Bilinear control for evolution equations

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Webinar

Control in Time of Crisis October 6, 2020

organised by: E. Fernández-Cara, Sylvain Ervedoza, A. Mercado, and L. de Teresa

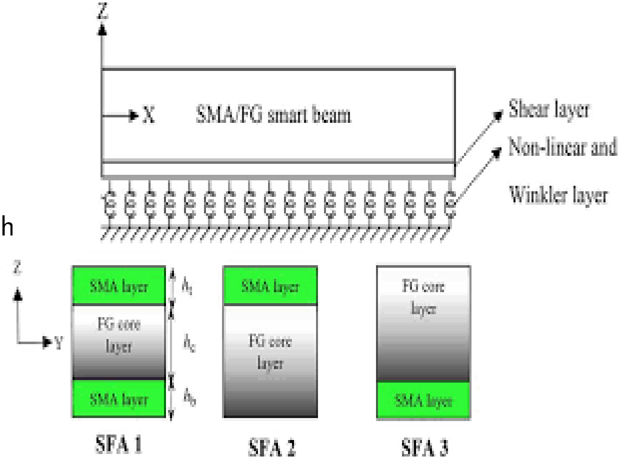
Introduction to bilinear control systems

Motivations

Bilinear controls enter the system equations as coefficients changing (at least some of) the principal parameters of the process at hand

Examples

- by embedded *smart* alloys, the natural frequency response of a beam can be changed
- the rate of a chemical reaction can be altered by various catalysts and/or by th speed at which the reaction ingredients are mechanically mixed

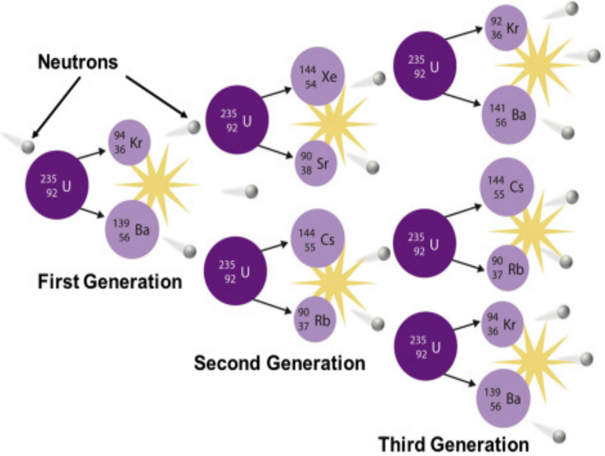


A simplified model of a nuclear chain reaction

A chain reaction refers to a process in which neutrons released in fission produce an additional fission in at least one further nucleus. This nucleus in turn produces neutrons, and the process repeats. The process may be controlled (nuclear power) or uncontrolled (nuclear weapons).

$$\boxed{u_t = a^2 \Delta u + v(t,x) u}$$

- $u(t,x) \geq 0$ neutron density in the reaction
- $v(t,x) > 0$ neutron amount in the surrounding medium
- $v(t,x)u$ neutrons provided by the collision of the particles in the reaction with the surrounding medium



Scalar-input bilinear control systems

Systems where control enters as a coefficient

$$\begin{cases} u'(t) + Au(t) + \mathbf{p}(t)Bu(t) = 0 \quad t \in [0, T] \\ u(0) = u_0 \in X \end{cases}$$

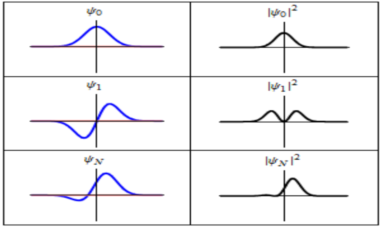
- the state space $(X, \langle \cdot, \cdot \rangle)$ is a Hilbert space
- $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on X
- $B : D(B) \subset X \to X$ is a linear operator on X
- control $\mathbf{p} : [0, T] \to \mathbb{R}$ is scalar function

Schrödinger equation

The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system

$$\boxed{i\psi_t = -\Delta\psi - p(t)\mu(x)\psi}$$

- ψ wave function of a particle
- p amplitude of the electric field
- μ dipolar moment of the particle



Fokker-Planck equation

The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as in Brownian motion

Let X_t be a $1D$ duffusion process in $(\Omega, \mathcal{F}, \mathbb{P})$

$$dX_t = b(t, X_t)dt + \sigma dW_t$$

where W_t is the standard Wiener process

The probability density of X_t

$$\mathbb{P}(\alpha \leq X_t \leq \beta) = \int_\alpha^\beta u(t,x)dx$$

satisfies the Fokker-Planck equation

$$\boxed{u_t(t,x) - \frac{\sigma^2}{2} u_{xx}(t,x) + \big(b(t,x)u(t,x)\big)_x = 0}$$

Bilinear control problem: to control FP by a drift term of the form $b(t,x) = p(t)\mu(x)$

What are the difficulties?

The map $\mathbf{\Phi} : \mathbf{p} \mapsto u$ is

Additive control:

$$\begin{cases} u' + Au + B\mathbf{p} = 0 \\ u(0) = u_0 \end{cases}$$

$\downarrow$

linear

$$u(t) = e^{-tA}u_0 - \int_0^t e^{(s-t)A}Bp(s)ds$$

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases}$$

$\downarrow$

nonlinear

$$u(t) = e^{-tA}u_0 - \int_0^t p(s)e^{(s-t)A}Bu(s)ds$$

An obstruction to exact controllability

Bilinear control:

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \\ u(0) = u_0 \end{cases} \tag{1}$$

Let $u_0 \in X$ and denote by $u(\cdot; p, u_0)$ the unique solution of (1) for $p \in L^1_{loc}(0, \infty)$.

Theorem (Ball, Marsden, Slemrod 1982)

Let $B \in \mathcal{L}(X)$. If $\dim X = \infty$, then the attainable set from u_0

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^1_{loc}(0, \infty)\}$$

is contained in a countable union of compact subsets of X. So, $X \setminus S(u_0)$ is dense.

References

- J.M. Ball, J.E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. SIAM Journal on Control and Optimization

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Superexponential stabilizability

Notions of stabilizability

$$\begin{cases} u' + Au + \mathbf{p}Bu = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases} \tag{2}$$

Let $\bar{p} \in L^1_{loc}(0, \infty)$ and let $\bar{u}_0 \in X$

Definitions

- (2) is *locally stabilizable to $u(\cdot; \bar{u}_0, \bar{p})$* if $\exists\, \delta > 0$ such that for all $u_0 \in B_\delta(\bar{u}_0)$ there exists $p \in L^1_{loc}(0, \infty)$ such that
$$\lim_{t \to +\infty} ||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| = 0$$

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- (2) is *locally exponentially stabilizable to $u(\cdot; \bar{u}_0, \bar{p})$* if $\exists\, M, \delta, \rho > 0$ such that for all $u_0 \in B_\delta(\bar{u}_0)$ there exist $p \in L^1_{loc}(0, \infty)$ satisfying
$$||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t} \qquad \forall t > 0$$

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$$||u(t; u_0, p) - u(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho t} \qquad \forall t > 0$$
- (2) is *locally superexponentially stabilizable to $u(\cdot; \bar{u}_0, \bar{p})$* if for any $\rho > 0$ there exists $\delta > 0$ such that, $\forall\, u_0 \in B_\delta(\bar{u}_0)$, it holds that
$$||u(t; u_0, p) - \bar{u}(t; \bar{u}_0, \bar{p})|| \leq Me^{-\rho e^{\omega t}} \qquad \forall t > 0$$

  for some constants $\exists\, M, \omega > 0$

Assumptions

Let $(X, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space and $A : D(A) \subset X \to X$ a densely defined linear operator satisfying the following standing assumptions

$$
\begin{array}{ll}
(a) & A \text{ is self-adjoint} \\
(b) & \langle Ax, x \rangle \geq 0 \qquad \forall x \in D(A) \\
(c) & D(A) \subseteq X \text{ is compact}
\end{array}
\tag{SA}
$$

$$\downarrow$$

1. X has an orthonormal basis $\{\varphi_k\}_{k \in \mathbb{N}^*}$ of eigenvectors of A
2. the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}^*}$ of A are nonnegative and $\lambda_k \to +\infty$ as $k \to +\infty$
3. $-A$ generates a strongly continuous semigroup of contractions e^{-tA}

Preliminaries

Given $T > 0$, consider the bilinear control problem

$$\begin{cases} u'(t) + \boldsymbol{A}u(t) + p(t)\boldsymbol{B}u(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \tag{3}$$

where $\boldsymbol{B} \in \mathcal{L}(X)$ and $p \in L^2(0, T)$

Consider system (3) with $p = 0$:

$$\begin{cases} u'(t) + \boldsymbol{A}u(t) = 0, & t \in [0, T] \\ u(0) = \boldsymbol{\varphi_1} \end{cases}$$

The solution $\boxed{\psi_1(t) = e^{-\boldsymbol{\lambda_1} t}\boldsymbol{\varphi_1}}$ is called the **ground state solution**

Remark

If $\langle \boldsymbol{A}x, x\rangle \geq \nu |x|^2$ $(\nu > 0)$, *then* $p = 0$ *yields*

$$||u(t) - \psi_1(t)|| = ||e^{-t\boldsymbol{A}}u_0 - e^{-t\boldsymbol{A}}\boldsymbol{\varphi_1}|| \leq e^{-\nu t}||u_0 - \boldsymbol{\varphi_1}||$$

So, (3) *is exponentially (but not superexponentially) stabilizable to* ψ_1

Superexponential stabilizability

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases}$$

Theorem

Suppose there exists a constant $\gamma > 0$ *such that the eigenvalues of* A *fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

Let $B : X \to X$ *be a linear bounded operator with the following properties:*

$$\langle B\varphi_1, \varphi_k \rangle \neq 0 \quad \forall k \geq 1 \qquad \& \qquad \exists\, \tau > 0 \quad \text{such that} \quad \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \qquad (\star)$$

Then, $\forall\, \rho > 0$, $\exists\, R > 0$ *such that any* $u_0 \in B_R(\varphi_1)$ *admits a control* $p \in L^2_{loc}(0, \infty)$ *such that*

$$||u(t) - \psi_1(t)|| \leq M e^{-\rho e^{\omega t} - \lambda_1 t} \qquad \forall t \geq 0$$

where M *and* ω *are positive constants depending only on* A *and* B

Sketch of the proof ($\lambda_1 = 0$)

Fix $T > \tau$ where $\tau > 0$ is given by $(\star)$

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t \in [0, T] \\ u(0) = u_0 \end{cases} \qquad \begin{cases} \psi_1'(t) + A\psi_1(t) = 0, & t \in [0, T] \\ \psi_1(0) = \varphi_1 \end{cases}$$

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$$\psi_1(t) \equiv \varphi_1 \qquad v := u - \varphi_1$$

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$$t \in [0, T] \qquad (\text{where } T > \tau)$$

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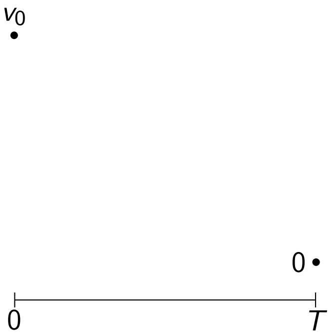

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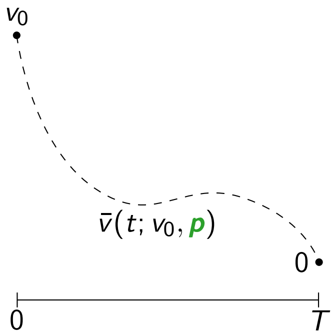

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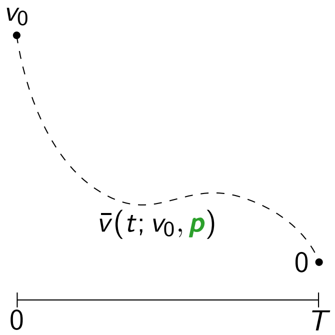

$$||p||_{L^2(0,T)} \leq \Lambda_T ||v_0||$$

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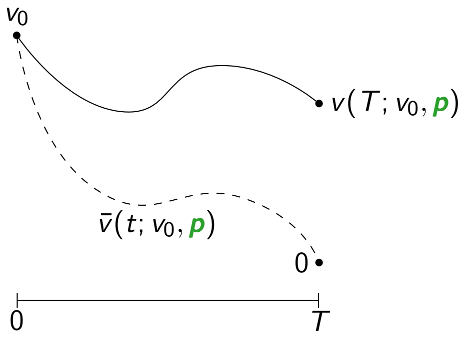

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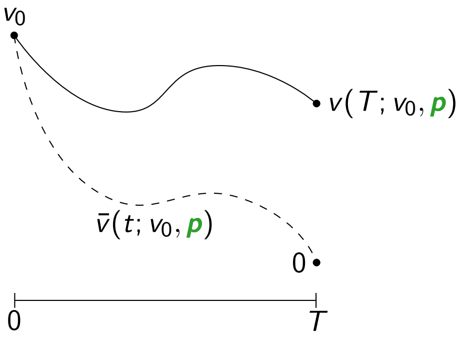

$$\|p\|_{L^2(0,T)} \leq \Lambda_T \|v_0\| \qquad \|(v - \bar{v})(T)\| = \|v(T)\| \leq K_T \|v_0\|^2.$$

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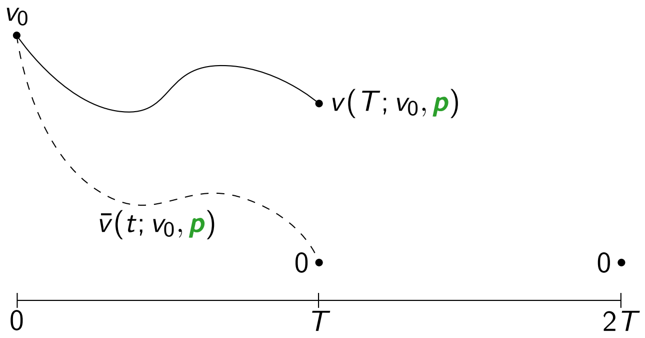
$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(T) = v_T, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(T) = v_T. \end{cases}$$

$$t \in [T, 2T] \qquad \text{(where } T > \tau\text{)}$$



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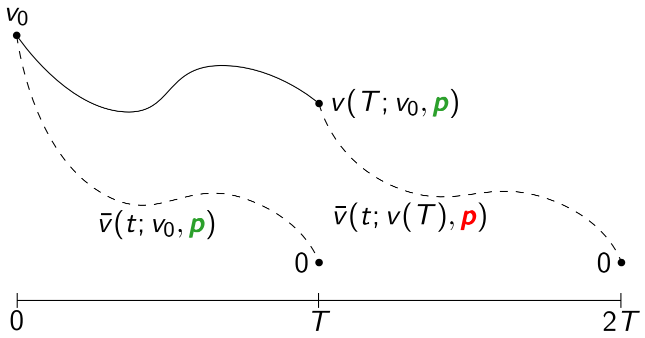

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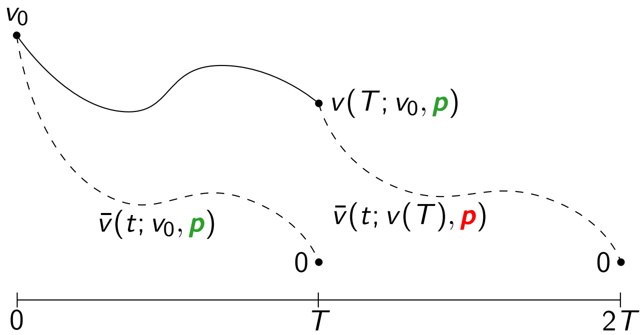
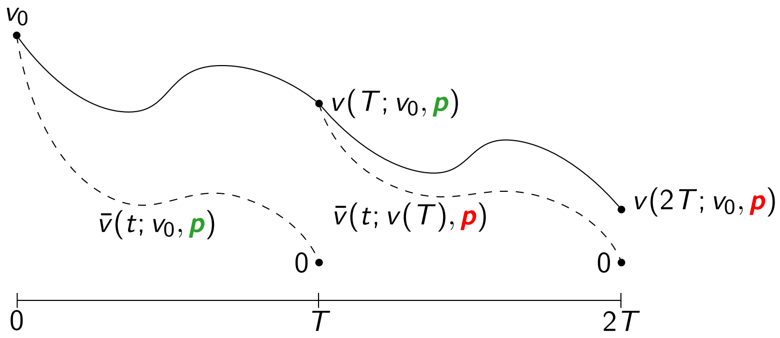


$$||p||_{L^2(T,2T)} \leq \Lambda_T ||v(T)||$$

Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(T) = v_T, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(T) = v_T. \end{cases}$$

$$t \in [T, 2T] \qquad (\text{where } T > \tau)$$

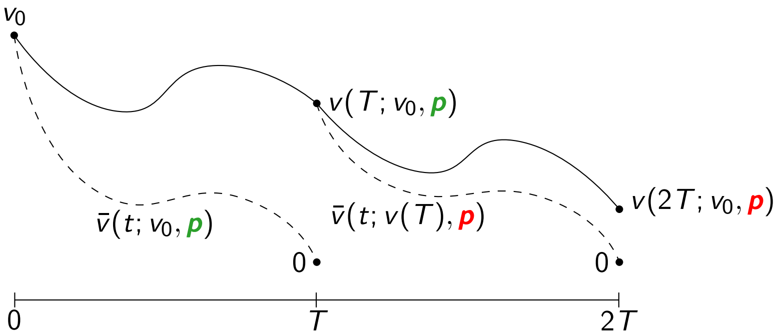


$$\|p\|_{L^2(T,2T)} \le \Lambda_T \|v(T)\|$$

Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(T) = v_T, \end{cases} \qquad \begin{cases} \bar v(t)' + A\bar v(t) + p(t)B\varphi_1 = 0, \\ \bar v(T) = v_T. \end{cases}$$

$$t \in [T, 2T]$$

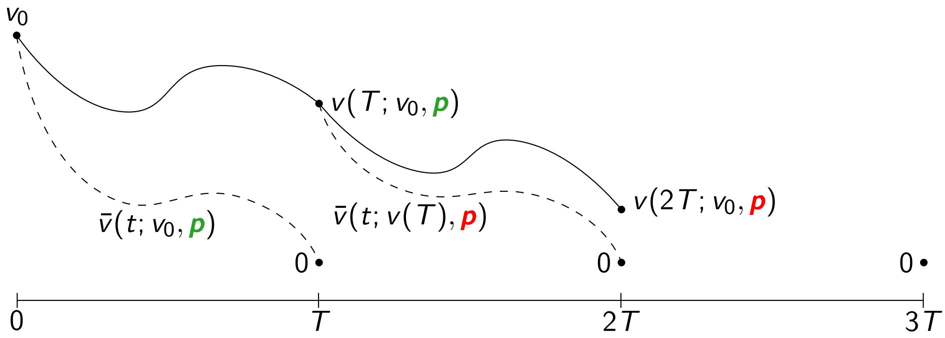


$$\|p\|_{L^2(T,2T)} \le C(T)\Lambda_T \|v(T)\|, \quad \|(v - \bar v)(2T)\| = \|v(2T)\| \le K_T \|v(T)\|^2 \le (1/K_T)(K_T\|v_0\|)^{2^2}$$

Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

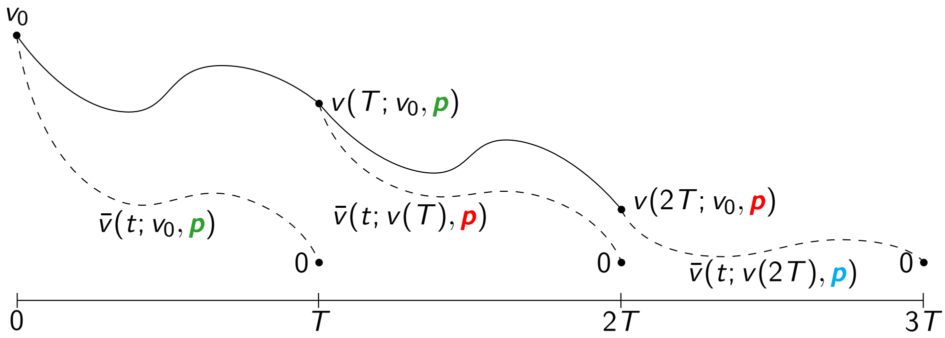
$$t \in [2T, 3T] \qquad (\text{where } T > \tau)$$



Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

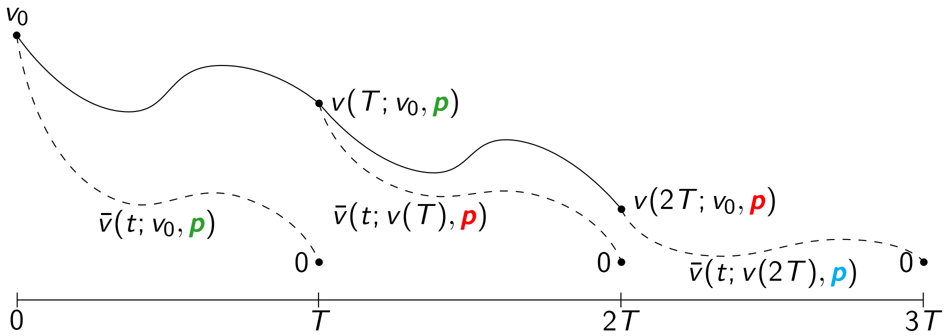
$$t \in [2T, 3T] \qquad (\text{where } T > \tau)$$



Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

$$t \in [2T, 3T] \qquad (\text{where } T > \tau)$$

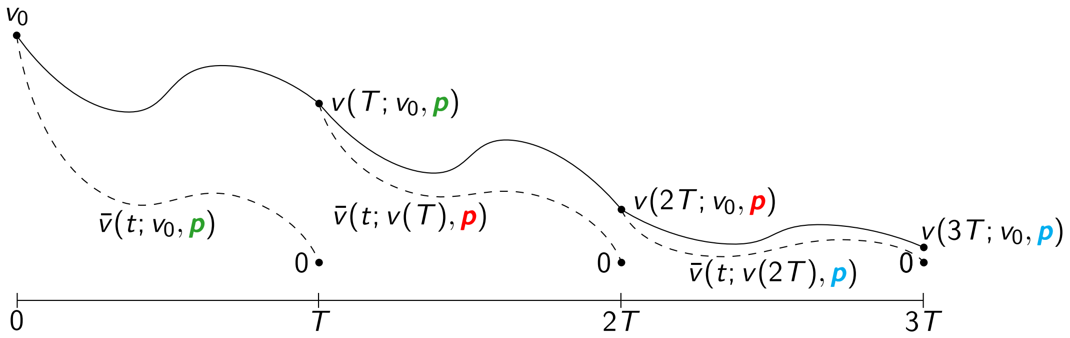


$$||p||_{L^2(2T,3T)} \leq \Lambda_T ||v(2T)||$$

Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

$$t \in [2T, 3T] \qquad (\text{where } T > \tau)$$

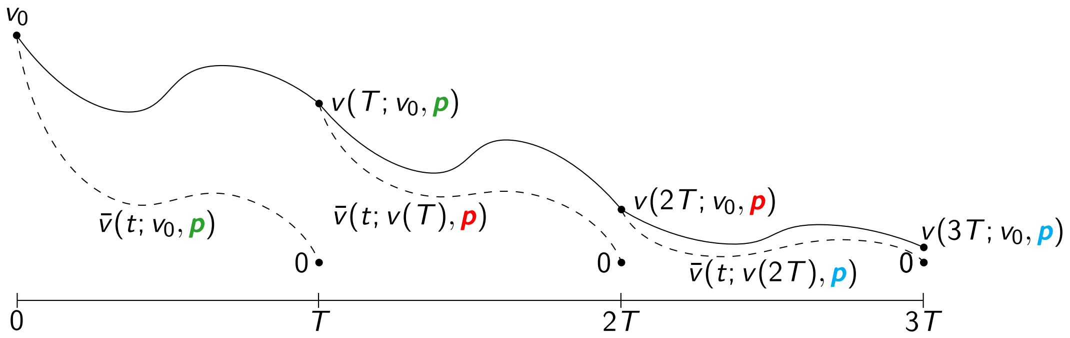


$$||p||_{L^2(2T,3T)} \leq \Lambda_T ||v(2T)||$$

Sketch of the proof ($\lambda_1 = 0$)

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \\ v(2T) = v_{2T}, \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0, \\ \bar{v}(2T) = v_{2T}. \end{cases}$$

$$t \in [2T, 3T] \qquad (\text{where } T > \tau)$$



$$||p||_{L^2(2T,3T)} \leq \Lambda_T ||v(2T)|| \qquad ||(v - \bar{v})(3T)|| = ||v(3T)|| \leq K_T ||v(2T)||^2 \leq (1/K_T)(K_T ||v_0||)^{2^3}$$

Solution of the linear problem

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0 \\ \bar{v}(0) = v_0 \end{cases}$$

Lemma

Let $T > \tau$. Then there exists a control $p \in L^2(0,T)$ such that $\bar{v}(T) = 0$. Moreover

$$||p||_{L^2(0,T)} \leq \Lambda_T ||v_0||$$

Since $\bar{v}(t) = e^{-tA}v_0 - \int_0^t e^{-(t-s)A}p(s)B\varphi_1 ds$,

$$\bar{v}(T) = 0 \iff \sum_{k\geq 1}\langle v_0, \varphi_k\rangle e^{-\lambda_k T}\varphi_k = \int_0^T p(s)\sum_{k\geq 1}\langle B\varphi_1, \varphi_k\rangle e^{-\lambda_k(T-s)}\varphi_k ds.$$

or

$$\int_0^T e^{\lambda_k s}p(s)ds = \frac{\langle v_0, \varphi_k\rangle}{\langle B\varphi_1, \varphi_k\rangle} \quad (k \geq 1)$$

The moment method

Thanks to the gap condition satisfied by the eigenvalues of A, there exists a biorthogonal family $\{\sigma_j\}_{j\geq 1}$ to t $\{e^{\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$. This means that

$$\forall k, j \geq 1, \quad \int_0^T \sigma_j(t) e^{\lambda_k t} dt = \delta_{jk}$$

Moreover

$$||\sigma_j||^2_{L^2(0,T)} \leq C^2_\gamma(T) e^{-2\lambda_j T} e^{C_\gamma \sqrt{\lambda_j}}, \quad \forall j \geq 1 \tag{P}$$

Therefore

$$p(s) = \sum_{k=1}^{\infty} \frac{\langle v_0, \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle} \sigma_k(s) \quad \Longrightarrow \quad \int_0^T e^{\lambda_k s} p(s) ds = \frac{\langle v_0, \varphi_k \rangle}{\langle B\varphi_1, \varphi_k \rangle}$$

(P) and $(\star)$ can be used to ensure that the above series converges in $L^2(0,T)$ and

$$||p||_{L^2(0,T)} \leq \Lambda_T ||v_0||$$

Proof of quadratic estimate for $v - \bar{v}$

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0 \\ v(0) = v_0 = u_0 - \varphi_1 \end{cases} \qquad \begin{cases} \bar{v}(t)' + A\bar{v}(t) + p(t)B\varphi_1 = 0 \\ \bar{v}(0) = v_0 \end{cases}$$

The difference $w(t) = v(t) - \bar{v}(t)$ satisfies

$$\begin{cases} w'(t) + Aw(t) + p(t)Bv(t) = 0, \quad t \in [0, T] \\ w(0) = 0 \end{cases}$$

So

$$\frac{1}{2}\frac{d}{dt}||w(t)||^2 \le |p(t)|||Bv(t)||||w(t)|| \le \frac{1}{2}||w(t)||^2 + C_B^2 \frac{1}{2}|p(t)|^2||v(t)||^2$$

Thus, by Gronwall's inequality and the previous estimate $||p||_{L^2(0,T)} \le \Lambda_T ||v_0||$, we obtain

$$\sup_{t\in[0,T]} ||w(t)||^2 \le C_B^2 e^T ||p||^2_{L^2(0,T)} \sup_{t\in[0,T]} ||v(t)||^2 \le K(T)^2 ||v_0||^4$$

Proof of stabilization ($\lambda_1 > 0$)

We introduce the operator

$$A_1 := \boldsymbol{A} - \boldsymbol{\lambda_1} I.$$

Notice that $A_1 : D(A_1) \subset X \to X$ is self-adjoint, accretive and $-A_1$ generates a strongly continuous semigroup of contractions. Its eigenvalues are given by

$$\mu_k = \boldsymbol{\lambda_k} - \boldsymbol{\lambda_1}, \qquad \forall k \in \mathbb{N}^*$$

(in particular, $\mu_1 = 0$) and A_1 has the same eigenfunctions as $\boldsymbol{A}$, $\{\boldsymbol{\varphi_k}\}_{k\geq 1}$. Moreover, the family $\{\mu_k\}_{k\geq 1}$ satisfies the same gap condition that is satisfied by the eigenvalues of $\boldsymbol{A}$

Proof of stabilization ($\lambda_1 > 0$)

We define $z(t) = e^{\lambda_1 t} u(t)$. Then

$$\begin{cases} z'(t) + A_1 z(t) + p(t) B z(t) = 0 & (t > 0) \\ z(0) = u_0. \end{cases}$$

So, we can apply the previous analysis to this problem:

$$\|z(t) - \varphi_1\| \leq M_T e^{-\rho e^{\omega_T t}} \qquad \forall t \geq 0$$

Therefore, returning to u,

$$\|u(t) - \psi_1(t)\| = \|e^{-\lambda_1 t} z(t) - e^{-\lambda_1 t} \varphi_1\| = e^{-\lambda_1 t} \|z(t) - \varphi_1\| \leq M_T e^{-(\rho e^{\omega_T t} + \lambda_1 t)} \quad \forall t \geq 0$$

j-null controllable pairs

Definition

Let $T > 0$. The pair $\{A, B\}$ is called j-null controllable in time T if there exists a constant $N(T) > 0$ such that for every $y_0 \in X$ one can find a control $p \in L^2(0, T)$ satisfying

$$||p||_{L^2(0,T)} \leq N(T)||y_0||,$$

and for which $y(T) = 0$, where $y(\cdot)$ is the solution of

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

The pair $\{A, B\}$ is called j-null controllable if there exists $T_0 > 0$ such that $\{A, B\}$ is j-null controllable in time T_0

Stabilizability to eigensolutions

Theorem

Let $\{A, B\}$ be a j-null controllable pair. Then, system

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0, & t > 0 \\ u(0) = u_0 \end{cases}$$

is superexponentially stabilizable to $\psi_j(t) = e^{-\lambda_j t}\varphi_j$, for any $j \geq 1$.

A sufficient conditions for j-null controllability

Theorem

Let $A : D(A) \subset X \to X$ be a densely defined linear operator satisfying modified $(A \geq -\sigma I)$ standing assumptions and such that there exists a constant $\alpha > 0$ for which its eigenvalues fulfill the gap condition

$$\sqrt{\lambda_{k+1} - \lambda_1} - \sqrt{\lambda_k - \lambda_1} \geq \alpha, \quad \forall\, k \in \mathbb{N}^*. \tag{15}$$

Let $B : X \to X$ be a bounded linear operator such that

$$\begin{aligned} &i) \quad \langle B\varphi_j, \varphi_k \rangle \neq 0, \quad \forall\, k \in \mathbb{N}^*, \\ &ii) \quad \exists\, \tau > 0 \,:\, \sum_{k \in \mathbb{N}^*} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_j, \varphi_k \rangle|^2} < +\infty. \end{aligned} \tag{16}$$

Then, the pair $\{A, B\}$ is j-null controllable.

A remark on the assumptions

The assumption

$$\langle B\varphi_j, \varphi_k\rangle \neq 0, \quad \forall\, k \in \mathbb{N}^* \tag{i}$$

is necessary for the null controllability of the linear system

$$\begin{cases} y'(t) + Ay(t) + p(t)B\varphi_j = 0, & t \in [0, T] \\ y(0) = y_0 \end{cases}$$

Indeed, the identity $y(T) = 0$ is equivalent to

$$\sum_{k\in\mathbb{N}^*} \langle y_0, \varphi_k\rangle e^{-\lambda_k T}\varphi_k = \int_0^T p(s) \sum_{k\in\mathbb{N}^*} \langle B\varphi_j, \varphi_k\rangle e^{-\lambda_k(T-s)}\varphi_k ds.$$

Since the family $\{\varphi_k\}_{k\in\mathbb{N}^*}$ forms an othonormal basis of X, it holds that

$$\langle y_0, \varphi_k\rangle = \int_0^T e^{\lambda_k s} p(s)\langle B\varphi_j, \varphi_k\rangle ds, \quad \forall\, k \in \mathbb{N}^*.$$

In particular, if (i) is violated, there exists some $\bar{k} \in \mathbb{N}^*$ such that $\langle B\varphi_j, \varphi_{\bar{k}}\rangle = 0$. Hence, in the $\bar{k}$-th direction we have that

$$\langle y_0, \varphi_{\bar{k}}\rangle = \int_0^T e^{\lambda_{\bar{k}} s} p(s)\langle B\varphi_j, \varphi_{\bar{k}}\rangle ds,$$

which yields $y_0 \in \varphi_{\bar{k}}^{\perp}$.

Exact controllability to eigensolutions

Exact controllability to the ground state solution

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases} \qquad (S)$$

Theorem

Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of A fulfill the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

Let $B : X \to X$ be a linear bounded operator satisfying the following

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then for any $T > 0$ there exists $R_T > 0$ such that, for any $u_0 \in B_{R_T}(\varphi_1)$, the solution to (S) can be steered to the ground state solution in time T by some control $p \in L^2(0, T)$

Notice that $(\star\star)$ is satisfied by all the above examples. Moreover

$$(\star\star) \qquad \Longrightarrow \qquad \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k \tau}}{|\langle B\varphi_1, \varphi_k \rangle|^2} < \infty \quad \forall \tau > 0$$

Idea of the proof ($\lambda_1 = 0$)

For fixed $0 < T$, we construct of the solution v of

$$\begin{cases} v'(t) + Av(t) + p(t)Bv(t) + p(t)B\varphi_1 = 0, \quad t \in [0, \frac{\pi^2}{6}T] \\ v(0) = v_0 = u_0 - \varphi_1. \end{cases} \tag{17}$$

in consecutive intervals of the form $[\tau_n, \tau_{n+1}]$ where

$$T_j := T/j^2 \quad \text{and} \quad \tau_n = \sum_{j=1}^{n} T_j, \qquad \forall n \in \mathbb{N}$$

By proving that

$$||v(\tau_n)|| \leq \left(e^{6C_K/T}||v_0||\right)^{2^n}$$

we obtain the conclusion as $n \to \infty$ since $\sum_{j=1}^{\infty} T_j = \frac{\pi^2}{6}T$

Global exact controllability on a strip

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 \quad (t > 0) \\ u(0) = u_0 \end{cases} \qquad (S)$$

Theorem

Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of A fulfill the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

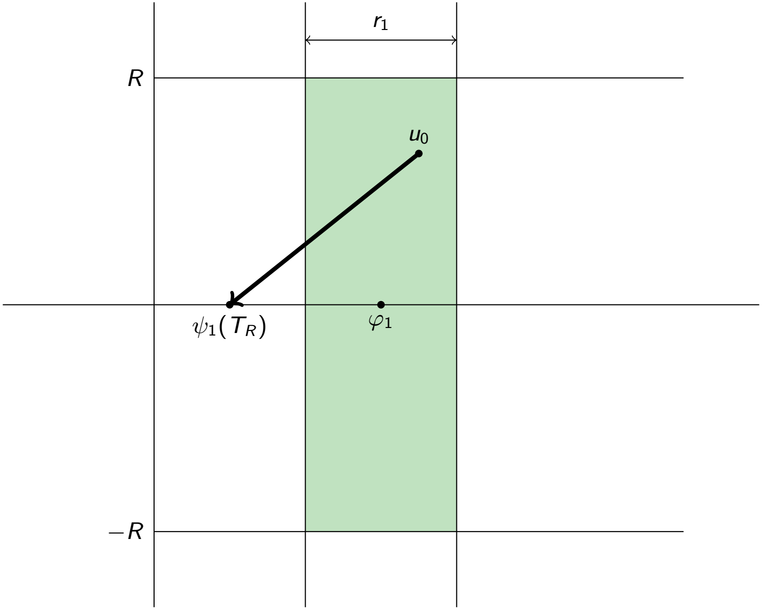
Let $B : X \to X$ be a linear bounded operator satisfying the following

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then there exists $r_1 > 0$ such that for all $R > 0$ there exists $T_R > 0$ such that for all $u_0 \in X$ in the strip

$$\begin{aligned} &|\langle u_0, \varphi_1 \rangle - 1| \leq r_1 \\ &||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R \end{aligned}$$

the solution to (S) can be steered to the ground state solution $\psi_1(t) = e^{-\lambda_1 t}\varphi_1$ in time T_R by some control $p \in L^2(0, T_R)$


r_1
R
u_0
$\psi_1(T_R)$
φ_1
$-R$

Global exact controllability outside $\varphi_1^{\perp}$

$$\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases} \qquad (S)$$

Corollary

Suppose there exists a constant $\gamma > 0$ *such that the eigenvalues of* A *fulfill the gap condition*

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

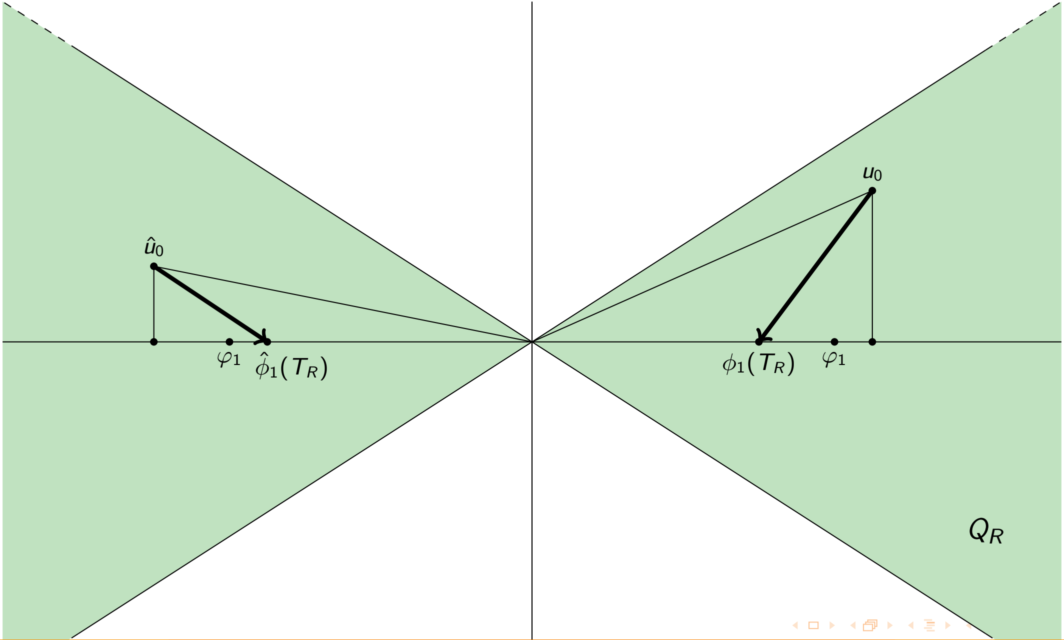
Let $B : X \to X$ *be a linear bounded operator satisfying the following*

$$\langle B\varphi_1, \varphi_1 \rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k \rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then for every $R > 0$ *there exists* $T_R > 0$ *such that for all* u_0 *satisfying*

$$||u_0 - \langle u_0, \varphi_1 \rangle \varphi_1|| \leq R |\langle u_0, \varphi_1 \rangle|$$

the solution to (S) *can be steered to* $\langle u_0, \varphi_1 \rangle \psi_1 =: \phi_1$ *in time* T_R *by some control* $p \in L^2(0, T_R)$



$\hat{u}_0$
φ_1
$\hat{\phi}_1(T_R)$
u_0
$\phi_1(T_R)$
φ_1
Q_R

Unbounded control operators

We allow for $B \notin \mathcal{L}(X)$ in $\begin{cases} u'(t) + Au(t) + p(t)Bu(t) = 0 & (t > 0) \\ u(0) = u_0 \end{cases}$ $\quad (S)$

Theorem

Suppose there exists a constant $\gamma > 0$ such that the eigenvalues of A fulfill the gap condition

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma \qquad \forall k \geq 1$$

Let $B : D(B) \subset X \to X$ be a linear operator such that $D(A^{1/2}) \subset D(B)$,

$$\|B\varphi\| \leq C\|\varphi\|_{D(A^{1/2})} \qquad \forall \varphi \in D(A^{1/2})$$

and

$$\langle B\varphi_1, \varphi_1\rangle \neq 0 \qquad \& \qquad \exists\, b, q > 0 \quad \text{such that} \quad \lambda_k^q |\langle B\varphi_1, \varphi_k\rangle| \geq b \quad \forall k > 1 \qquad (\star\star)$$

Then for any $T > 0$ there exists $R_T > 0$ such that, $\forall u_0 \in D(A^{1/2})$ satisfying $\|A^{1/2}(u_0 - \varphi_1)\| < R_T$, the solution to (S) can be steered to the ground state solution in time T by some control $p \in L^2(0, T)$

Applications

Example 1 (heat eqn, Dirichlet bc)

Consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\mu(x)u(t,x) = 0, & (t,x) \in (0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Example 1

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x) \end{cases} \tag{10}$$

where A and B are defined by

$$\begin{array}{ll} D(A) = H^2 \cap H_0^1(0,1), & A\varphi = -\frac{d^2\varphi}{dx^2} \\ B \in \mathcal{L}(X), & B\varphi = \mu\varphi \end{array} \tag{11}$$

- A is a **self-adjoint accretive** operator and $-A$ generates an **analytic** semigroup.
- eigenvalues and eigenvectors of A:

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2}\sin(k\pi x) \quad \forall k \geq 1$$

We want to study the superexponential stabilizability of (10)-(11) to the ground state solution $\psi_1 = e^{-\lambda_1 t}\varphi_1$

Example 1

- gap condition:

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$$

- estimate of the Fourier coefficients:

$$\begin{aligned}\langle B\varphi_1, \varphi_k\rangle &= \int_0^1 2\mu(x)\sin(\pi x)\sin(k\pi x)dx \\ &= \frac{4}{k^3}\Big((-1)^{k+1}\mu'(1) - \mu'(0)\Big) + \\ &\quad - \frac{\sqrt{2}}{(k\pi)^3}\int_0^1 (\mu(x)\varphi_1(x))''' \cos(k\pi x)dx\end{aligned}$$

If $\langle B\varphi_1, \varphi_k\rangle \neq 0 \ \forall k \in \mathbb{N}^*$ and $\mu'(1) \pm \mu'(0) \neq 0$, then we have

$$|\langle B\varphi_1, \varphi_k\rangle| \geq C\lambda_k^{-3/2}, \quad \forall k \in \mathbb{N}^*.$$

EXAMPLE: $\boxed{B\varphi(x) = x^2\varphi(x)}$

Example 1: conclusion

Therefore, our stabilizability and controllability results apply to

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)x^2u(t,x) = 0 & (t,x) \in (0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Example 2 (degenerate heat eqn)

Let $0 \leq \alpha < 2$ and consider the bilinear control system

$$\begin{cases} u_t - (x^\alpha u_x)_x + p(t)\mu(x)u = 0, & (t,x) \in (0,\infty) \times (0,1) \\ u(t,1) = 0, \quad \begin{cases} u(t,0) = 0, & \text{if } \alpha \in [0,1), \\ (x^\alpha u_x)(t,0) = 0, & \text{if } \alpha \in [1,2) \end{cases} \\ u(0,x) = u_0(x) \end{cases}$$

Example 2: weakly degenerate case ($\alpha < 1$)

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x). \end{cases}$$

where A and B are defined by

$$\begin{array}{ll} D(A) = \left\{ u \in H^1_{\alpha,0}(0,1) \ : \ x^\alpha \frac{du}{dx} \in H^1(0,1) \right\}, & A\varphi = -\frac{d}{dx}(x^\alpha \frac{du}{dx}) \\ B \in \mathcal{L}(X), & B\varphi = \mu\varphi \end{array}$$

where

$$H^1_{\alpha,0}(0,1) = \left\{ u \in L^2(0,1) \ : \ u \in AC([0,1]),\ x^{\alpha/2}\frac{du}{dx} \in L^2(0,1),\ u(0) = 0,\ u(1) = 0 \right\}$$

Then

- A is a **self-adjoint accretive** operator and $-A$ generates a C^0-semigroup of contractions

Example 2: gap condition ($\alpha < 1$)

For any $\nu \geq 0$, we denote by

- J_ν the Bessel function of the first kind and order ν
- $j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,k} < \ldots$ the sequence of all positive zeros of J_ν

Set

$$\nu_\alpha := \frac{1-\alpha}{2-\alpha}, \qquad k_\alpha := \frac{2-\alpha}{2}.$$

Then eigenvalues and eigenvectors of A are given by :

$$\lambda_{\alpha,k} = k_\alpha^2 j_{\nu_\alpha,k}^2 \qquad (k \geq 1)$$

$$\varphi_{\alpha,k}(x) = \frac{\sqrt{2k_\alpha}}{|J'_{\nu_\alpha}(j_{\nu_\alpha,k})|} x^{(1-\alpha)/2} J_{\nu_\alpha}\left(j_{\nu_\alpha,k} x^{k_\alpha}\right) \qquad (k \geq 1)$$

So, the gap condition is satisfied :

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = k_\alpha \left(j_{\nu_\alpha,k+1} - j_{\nu_\alpha,k}\right) \geq k_\alpha \left(j_{\nu_\alpha,2} - j_{\nu_\alpha,1}\right) \geq \frac{7}{16}\pi$$

Example 2: conclusion

- Taking $\boxed{\mu(x) = x^{2-\alpha}}$ a long computation shows that

$$|\langle \mu\varphi_1, \varphi_k \rangle| \geq \frac{C}{\lambda_k^{3/2}} \quad \forall k \geq 1$$

  for some $C > 0$

Therefore, our stabilizability and controllability results apply to

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)x^{2-\alpha}u(t,x) = 0, \quad (t,x) \in (0,\infty) \times (0,1) \\ u_x(t,0) = u_x(t,1) \\ u(0,x) = u_0(x) \end{cases}$$

Example 3 (Fokker-Planck equation with Dirichlet b.c.)

Consider the bilinear control system

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\big(\mu(x)u(t,x)\big)_x = 0, & (t,x) \in (0,\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Example 3

Let $X = L^2(0,1)$ and consider the bilinear control system

$$\begin{cases} u_t(t) + Au(t) + p(t)Bu(t) = 0, & t \in (0,\infty) \\ u(0) = u_0(x) \end{cases} \tag{10}$$

where A and B are defined by

$$\begin{aligned} D(A) &= H^2 \cap H_0^1(0,1), & A\varphi &= -\frac{d^2\varphi}{dx^2} \\ D(B) &= \left\{\varphi \in X \ : \ \frac{d}{dx}(\mu\varphi) \in X\right\}, & B\varphi &= \frac{d}{dx}(\mu\varphi) \end{aligned} \tag{11}$$

Observe that $D(A^{1/2}) = H_0^1(0,1) \subset D(B)$ if $\mu \in C^1([0,1])$ and

- the eigenvalues and eigenvectors of A satisfy the gap condition:

$$\lambda_k = (k\pi)^2, \quad \varphi_k(x) = \sqrt{2}\sin(k\pi x) \quad \forall k \geq 1$$

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = (k+1)\pi - k\pi = \pi \qquad \forall k \geq 1$$

- $\|B\varphi\| \leq \left(\|\mu\|_\infty^2 + \|\mu'\|_\infty^2\right)^{1/2} \|\varphi\|_{D(A^{1/2})}$ for any $\varphi \in D(A^{1/2})$

Example 3

- estimate of the Fourier coefficients:

$$\begin{aligned}\langle B\varphi_1, \varphi_k\rangle &= \sqrt{2}\int_0^1 (\mu\varphi_1)'(x)\sin(k\pi x)dx \\ &= \frac{2}{k}\Big((-1)^k\mu(1) + \mu(0)\Big) \\ &\quad + \frac{\sqrt{2}}{k\pi}\int_0^1 (\mu\varphi_1)''(x)\cos(k\pi x)dx\end{aligned}$$

If $\langle B\varphi_1, \varphi_k\rangle \neq 0\ \forall k \in \mathbb{N}^*$ and $\mu(1) \pm \mu(0) \neq 0$, then we have

$$|\langle B\varphi_1, \varphi_k\rangle| \geq C\lambda_k^{-1/2}, \quad \forall k \geq 1$$

EXAMPLE: $\boxed{B\varphi(x) = \frac{d}{dx}(x^n\varphi(x))}$ for any $n \geq 1$

Example 3: control of Fokker-Planck equation

Our abstract result guarantees that, for any $n \geq 1$ and $T > 0$,

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) + p(t)\big(x^n u(t,x)\big)_x = 0, & (t,x) \in (0,T) \times (0,1) \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

is locally exactly controllable to $\psi_1(T) = e^{-\lambda_1 T}\varphi_1$ by some control p
Consequently, the density of the stochastic process

$$dX_t = p(t) X_t^n dt + \sigma dW_t$$

can be locally controlled to $\psi_1(T)$ for any $T > 0$

Thank you