Classical and semiclassical observability for the Baouendi-Grushin operator

(Based on collaborations with N. Burq and C. Letrouit)

Chenmin SUN (Laboratoire AGM)

Control in times of crisis
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Outline

General Context I: Control for the Schödinger equation

Linear Schrödinger equation:

$$i\partial_t u = Pu, \quad u|_{t=0} = u_0 \in L^2(M)$$

P is a self-adjoint operator. The solution is given by the unitary group e^{-itP} on L^2:

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Exact controllability: $\forall\ u_0, u_T \in L^2(M)$, $\exists\ f \in L^2((0,T); L^2(M))$, supported in $(0,T) \times \omega$, such that the solution u of

$$i\partial_t u = Pu + f, \quad u|_{t=0} = u_0$$

satisfies $u|_{t=T} = u_T$.

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By duality and the time reversibility, the exact-controllability is equivalent to the following observability:
Observability: For ω and $T>0$, $\exists C(T,\omega) > 0$, such that

$$\|u_0\|^2_{L^2(M)} \leq C(T,\omega)\int_0^T \|\mathbf{1}_\omega e^{-itP}u_0\|^2_{L^2(M)}dt.$$

General Context II: Semi-classical observability

Assume that $P \geq 0$ is self-adjoint (with a compact resolvent)

$$P\varphi_j = \lambda_j^2 \varphi_j, \quad 0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \leq \lambda_j^2 \to +\infty.$$

Quasi-modes of order $o(h^\alpha)$: $(h^2 P - 1)u_h = o_{L^2}(h^\alpha)$.

$$u_h = \sum_{\lambda_j \in [h^{-1}-c(h), h^{-1}+c(h)]} c_j \varphi_j, \quad c(h) \sim o(h^{\alpha-1}).$$

Concentration: Can u_h concentrate somewhere asymptotically $(h \to 0)$?

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Semi-classical observability: (resolvent estimate)

$$\|u\|_{L^2(M)} \leq C\|u\|_{L^2(\omega)} + Ch^{-\alpha}\|(h^2 P - 1)u\|_{L^2(M)}$$

This estimate excludes the possibility of quasi-modes of order $o(h^\alpha)$ concentrated outside ω .

Baouendi-Grushin operator

Baouendi-Grushin operator :

$$\Delta_\gamma = \partial_x^2 + |x|^{2\gamma}\partial_y^2, \quad \gamma > 0$$

with domain

$$D(\Delta_\gamma) = \{f \in L^2(M) : \partial_x^2 f, |x|^{2\gamma}\partial_y^2 f \in L^2(M), f|_{\partial M} = 0\},$$

where

$$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad M = (-1,1)_x \times \mathbb{T}_y.$$

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$$\Delta_\gamma = X_1^2 + X_2^2, \quad X_1 = \partial_x,\ X_2 = |x|^\gamma \partial_y.$$

- If $\gamma \in \mathbb{N}$: γ brackets to recover $T_{(0,y)}M$

$$[\partial_x, x\partial_y] = \partial_y, \quad \mathrm{span}\{\partial_x, x\partial_y, [\partial_x, x\partial_y]\}|_{(0,y)} = \mathbb{R}^2 = T_{(0,y)}M.$$

- Loss of regularity of order $\frac{\gamma}{\gamma+1}$:

$$\|f\|_{H^{\frac{1}{\gamma+1}}(M)} \lesssim \|\nabla_\gamma f\|_{L^2(M)},$$

where the associated gradient $\nabla_\gamma = (\partial_x, |x|^\gamma \partial_y)$.

Observability for the classical Schrödinger

Theorem (Lebeau)

If ω satisfies (GCC), then for any $T > 0$, the observability holds:

$$\|u_0\|^2_{L^2(\Omega)} \leq C_T \int_0^T \|e^{it\Delta} u_0\|^2_{L^2(\omega)} dt.$$

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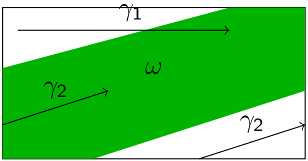
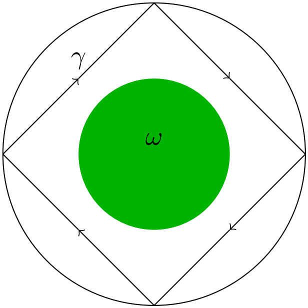
ω satisfies (GCC) if there exists $T_0 > 0$, such that all generalized geodesics of length $T > T_0$ intersects with ω.



GCC is satisfied



GCC is not satisfied

Classical Schrödinger on $\mathbb{T}^2$

(GCC) is not necessary in general !

Theorem (Jaffard, Burq-Zworski, Bourgain-Burq-Zworski, Anantharaman-Macià)

Assume that $V \in L^2(\mathbb{T}^2)$ *and let* $P = -\Delta + V(x)$*. For any* $T > 0$ *and any non-empty open set* $\omega \subset \mathbb{T}^2$*, there exists* $C(T,\omega) > 0$*, such that for all* $u_0 \in L^2(\mathbb{T}^2)$*,*

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C(T,\omega) \int_0^T \|e^{itP} u_0\|^2_{L^2(\omega)} dt.$$

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Assume that $V \in L^2(\mathbb{T}^2)$ and let $P = -\Delta + V(x)$. For any $T > 0$ and any non-empty open set $\omega \subset \mathbb{T}^2$, there exists $C(T, \omega) > 0$, such that for all $u_0 \in L^2(\mathbb{T}^2)$,

$$\|u_0\|^2_{L^2(\mathbb{T}^2)} \leq C(T, \omega) \int_0^T \|e^{itP} u_0\|^2_{L^2(\omega)} dt.$$

Semi-classical view-point

- If ω satisfies (GCC), Lebeau's theorem can be reformulated as

$$\|u\|_{L^2} \leq C \|u \mathbf{1}_\omega\|_{L^2} + \frac{C}{h} \|(h^2 \Delta + 1) u\|_{L^2}.$$

- If $\emptyset \neq \omega \subset \mathbb{T}^2$, at least we have

$$\|u\|_{L^2(\mathbb{T}^2)} \leq C \|u \mathbf{1}_\omega\|_{L^2(\mathbb{T}^2)} + o(h^{-2}) \|(h^2 \Delta + 1) u\|_{L^2(\mathbb{T}^2)}.$$

Some notations

For a given PDE and ω, we denote by

$$T_{ob}(\omega) := \inf \left\{ T > 0 : \text{observability holds for } \omega \text{ at } T \right\}$$

$$T_{ob}(\omega) = \infty, \text{ if never observable in any finite } T > 0.$$

$$T_{ob}(\omega) = 0, \text{ if observability holds for any } T > 0.$$

Example

- For the classical Schrödinger equation on the torus $\mathbb{T}^2$ and $\omega \neq \emptyset$, $T_{ob}(\omega) = 0$.
- For the classical wave equation, if ω satisfies (GCC), then $T_{ob} = T_{GCC}$. If ω does not satisfy (GCC), $T_{ob} = +\infty$.

Bouendi-Grushin Schrödinger equation: $\gamma = 1$

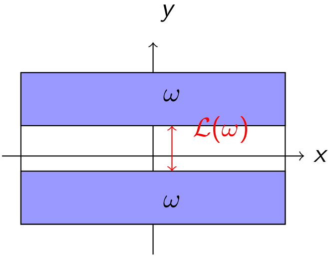
Consider the Schödinger equation

$$i\partial_t u + (\partial_x^2 + x^2\partial_y^2)u = 0.$$

Let ω be of the form $(-1,1)_x \times I$, where $I \subset \mathbb{T}$ is a finite union of intervals. For such ω, we define $\mathcal{L}(\omega)$:

$$\mathcal{L}(\omega) := \sup\{s : \exists y_1, y_2 \in \mathbb{T}, \mathrm{dist}_{\mathbb{T}}(y_1, y_2) = s, [(0,y_1),(0,y_2)] \cap \omega = \emptyset\}$$

the length of largest interval in $\Omega \setminus \omega \cap \{x = 0\}$.



Theorem (N. Burq-S. '19)

For $s = \gamma = 1$, in the above geometry, $T_{ob} = \mathcal{L}(\omega)$.

Vertical transport: Why $T_{ob}(\omega) \geq \mathcal{L}(\omega)$?

Consider the Grushin Schödinger equation on $\mathbb{R}^2$:

$$i\partial_t u + \partial_x^2 u + x^2 \partial_y^2 u = 0, \quad (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

Taking the Fourier transform in y, we find

$$(i\partial_t + \underbrace{\partial_x^2 - \eta^2 x^2}_{\text{semi-classical harmonic oscillator}})\widehat{u}(t, x, \eta) = 0$$

Thus we have the explicit formula

$$u(t, x, y) = (2\pi)^{-1/2} \sum_{m=0}^{\infty} \int_{\mathbb{R}} e^{-it(2m+1)|\eta| + iy\eta} \widehat{f}_m(\eta) h_m(\sqrt{|\eta|}x) d\eta,$$

where $h_m(x)$ is the m-th Hermite function.
Specifically, for initial data concentrated on modes $m = 0$ and $\eta \sim N > 0$, we have the solution

$$u(t, x, y) = f(x, y - t)$$

satisfies a transport equation in y.

Stronger degeneracy: $\gamma > 1$

With stronger degeneracy, the Schrödinger equation $i\partial_t u + \Delta_\gamma u = 0$ with $\gamma > 1$ is never observable, i.e. $T_{ob} = +\infty$.

We are not able to detect the observability at the finite classical time scale!

How about the semi-classical scale?

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We are not able to detect the observability at the finite classical time scale!
How about the semi-classical scale?

Theorem (Letrouit-S., '20)

Assume that $\gamma \geq 1$ and ω contains a horizontal strip. There exist $C > 0, 0 < h_0 \ll 1$ such that for any $v \in D(\Delta_\gamma)$ and $0 < h < h_0$, there holds

$$\|v\|_{L^2(M)} \leq C\|v\|_{L^2(\omega)} + Ch^{-(\gamma+1)}\|(h^2\Delta_\gamma + 1)v\|_{L^2(M)}. \qquad (1)$$

- Consequently, quasi-modes of order $o(h^{\gamma+1})$ cannot be too "small" on ω !
- This resolvent estimate is optimal.
- Cannot expect similar estimates for $\gamma < 1$. When $\gamma = 0$ and ω is a horizontal strip, there exist quasi-modes of order $o(h)$ concentrated along a geodesic ray parallel to ω.

Consequence I: Threshold of the Schrödinger observability

Consider the Schrödinger-type equation:

$$i\partial_t u - (-\Delta_\gamma)^s u = 0. \tag{2}$$

Recall that (2) is observable (since it is time-reversible) in ω at T if

$$\|u(0,\cdot)\|^2_{L^2(M)} \leq C_{T,\omega} \int_0^T \|u(t,\cdot)\|^2_{L^2(\omega)} dt$$

and T_{ob} is the inf. of all observable time $T > 0$.

Theorem (Letrouit-S.)

Assume that $\gamma \geq 1$ and ω contains a horizontal strip $(-1,1)_x \times I_y$. Then

(1) *If $s < \frac{\gamma+1}{2}$, then $T_{ob} = \infty$;*

(2) *If $s > \frac{\gamma+1}{2}$, then T_{ob}=0;*

(3) *If $s = \frac{\gamma+1}{2}$, then $0 < T_{ob} < \infty$.*

- We do not compute the exact value T_{ob} when $s = \frac{\gamma+1}{2}$ for general s and γ.

Non-observability for Schrödinger: The case $s < \frac{\gamma+1}{2}$

Consider the easier geometry $(x, y) \in \mathbb{R}^2$. Taking the Fourier transform in y, we can construct a sequence of explicit solutions to

$$i\partial_t u_h - (-\Delta_\gamma)^s u_h = 0 :$$

$$u(t,x,y) = \int_{\mathbb{R}} \chi(h\eta) e^{iy\eta - it\mu_0|\eta|^{\frac{2s}{\gamma+1}}} |\eta|^{\frac{1}{2(\gamma+1)}} \phi_\gamma(|\eta|^{\frac{1}{\gamma+1}} x) d\eta,$$

where $\phi_\gamma(\cdot)$ is the ground state of $-\partial_x^2 + |x|^{2\gamma}$ satisfying

$$(-\partial_x^2 + |x|^{2\gamma})\phi_\gamma = \mu_0 \phi_\gamma.$$

The group velocity of the special solution is

$$\partial_\eta(y\eta - t\mu_0|\eta|^{\frac{2s}{\gamma+1}}) = y - \frac{2s}{\gamma+1} t\mu_0 |\eta|^{\frac{2s}{\gamma+1}-1} \mathrm{sign}(\eta).$$

Consequence II: Damped wave equation

The damped wave equation:

$$\partial_t^2 u - \Delta_\gamma u + b(x,y)\partial_t u = 0, \quad b(x,y) \geq 0, b \in L^\infty$$

is well-posed as a semi-group $e^{t\mathcal{A}}$ on $H^1_\gamma \times L^2$, where

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta_\gamma & -b \end{pmatrix}$$

The free energy

$$E[u(t)] := \int_M |\partial_t u|^2 + |\nabla_\gamma u|^2$$

is decreasing in time:

$$\frac{d}{dt}E[u(t)] = -\int_M b|\partial_t u|^2 \leq 0.$$

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One can verify that $i\mathbb{R} \cap \mathrm{Spec}(\mathcal{A}) = \emptyset$.

Theorem (Borichev-Tomilov)

There is an equivalence between:

- $\|(\mathcal{A} - i\lambda \mathrm{Id})^{-1}\|_{\mathcal{L}(H^1_\gamma \times L^2)} = O(|\lambda|^\alpha)$, $\lambda \in \mathbb{R}$ *and* $|\lambda| \to \infty$;
- $E[u(t)] = O(t^{-\frac{1}{\alpha}})\|\mathcal{A}(u(0), \partial_t u(0))\|_{H^1_\gamma \times L^2}$, $t \to +\infty$.

Damped wave equation

Corollary

There exits $C_0 > 0$, such that for all $\lambda \in \mathbb{R}, |\lambda| \geq 1$, we have

$$\|(\mathcal{A} - i\lambda \mathrm{Id})^{-1}\|_{\mathcal{L}(H^1_\gamma \times L^2)} \leq C_0 |\lambda|^{2\gamma}.$$

For any sequence $(v_\lambda) \in L^2, \|v_\lambda\|_{L^2} = 1$, we have

$$\|(-\Delta_\gamma - \lambda^2 + i\lambda b)v_\lambda\|_{L^2} \geq C_0' |\lambda|^{1-2\gamma}.$$

Applying Borichev-Tomilov, we have

Corollary

Assume that $\gamma \geq 1$ and $b = \mathbf{1}_\omega$. Then for $(u_0, u_1) \in D(\mathcal{A})$, the solution of the damped wave equation with the initial data (u_0, u_1) satisfies

$$(E[u(t)])^{\frac{1}{2}} \leq \frac{C}{(1+t)^{\frac{1}{2\gamma}}} \|(u_0, u_1)\|_{D(\mathcal{A})}.$$

Comparison for the heat equation

$\partial_t u - \Delta_\gamma u = 0$		
	ω vertical	ω horizontal
$0 < \gamma < 1$	$T_{ob} = 0$ [1]	$T_{ob} = 0$ [1]
$\gamma = 1$	$0 < T_{ob} < \infty$ [1][2][3]	$T_{ob} = +\infty$ [4]
$\gamma > 1$	$T_{ob} = \infty$ [1]	$T_{ob} = \infty$ [1]

Remark

For $\gamma = 1$, the explicit value is obtained in [2],[3]. We refer [5] for non-rectangular ω.

References:

[1] Beauchard-Cannarsa-Guglielmi, [2] Beauchard-Dardé-Ervedoza, [3] Beauchard-Miller-Morancey, [4] Koenig, [5] Koenig-Duprez

Conclusion and perspectives

Observability for the Schrödinger:

$i\partial_t u - (-\Delta_\gamma)^s u = 0$		
	ω vertical	ω horizontal
$s = \frac{\gamma+1}{2}, \gamma \geq 1$	$T_{ob} = \infty$	$0 < T_{ob} < \infty$
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$\gamma = 0, s = 1$	$T_{ob} = 0$	$T_{ob} = 0$
$0 < \gamma < 1, s = 1$	?	?

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Energy decay rate for the damped wave:

Log decay: Laurent-Léautaud ('20)

Possible better decay: $(u, \partial_t u)|_{t=0} \in D(\mathcal{A})$

$\partial_t^2 u - \Delta_\gamma u + b\partial_t u = 0$		
	$b = \mathbf{1}_\omega$, ω vertical	$b = \mathbf{1}_\omega$, ω horizontal
$\gamma \geq 1$	$\log(1+t)^{\frac{1}{\gamma+1}}$ $(\gamma \in \mathbb{N})$	$(1+t)^{-\frac{1}{2\gamma}}$ (Optimal?)
$0 < \gamma < 1$	?	?

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$0 < \gamma < 1$	?	?

Other questions: General Step 2 sub-elliptic operator P ?

Fermanian Kammerer-Letrouit ('20): quotients of Heisenberg-type groups

More technical details are provided below for interesting readers

Thank you for your attention!

Sketch of the proof of Resolvent estimate: Different regimes

For $(h^2\Delta_\gamma + 1)u = f$, we want to prove that

$$\|u\|_{L^2(M)} \leq C\|u\|_{L^2(\omega)} + Ch^{-(\gamma+1)}\|f\|_{L^2(M)}.$$

By contradiction, assume that $(h_n^2\Delta_\gamma + 1)u_n = f_n$,

$$\|u_n\|_{L^2(M)} \sim 1,\ \|u_n\|_{L^2(\omega)} = o(1),\ \|f_n\|_{L^2(M)} = o(h_n^{\gamma+1}),\ h_n \to 0.$$

- W.L.O.G., we may assume that $|h^2\Delta_\gamma| \sim 1$, i.e. $u_n = \psi(h_n^2\Delta_\gamma)u_n$, $\psi \equiv 1$ near 1, since $(1-\psi(h_n^2\Delta_\gamma))(h_n^2\Delta_\gamma + 1)$ is elliptic.
- Subelliptic estimate: $|D_y| \lesssim h^{-(\gamma+1)}$.

Different regimes

(a) Degenerate regime: $h^{-1} \ll |D_y| \lesssim h^{-(\gamma+1)}$ (leads to $|x| \ll 1$).

(b) GCC regime: $|D_y| \sim h^{-1}$.

(c) Horizontal propagation regime: $|D_y| \ll h^{-1}$.

Degenerate regime: $h^{-1} \ll |D_y| \lesssim h^{-(\gamma+1)}, |x| \ll 1$

- Key observation:

$$[\Delta_\gamma, x\partial_x + (\gamma+1)y\partial_y] = 2\Delta_\gamma.$$

- Inserting suitable truncation $\chi(y)$ to make the vector field $x\partial_x + (\gamma+1)y\partial_y$ well-defined. Choose χ such that

$$\mathrm{supp}\chi'(y) \subset \omega.$$

- Computing

$$\left([h_n^2\Delta_\gamma + 1, \chi_1(x)\chi(y)(x\partial_x + (\gamma+1)y\partial_y)]u_n, u_n\right)_{L^2(M)}$$

to obtain the estimate

$$\|h_n\nabla_\gamma u_n\|^2_{L^2(M)} \leq O(1)\underbrace{\|h_n\nabla_\gamma u_n\|^2_{L^2(\omega)}}_{\text{observation term}} + \underbrace{\|h_n\nabla_\gamma u_n\|_{L^2(\mathrm{supp}(\chi_1'))}}_{\text{small}}$$
$$+ O(1)\|f_n\|_{L^2(M)}(\|\partial_x u_n\|_{L^2(M)} + \|\partial_y u_n\|_{L^2(M)}) + O(h).$$

- $\|\partial_y u_n\|_{L^2(M)} \lesssim h^{-(\gamma+1)}, \|\partial_x u_n\|_{L^2(M)} \lesssim h^{-1}, \implies \|h_n\nabla_\gamma u_n\|_{L^2(M)} = o(1)$.
- Analysis is simpler compared to the time-dependent case.

GCC regime: $|D_y| \sim h^{-1}$

- Standard propagation argument: (Egorov, defect-measure, positive commutator). Check that the Hamiltonian flow associated with the symbol

$$p(x, y; \xi, \eta) = \xi^2 + |x|^{2\gamma}\eta^2$$

satisfies (GCC). We need $\gamma \geq 1$ to define the flow !

Horizontal propagation regime I: $h^{-\epsilon} \lesssim |D_y| \ll h^{-1}$

- Positive commutator method with

$$[h^2\Delta_\gamma + 1, \chi(y)y\partial_y] = 2\chi(y)|x|^{2\gamma}(h\partial_y)^2 + 2y\chi'(y)|x|^{2\gamma}(h\partial_y)^2 + h^2\chi''(y)y|x|^{2\gamma}\partial_y + 2h^2\chi'(y)|x|^{2\gamma}\partial_y.$$

  Using this, we obtain that

$$\||x|^\gamma h_n\partial_y u_n\|_{L^2}^2 \le \underbrace{C\|\chi'(y)^{1/2}|x|^\gamma h_n\partial_y u_n\|_{L^2}^2}_{\text{observation term}} + \underbrace{Ch_n^{-1}\|f_n\|_{L^2}\|h_n\partial_y u_n\|_{L^2}}_{\text{dangerous}} + Ch\||x|^\gamma h_n\partial_y u_n\|_{L^2}\|u_n\|_{L^2} + O(h^2).$$

- Using the horizontal propagation: $\|\partial_y u_n\|_{L^2(M)} \le C\||x|^\gamma\partial_y u_n\|_{L^2(M)} + o(h^{\gamma-1})$.
- Replacing $h_n\partial_y u_n$ by $|x|^\gamma h_n\partial_y u_n$,...

Horizontal propagation regime II: $|D_y| \ll h^{-\epsilon}$

Normal form method in the spirit of Burq-Zworski:

- Idea: Find a pseudo $q(x, h_n D_x)$ and

$$v_n := (1 + h_n q(x, h_n D_x) D_y^2) u_n$$

such that

$$h^2 \partial_x^2 v_n + h^2 M \partial_y^2 v_n + v_n = o_{L^2}(h_n^2),$$

where

$$M = \frac{1}{2} \int_{-1}^{1} |x|^{2\gamma} d\gamma.$$

The choice of $q(x, \xi)$ is

$$q(x, \xi) = \frac{1}{2i\xi} \int_{-1}^{x} (M - |z|^{2\gamma}) dz$$

- Using the resolvent estimate for the usual Laplacian on $\mathbb{T}^2$: Jaffard, Burq-Zworski, Anantharaman-Macià, $\forall \widetilde{\omega} \neq \emptyset$,

$$\|v\|_{L^2(\mathbb{T}^2)} \leq C\|v\|_{L^2(\widetilde{\omega})} + o(h^{-2})\|(h^2\partial_x^2 + h^2 M \partial_y^2 + 1)v\|_{L^2(\mathbb{T}^2)}.$$