standard energy estimate only gives the exponent  $\lambda$  in (1).

However the above strategy does not seem lead to stabilization in finite time. This is due to the fact that u(t, y) is small along the trajectories starting from the time 0 but might be quite large for a given y and  $t \to T_-$ . In fact  $0 \in L^2(0, 1)$  is (probably) not stable with this feedback law.

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

We look for time-varying feedback laws  $(t, y) \in \mathbb{R} \times L^2(0, 1) \mapsto u(t, y) \in \mathbb{R}$  satisfying the following three properties.

 $(\mathcal{P}_1)$ . The feedback law u is T-periodic with respect to time:

1) 
$$u(t,y) = u(t+T,y)$$
 for every  $(t,y) \in \mathbb{R} \times L^2(0,1)$ .

 $(\mathcal{P}_2).$  There exists a strictly increasing sequence  $(t_n)_{n\in\mathbb{N}}$  of real numbers such that

$$(2) t_0 = 0,$$

(3) 
$$\lim_{n \to +\infty} t_n = T,$$

(4) 
$$u$$
 is of class  $C^1$  in  $[t_n, t_{n+1}) \times L^2(0, 1)$  for every  $n \in \mathbb{N}$ .

 $(\mathcal{P}_3).$  The map u vanishes on  $\mathbb{R}\times\{0\}$  and there exists a continuous function  $M:[0,T)\to[0,+\infty)$  such that

(5) 
$$|u(t, y_2) - u(t, y_1)| \leq M(t)|y_2 - y_1|_{L^2}$$
  
 $\forall (t, y_1, y_2) \in [0, T) \times L^2(0, 1) \times L^2(0, 1).$ 

## Proposition

Assume that F satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$ . Let  $0 \leq s < T$ . Then, for every  $y^0 \in L^2(0,1)$ , there exists a unique solution  $y \in C^0([s,T); L^2(0,1))$  of

(1) 
$$\begin{cases} y_t(t,x) = y_{xx}(t,x) & \text{for } (t,x) \in (s,\tau) \times [0,1], \\ y(t,0) = 0, \ y(t,1) = u(t,y(t,\cdot)) & \text{for } t \in (s,\tau), \\ y(s,\cdot) = y^0 & \text{for } x \in [0,1]. \end{cases}$$

▲□▶ ▲□▶ ▲三▶ ▲三▶ - 三 - のへで

#### Proposition

Assume that F satisfies Properties  $(\mathcal{P}_1)$ ,  $(\mathcal{P}_2)$ , and  $(\mathcal{P}_3)$  and that there exist C > 0 and  $\overline{T} \in (0,T)$  such that

$$(\mathcal{P}_4) |u(t,y)| \leq C|y|_{L^2}^{1/2}, \, \forall \, (t,y) \in [\bar{T},T) \times L^2(0,1).$$

Then, for every  $s \in \mathbb{R}$  and for every  $y^0 \in L^2(0,1)$ , there exists a unique solution  $y \in C^0([s,+\infty);L^2(0,1))$  of

(1) 
$$\begin{cases} y_t(t,x) = y_{xx}(t,x) & \text{for}(t,x) \in (s,+\infty) \times [0,1], \\ y(t,0) = 0, \ y(t,1) = u(t,y(t,\cdot)) & \text{for} \ t \in (s,+\infty), \\ y(s,\cdot) = y^0 & \text{for} \ x \in [0,1]. \end{cases}$$

▲日▶ ▲□▶ ▲ヨ▶ ▲ヨ▶ ヨー シタの

Notation  $\phi(t, s, y^0) := y(t, \cdot).$ 

#### Theorem (JMC and H.-M. Nguyen (2015))

Let T > 0 and  $\Gamma > 0$ . There exists a time-varying feedback laws  $(t, y) \in \mathbb{R} \times L^2(0, 1) \mapsto u(t, y) \in \mathbb{R}$  satisfying Properties ( $\mathcal{P}_1$ ), ( $\mathcal{P}_2$ ), ( $\mathcal{P}_3$ ) and ( $\mathcal{P}_4$ ) such that

(1) 
$$\Phi(t+T,t,y^0) = 0$$
 for every  $(t,y^0) \in \mathbb{R} \times L^2(0,1)$   
such that  $|y^0|_{L^2} \leqslant \Gamma$ 

and such that the following uniform stability condition

(2) 
$$\begin{cases} \forall \varepsilon > 0, \exists \eta > 0 \text{ such that, } \forall t' \in \mathbb{R}, \forall t \in [t', +\infty), \\ \text{and } \forall y^0 \in L^2(0, 1), (|y^0|_{L^2} \leqslant \eta) \Rightarrow (|\Phi(t, t', y^0)| \leqslant \varepsilon) \end{cases}$$

holds. In particular our heat equation is small-time stabilizable by means of time-varying feedback laws.

#### Open problem

Is it possible to stabilize in finite time (or even in small time) the heat equation by means of stationary feedback laws?

May be one can try to use the kernel  $k_{\lambda(y)}$  instead of the kernel  $k_{\lambda(t)}$  with  $\lambda(y)$  converging to  $+\infty$  as  $y \to 0$ .

▲日▶ ▲□▶ ▲ヨ▶ ▲ヨ▶ ヨー シタの

## Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- 3 Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws



(1) 
$$y_t + y_x + y_{xxx} + yy_x = 0, t \in [0, T], x \in [0, L],$$
  
(2)  $y(t, 0) = y(t, L) = 0, y_x(t, L) = u(t), t \in [0, T].$ 

where, at time  $t\in[0,T],$  the control is  $u\in\mathbb{R}$  and the state is  $y(t,\cdot)\in L^2(0,L).$ 

# Theorem (L. Rosier (1997))

For every T>0, the linearized control system is controllable in time T (in  $L^2(0,L)$ ) if and only

$$L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, \, k \in \mathbb{N}^*, \, l \in \mathbb{N}^* \right\}.$$

▲□▶ ▲□▶ ▲三▶ ▲三▶ - 三 - のへで

# Theorem (L. Rosier (1997))

For every T > 0, the KdV control system is locally controllable in time T if  $L \notin \mathcal{N}$ .



(1) 
$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \ t \in (0,T), \ x \in (0,L), \\ y(t,0) = u(t), \ y(t,L) = 0, \ y_x(t,L) = 0, \ t \in (0,T). \end{cases}$$

For every L > 0, the control system (1) is locally null controllable in small time: L. Rosier (2004).

## Theorem (E. Cerpa and JMC (2013))

For every  $\lambda > 0$ , there exist C > 0, r > 0 and a feedback law  $y \mapsto u(y)$  such that, for this feedback law,

(2) 
$$(|y(0)|_{L^2(0,L)} \leq r) \Rightarrow (|y(t)|_{L^2(0,L)} \leq Ce^{-\lambda t} |y(0)|_{L^2(0,L)}, \forall t > 0.)$$

▲日▶ ▲□▶ ▲ヨ▶ ▲ヨ▶ ヨー シタの

# Proof: With M. Krstic's backstepping approach

We look for a transformation  $y\in L^2(0,L)\mapsto z\in L^2(0,L)$  defined by

(1) 
$$z(x_1) := y(x_1) - \int_{x_1}^L k(x_1, x_2) y(x_2) dx_2,$$

such that the trajectory y of

(2) 
$$y_t + y_x + y_{xxx} = 0, \ y(t,0) = u(t), \ y(t,L) = 0, \ y_x(t,L) = 0,$$

with the feedback law  $u(t) := \int_0^L k(0, x_2)y(t, x_2)dx_2$  is mapped into the trajectory z = z(t, x), solution of the linear system

(3) 
$$z_t + z_x + z_{xxx} + \lambda z = 0, \ z(t,0) = 0, \ z(t,L) = 0, \ z_x(t,L) = 0.$$

Note that, for (3), one has (just multiply (3) by z and do some integrations by parts):

(4) 
$$|z(t)|_{L^{2}(0,L)} \leq e^{-\lambda t} |z(0)|_{L^{2}(0,L)}, \forall t \ge 0.$$

This property for the transformation  $y \mapsto z$  holds if (and only if)

(1) 
$$\begin{cases} k_{111} + k_1 + k_{222} + k_2 &= -\lambda k, & \text{for } 0 < x_1 < x_2 < L, \\ k(x_1, L) &= 0, & \text{in } [0, L], \\ k(x_1, x_1) &= 0, & \text{in } [0, L], \\ k_1(x_1, x_1) &= \frac{\lambda}{3}(L - x_1), & \text{in } [0, L]. \end{cases}$$

with  $k_i := \partial_{x_i}k$ ,  $k_{iii} := \partial^3_{x_ix_ix_i}k$ . Moreover, if k is smooth enough (Lipschitz is sufficient), one can check that the same feedback law provides for the initial nonlinear KdV control system (local) asymptotic stability with an exponential decay rate at least equal to  $\lambda$ . The proof of the existence of k follows related arguments to the ones introduced by D. Bošković, M. Krstic and W. Liu in 2001 for the existence of k for the heat equation  $y_t - y_{xx} = 0$ .

Shengquan Xiang recently proved the following estimates for  $\lambda \geqslant 1$ 

(1) 
$$|k_{\lambda}|_{C^{0}(\mathcal{T})} \leq e^{(1+L)^{2}\sqrt{\lambda}} \text{ and } |l_{\lambda}|_{C^{0}(\mathcal{T})} \leq e^{(1+L)^{2}\sqrt{\lambda}}$$

where  $l_{\lambda}$  is the kernel of the inverse transform. It allowed him to give a new proof of the null controlloability in small-time and the small-time stabilization thanks to these estimates. One may wonder if one could replace  $\sqrt{\lambda}$  by  $\lambda^{1/3}$  in (1).

(1) 
$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \ t \in (0,T), \ x \in (0,L), \\ y(t,0) = 0, \ y(t,L) = 0, \ y_x(t,L) = u(t) \ t \in (0,T). \end{cases}$$

We assume that

(2) 
$$L \notin \mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}}, k \in \mathbb{N}^*, l \in \mathbb{N}^* \right\}.$$

Then the linearized control system around 0 is controllable and the nonlinear control system is locally controllable in small-time. We are interested in the rapid exponential stabilization of the nonlinear system.

# Rapid exponential stabilization of the initial KdV-control system

# Theorem (JMC and Q. Lü (2013))

Let us assume that  $L \notin \mathcal{N}$  For every  $\lambda > 0$ , there exist C > 0, r > 0 and a feedback law  $y \mapsto u(y)$  such that, for this feedback law,

(1) 
$$(|y(0)|_{L^2(0,L)} \leq r) \Rightarrow (|y(t)|_{L^2(0,L)} \leq Ce^{-\lambda t} |y(0)|_{L^2(0,L)}, \forall t > 0.)$$

▲□▶ ▲□▶ ▲三▶ ▲三▶ - 三 - のへで

# Proof of the rapid exponential stabilizability

Unfortunately the backstepping approach (i.e. Fredholm transformations of the second kind) is not working. We need to use more general transformations:  $y \in L^2(0,L) \mapsto z \in L^2(0,L)$  is now defined by

(1) 
$$z(x_1) := y(x_1) - \int_0^L k(x_1, x_2) y(x_2) dx_2.$$

(Every linear transformation  $y \in L^2(0,L) \mapsto z \in L^2(0,L)$  can been written in this form). Again, we want that the trajectory y of

(2) 
$$y_t + y_x + y_{xxx} = 0, \ y(t,0) = 0, \ y(t,L) = 0, \ y_x(t,L) = u(t),$$

with the feedback law  $u(t) := \int_0^L k_{x_1}(0, x_2)y(t, x_2)dx_2$  is mapped into the trajectory z = z(t, x), solution of the linear system

(3) 
$$z_t + z_x + z_{xxx} + \lambda z = 0, \ z(t,0) = 0, \ z(t,L) = 0, \ z_x(t,L) = 0.$$

This property for the transformation  $y \mapsto z$  holds if (and only if)

(1) 
$$\begin{cases} k_{111} + k_1 + k_{222} + k_2 + \lambda k = \lambda \delta(x_1 - x_2), & \text{on } (0, L)^2, \\ k(x_1, 0) = k(x_1, L) = k_2(x_1, 0) = k_2(x_1, L) = 0 & \text{on } (0, L), \\ k(0, x_2) = k(L, x_2) = 0 & \text{on } (0, L), \end{cases}$$

where  $\delta(x_1 - x_2)$  is the Dirac mass on the diagonal of the square  $[0, L] \times [0, L]$ .

Next step: Prove the existence of a solution to the kernel equation (1).

・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・
 ・

Let us define an unbounded linear operator  $A:D(A)\subset L^2(0,L)\to L^2(0,L)$  as follows.

(1) 
$$D(A) := \{\varphi; \varphi \in H^3(0, L), \varphi(0) = \varphi(L) = 0, \varphi_x(0) = \varphi_x(L)\},$$
  
(2)  $A\varphi := -\varphi_{xxx} - \varphi_x.$ 

The operator A is a skew-adjoint operator and has compact resolvent. Denote by  $\{i\mu_j\}_{j\in\mathbb{Z}}$ ,  $\mu_j\in\mathbb{R}$ , the eigenvalues of A, which are organized in the following way:

$$(3) \qquad \ldots \leqslant \mu_{-2} \leqslant \mu_{-1} < 0 \leqslant \mu_0 \leqslant \mu_1 \leqslant \mu_2 \leqslant \ldots$$

Since the control is of dimension 1 and the linearized control system is controllable, all these eigenvalues are simple. Let us write  $\{\varphi_j\}_{j\in\mathbb{Z}}$  for the corresponding eigenfunctions with  $|\varphi_j|_{L^2(0,L)} = 1$   $(j \in \mathbb{Z})$ . It is well known that  $\{\varphi_j\}_{j\in\mathbb{Z}}$  constitutes an orthonormal basis of  $L^2(0,L)$ .

The idea is to search  $\boldsymbol{k}$  in the following form

(1) 
$$k(x_1, x_2) = \sum_{j \in \mathbb{Z}} \psi_j(x_1) \varphi_j(x_2).$$

. . .

Then prove that  $y\in L^2(0,L)\mapsto z\in L^2(0,L)$  defined by

(2) 
$$z(x_1) := y(x_1) - \int_0^L k(x_1, x_2) y(x_2) dx_2$$

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 ● のへで

is invertible.

We consider the following linear control system in finite dimension

(1) 
$$\dot{y} = Ay + Bu,$$

where the state is  $y \in \mathbb{R}^n$  and the control is  $u \in \mathbb{R}$ . We assume that

(2) the control system (1) is controllable.

Let  $\lambda \in \mathbb{R}$ . Let  $GL(n, \mathbb{R})$  be the set of invertible elements of  $\mathbb{R}^{n \times n}$ . We are looking for  $T \in GL(n, \mathbb{R})$  and  $K \in \mathbb{R}^{1 \times n}$  such that, if y = Tz and u = Kz + v, then (1) is equivalent to

(3) 
$$\dot{z} = (A - \lambda \mathsf{Id})z + Bv,$$

where Id is the identity matrix in  $\mathbb{R}^{n \times n}$ . Clearly, if such T and K exists for every  $\lambda \in \mathbb{R}$  the control system  $\dot{y} = Ay + Bu$  satisfies the rapid exponential stabilization property.

The equivalence between  $\dot{y} = Ay + Bu$  and  $\dot{z} = (A - \lambda Id)z + Bv$  with y = Tz and u = Kz + v holds if and only if

(1) 
$$AT + BK = TA - \lambda T,$$
  
(2) 
$$TB = B.$$

One has the following theorem.

## Proposition (JMC (2015))

If  $\dot{y} = Ay + Bu$  is controllable, there exists one and only one  $(T, K) \in GL(n, \mathbb{R}) \times \mathbb{R}^{1 \times n}$  such that (1) and (2) hold.

## Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- 3 Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws







# 1-D hyperbolic systems

Our hyperbolic control system is

(1) 
$$y_t + A(y)y_x = S(y), \ (t,x) \in [0,T] \times [0,L],$$

where, at time  $t \in [0, T]$ , the state is  $x \in [0, L] \mapsto y(t, x) \in \mathbb{R}^n$ . Let  $y^* \in \mathbb{R}^n$  be fixed. Assume that  $\Lambda := A(y^*)$  has n distinct real non zero eigenvalues: after a suitable linear change of variables  $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, -\lambda_{k+1}, \ldots, -\lambda_n)$  with  $-\lambda_n < \ldots < -\lambda_{k+1} < 0 < \lambda_k < \ldots < \lambda_1$ . for some  $k \in \{0, \cdots, n\}$ . The term S(y) is the source term. We assume that  $S(y^*) = 0$ . In this section we even assume that S = 0. The case  $S \neq 0$  is considered in the next section (one speaks in this case of hyperbolic balance laws). Let m := n - k. For  $y \in \mathbb{R}^n$ , let  $y_- \in \mathbb{R}^m$  and  $y_+ \in \mathbb{R}^{n-m}$  be such that

$$y = \begin{pmatrix} y_-\\ y_+ \end{pmatrix}$$

The control is part of  $y_+(t,0)$  and part of  $y_-(t,L)$ 

(1) 
$$\binom{y_{-}(t,1)}{y_{+}(t,0)} = G \binom{y_{-}(t,0)}{y_{+}(t,1)}, t \in [0,+\infty),$$

where

(i)  $y_{-} \in \mathbb{R}^{m}$  and  $y_{+} \in \mathbb{R}^{n-m}$  are defined by

(2) 
$$y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix},$$

(ii) the map  $G: \mathbb{R}^n \to \mathbb{R}^n$  is such that

(3) 
$$\begin{pmatrix} y_{-}^{*}(1) \\ y_{+}^{*}(0) \end{pmatrix} = G \begin{pmatrix} y_{-}^{*}(0) \\ y_{+}^{*}(1) \end{pmatrix}.$$

Part of G is fixed, part of G can be chosen in order to achieve the exponential stability of  $y^*$ .

# Shower



# Musical wind instruments





◆□▶ ◆□▶ ◆臣▶ ◆臣▶ 臣 のへぐ


#### Electrical transmission lines



### Chromatography



### Heat exchangers



# Navigable rivers (see above)



# Control for navigable rivers



## Control for navigable rivers



▲□▶ ▲□▶ ▲ = ▶ ▲ = ▶ ▲ □▶

### Successive pools of a navigable river



▲□▶ ▲□▶ ▲ 三▶ ▲ 三 ● ●

#### More examples (commercial break)

Progress in Nonlinear Differential Equations and Their Applications Subseries in Control

Georges Bastin Jean-Michel Coron

Stability and Boundary Stabilization of 1-D Hyperbolic Systems

🕲 Birkhäuser

G. Bastin and JMC, Stability and Boundary Stabilization of 1-D Hyperbolic Systems, 2016, PNLDE Subseries in Control, Birkhaüser. This is a new Subseries; Please submit books to this Subseries.

▲□▶ ▲□▶ ▲□▶ ▲□▶ □ □ のへで

We consider the simplest hyperbolic equation

(1) 
$$y_t - \lambda y_x = 0, x \in (0, 1),$$

under the boundary conditions

(2) 
$$y(t,1) = u(t),$$

where  $\lambda \in (0, +\infty)$ . The control is  $u(t) \in \mathbb{R}$ . The goal is to produce a feedback law  $y \in L^2(0, L)^2 \mapsto u(y) \in \mathbb{R}$  such that, for the closed-loop system, one has finite-time stability in the optimal time, which is

(3) 
$$T_{\mathsf{opt}} := \frac{1}{\lambda}.$$

It is very simple: just taking u(y) = 0 solves the problem.





◆□◆ ▲□◆ ▲目◆ ▲目◆ ▲□◆



・ロト・(四ト・(日下・(日下・(日下)

# A tutorial example showing that u = 0 is not always the best choice

We consider the control system

(1) 
$$y_t^1 + \lambda_1 y_x^1 = 0, \ y_t^2 - \lambda_2 y_x^2 = 0, \ y_t^3 - \lambda_3 y_x^3 = 0, \ x \in (0,1), \ t > 0,$$

(2) 
$$y^1(t,0) = ay^2(t,0) + by^3(t,0), \ y^2(t,1) = u_1(t), \ y^3(t,1) = u_2(t),$$

(3) 
$$y^1(0,x) = y_0^1(x), \ y^2(0,x) = y_0^2(x), \ y^3(0,x) = y_0^3(x),$$

where, at time  $t \ge 0$ , the control is  $(u_1(t), u_2(t))^{\mathrm{tr}} = (y^2(t, 1), y^3(t, 1)) \in \mathbb{R}^2$  and the state is  $y(t, \cdot) = (y^1(t, \cdot), y^2(t, \cdot), y^3(t, \cdot))^{\mathrm{tr}} \in L^2(0, 1)^3$ . The real numbers a and b are given. We assume that

$$(4) 0 < \lambda_1, 0 < \lambda_2 < \lambda_3.$$

Let

If one uses the controls  $u_1(t)=u_2(t)=0$ , one gets that

$$(1) y(T_1, \cdot) = 0,$$

with

(2) 
$$T_1 := \tau_1 + \tau_2.$$

Moreover if  $a \neq 0$  and b = 0,  $T_1$  is optimal: there are initial data such that, whatever are the controls one cannot reach 0 at a time smaller than  $T_1$  (see Li Tatsien's book in 2010 for more general situations). However, as proved by L. Hu in 2015 (in a more general framework), if  $b \neq 0$  one can steer the control system to 0 in time

(3) 
$$T_0 := \max\{\tau_1 + \tau_3, \tau_2\} < T_1.$$

Let us do it with a feedback law. Note that  $u_1 = u_2 = 0$  is not working if  $a \neq 0$ : in this case whatever is  $0 < T < T_1$  there are initial data such, with these controls, that  $y(T, \cdot) \neq 0$ . The idea is that we would like to have

(4) 
$$(y^1(t,0) =) ay^2(t,0) + by^3(t,0) = 0.$$

as fast as possible. For that, we point out that

(5) 
$$ay^2(t,0) + by^3(t,0) = ay^2(t-\tau_3,\tau_3/\tau_2) + by^3(t-\tau_3,1).$$

Hence the idea is to use the feedback law

(1) 
$$u^{1}(t) = y^{2}(t,1) = 0, \ u^{2}(t) = y^{3}(t,1) = -(a/b)y^{2}(t,\tau_{3}/\tau_{2}).$$

◆□▶ ◆□▶ ◆三▶ ◆三▶ ・三 ● のへで

With this feedback law, one has

(2) 
$$y^1(t,0) = 0, \forall t > \tau_3,$$

(3) 
$$y^1(t,x) = 0, \forall t > \tau_3 + \tau_1 x,$$

(4) 
$$y^2(t,x) = 0$$
, if  $t > \tau_2 - x\tau_2$ ,  
(5)  $y^3(t,1) = 0, \forall t > \tau_2 - \tau_3$ ,

(6) 
$$y^3(t,x) = 0, \forall t > \tau_2 - \tau_3 x.$$

In particular

(7) 
$$y(t,\cdot) = 0, \forall t \ge T_0.$$



▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●



▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●



▲□▶ ▲□▶ ▲□▶ ▲□▶ ▲□ ● ● ●



▲□▶ ▲□▶ ▲□▶ ▲□▶ □ □ ○ ○ ○

#### Comparaison of the two feedback laws

 $y^2(t,1) = 0$  in both cases.



### The general case

We consider the  $n \times n$  hyperbolic system

(1) 
$$y_t + \Lambda y_x = 0, t > 0, x \in (0, 1),$$

with  $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k, -\lambda_{k+1}, \ldots, -\lambda_n)$ . We assume that  $-\lambda_n < \ldots < -\lambda_{k+1} < 0 < \lambda_k < \ldots < \lambda_1$ . The controls are

(2) 
$$u_1(t) = y^{k+1}(t,1), \dots, u_m(t) = y^n(t,1).$$

with m := n - k. On the boundary x = 0 the boundary condition is

(3) 
$$(y^1, \dots, y^k)^{\text{tr}}(t, 0) = B(y^{k+1}, \dots, y^n)^{\text{tr}}(t, 0)$$

with  $B \in \mathbb{R}^{k \times m}$ . Let  $\tau_i := 1/\lambda_i$ . Let us define  $T_0 > 0$  by

(4) 
$$T_0 := \max\{\tau_1 + \tau_{m+1}, \dots, \tau_k + \tau_{m+k}, \tau_{k+1}\} \text{ if } m \ge k,$$

(5) 
$$T_0 := \max\{\tau_{k+1-m} + \tau_{k+1}, \dots, \tau_k + \tau_{k+m}\} \text{ if } m < k.$$

Then one has the following theorem.

#### Theorem (JMC and Hoai-Minh Nguyen (2018))

Assume that, for every  $1 \le i \le \min\{k, m-1\}$ , the  $i \times i$  matrix formed from the last i columns and the last i rows of B is invertible. Then there exists a linear feedback which yields the null-controllability at the time  $T_0$ . Moreover, for any  $T < T_0$ , there exists an initial datum such that  $y(T, \cdot) \neq 0$  for every control.

#### Remark

This result also improves the optimal time for null controllability. The best prior estimate for the null controllability was the time given by L. Hu (2015) in the case  $m \ge k$ :  $T_1 := \max\{\tau_k + \tau_{m+1}, \tau_{k+1}\}$ . One always have  $T_0 \le T_1$  and this inequality is strict if k > 1.

#### Motivation of the stabilization/rapid stabilization problems

- 2 Some results in finite dimension
- 3 Small-time stabilization of 1-D linear heat equations
- 4 Rapid exponential stabilization of Korteweg-de Vries equations

- 5 Finite-time stabilization of 1-D linear hyperbolic systems
- 6 Finite-time stabilization of 1-D linear balance laws