Propagation of smallness and control for heat eq

Joint with. I. Moyano.

I Classical question $M$, $\partial M$,

$$\Delta = \mathrm{div}_\varkappa \nabla_g = \frac{1}{\varkappa} \sum_{ij} \frac{\partial}{\partial x_i} g^{ij}(x) \varkappa(x) \frac{\partial}{\partial x_j}$$

$\varkappa(x) > c > 0$, $g(x) > c\, Id$

$$\partial_t u - \Delta u = f 1_G, \quad \mathbb{R}^+ \times M.$$

$$\begin{cases} u|_{\partial M} = 0 \\ \text{or} \\ \partial_\nu u|_{\partial M} = 0 \end{cases} \qquad u|_{t=0} = u_0 \in L^2(M)$$

$$G \subset [0,T] \times M$$

Q: $u_0$ given: $\exists f$ : $u|_{t>T} \equiv 0$ ?

A Yes when:

i) $G = (0,T) \times \omega$, $\omega$ open $\subset M$.

(Lebeau Robbiano / Fursikov Imanuvilov 95') Carleman estimates:

$\alpha, g, C^\infty \to$ Lip. ($\partial M = \phi$)

ii) $G = F_t \times \omega_x$, $|F| > 0$ $F \subset [0,T]$

$\omega_x$ open.

Phung Wang 13'

iii) $|G| > 0$ in $[0,T] \times M$.

Apraiz Escauriaza Wang Zhang.

$\Delta = \Delta$ euclidean complex analysis ingredients!

Th (N. B. I. Moyano 19')

Yes $g, \alpha \in \text{Lip}$. $\boxed{|G| > 0}$

$M \in W^{2,\infty}$

Difficult part = Logunov Malliniakova

$$G = \bigcup_t \{t\} \times \tilde{G}_t .$$

$$|G_t|_x \geq c > 0$$

## II Spectral estimate.

$$-\Delta e_n = \lambda_n^2 e_n$$

$$\Pi_\lambda u = 1_{\sqrt{-\Delta} \leq \lambda} u = \sum_{\lambda_n \leq \lambda} u_n e_n .$$

$$u = \sum_n u_n e_n$$

Th: $\exists C^{|E|>0}$ ; $\forall \lambda$ $\forall u \in L^2$

$$\|\Pi_\lambda u\|_{L^2(M)} \leq C e^{C\lambda} \|\Pi_\lambda u\|_{\underline{L^2(E)}}$$

$$C(|E|)$$

$L^\infty(E)$

Q: $|E| = 0$ ?

Th OK if .

i) ~~w~~ E open . (Lebeau Jerison . ) 95-99'
Lebeau Robbiano ) .
(Carleman Inequalities) . real
$g, x \in C^\infty$ ( Lip ) analysis .

ii) $\Delta = \Delta_{end}$ $|E| > 0$ complex analysis . 60'
Zygmund 30', Nazarov Veselic 70' .

Th 2019' (N. B, I. Moyano) . real analysis .
$|E| > 0$ . $x, g \in$ lips . $+ \partial M$, Dor N .

Key point to prove null control for heat .

Th : $\|\Pi_\Lambda u\|_{L^\infty(M)} \leq C e^{C\Lambda} \|\Pi_\Lambda u\|_{L^\infty(E)}$.

$\mathcal{H}^{d-\delta}(E) > 0$.

$\exists \delta > 0$ ; if $\exists C$ ; $\forall u, \Lambda$

$|E| > 0$, $\mathcal{H}^{d-\delta}(E) > 0$

Th (N.B (M) $t_n \to T$ not too fast

$t_{n+1} - T > c(t_n - T)$ , $c > 0$

$\mathcal{H}^{d-\delta}(E) > 0$, $\overline{E}_n = E_n$, $\mu_n$ measures supported on $E_n$

$(\partial_t + \Delta) u = \sum_n \delta_{t = t_n} \otimes \mu_n$ $\subset H^{-N}(M)$

$t \to t_n$ $u(t_n + 0) = u(t_n - 0) + \mu_n$

$\forall u_0 \in L^2$, $\exists \mu_n$ ; s.t solution

$u|_{t>T} \equiv 0$

$$f = \sum_n \delta_{t=t_n} \otimes \mu_n .$$

supported $\bigcup_n \{t_n\} \times E = \sigma$.

$\dim(\sigma) = 0_t + d - \delta_x < d$.

usual results:

$\dim(\sigma) = \dim((0,T) \times \omega) = d+1$.

$\boxed{\delta + 1}$

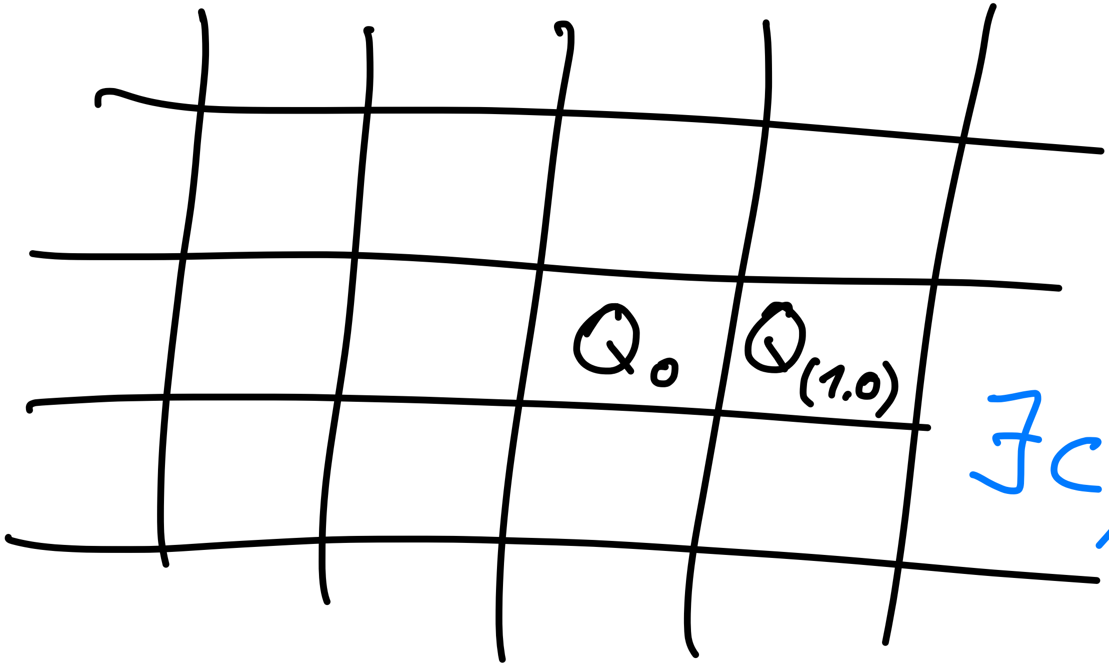
Q: What happens $\mathbb{R}^d$ ?

$M \to \mathbb{R}^d$

$$\Pi_n = 1_{\sqrt{-\Delta} < n}$$

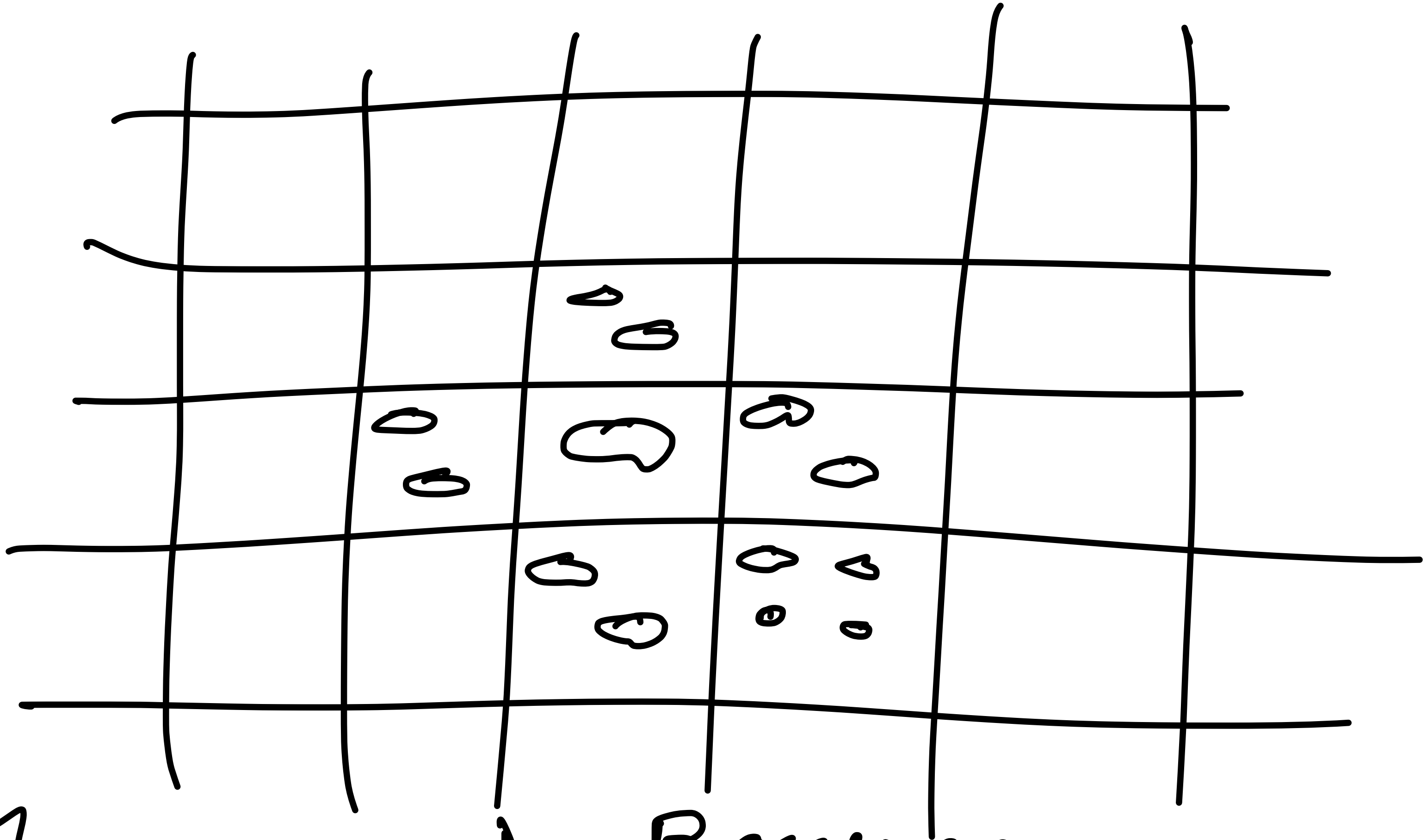
$$\|\Pi_n u\|_{L^2(\mathbb{R}^d)} \leq C e^{Cn} \|\Pi_n u\|_{L^2(\omega)}$$

A: $\mathbb{R}^d = \bigcup_{p \in \mathbb{Z}^d} Q_p$ , $Q_p = \prod_{j=1}^{d} [p_j, p_{j+1}[$




$$\boxed{|\omega \cap Q_p| \geq c > 0.}$$

$\exists c; \forall p$



Moyano le Rousseau

Th. A. spectral estimate true.

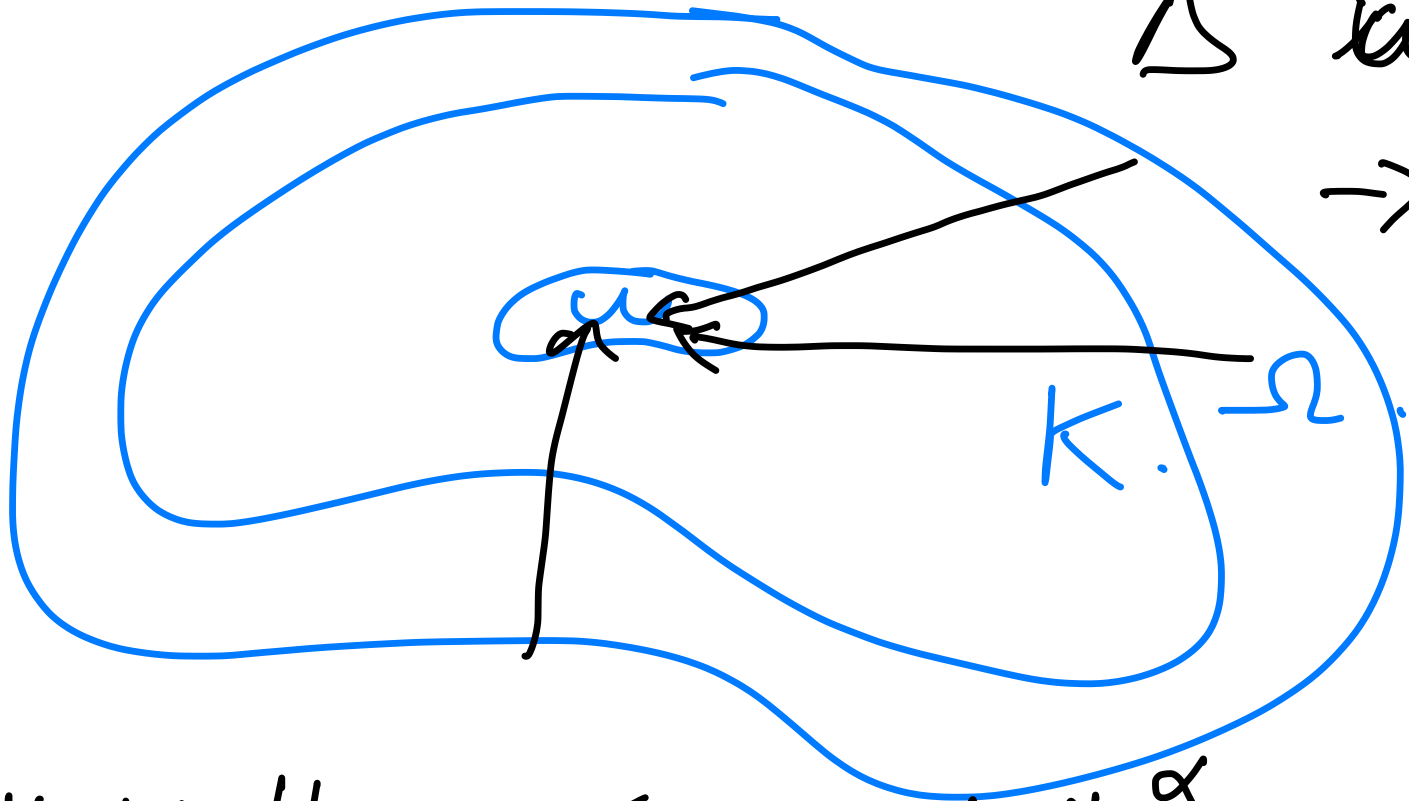
i) $\Delta = \Delta_{eucl}$.

ii) 19' Lebeau Moyano ; Carleman $\bar{\partial}$

$\Delta_g$ , $g = Id + \tilde{g}$ , $\tilde{g}$ real analytic + small at $\infty$ :

Th : OK. $g, x$ Lips. unif$^{ly}$ on $\mathbb{R}^d$.

ass A: drop "small perturbation assumption"




$\Delta\, u(x) = 0.$

$\to U(\varepsilon x) = U_\varepsilon(x)$

$\Delta_\varepsilon U_\varepsilon = 0$

$$\|U\|_{L^\infty(K)} \lesssim C\, \|u\|^{\alpha}_{L^\infty(\omega)}\, \|u\|^{1-\alpha}_{L^\infty(\Omega)}.$$

# III Logunov Malinnikova Propagation of smallness. Harmonic functions

$\Delta = \operatorname{div} g \nabla$, $g \in \mathrm{Lip}$.

$\Delta U = 0$. $\Omega \subset \mathbb{R}^n$

$\overline{K} \subset \Omega$, $G \subset \Omega$.

$|G| > 0$, $\boxed{\mathcal{H}^{d-\delta}(G) > 0 \ \exists\, \delta > 0}$

Then $$\|U\|_{L^2(K)} \le C \|U\|^{\alpha}_{L^2(G)} \|U\|^{1-\alpha}_{L^2(\Omega)}$$

$$L^\infty(K) \qquad\qquad L^\infty(G) \qquad\qquad L^\infty(\Omega)$$

$\alpha > 0$


$+\ U \to \nabla U$. $\mathcal{H}^{d-\delta-1}(G) > 0$.

$$\boxed{\Pi_\Lambda u} \to U = \frac{\operatorname{sh}(\sqrt{-\Delta}\, t)}{\sqrt{-\Delta}} \Pi_\Lambda u$$

$$u = \sum_n u_n e_n \qquad -\Delta_g e_n = \frac{1}{\varkappa} \partial_i g^{ij} \varkappa \partial_j e_n$$

$$= \lambda_n^2 e_n$$

$$\Pi_\Lambda u = \sum_{\lambda_n \leq \Lambda} u_n e_n(x)$$

$$U(t,x) = \sum_{\lambda_n \leq \Lambda} \frac{\operatorname{sh}(\lambda_n t)}{\lambda_n} u_n e_n(x)$$

$e^{\lambda}$

$$\varkappa \partial_t^2 U + \partial_i g^{ij} \varkappa \partial_j U = 0$$

$$\partial_t \varkappa(x) \partial_t + \underline{\qquad\qquad} U = 0 = \Delta_{t,x} U$$

$$\|\nabla u\|_{L^\infty(K)} \leq C \|\nabla \cdot u\|^{\alpha}_{L^\infty(G)} \, \|u\|^{1-\alpha}_{L^\infty(\Omega)}$$

$$\|\Pi_n u\|_{L^\infty(B(x_0,1))} \qquad \|\Pi_n u\|^{\alpha}_{L^\infty(E)}$$

$$\Omega = [-1, 1] \times M$$

$$K = \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times B(x_0, 1)$$

$$G = \{0\}_t \times E_x$$

$$\nabla = \nabla_{t,x}, \quad \partial_t u \big|_{t=0} = \Pi_n u .$$

$$\|\nabla u\|_{L^\infty((-1,1)\times B(x_0,1))} \geq \|\partial_t u\|_{L^\infty(\{0\}\times B)}$$

$$= \|\Pi_n u\|_{L^\infty(B(x_0,1))}$$

$$\|u\|_{L^\infty(-1,1\times M)} \lesssim \Lambda^N C e^{C\Lambda} \|\Pi_\Lambda u\|_{L^2} \lesssim C e^{C'\Lambda} \|\Pi_\Lambda u\|_{L^2(M)}.$$

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$$\|\Pi_\Lambda u\|_{L^\infty(B(x_0,1))} \lesssim C e^{C\Lambda} \|\Pi_\Lambda u\|^{\alpha}_{L^\infty(E)} \times \|\Pi_\Lambda u\|^{1-\alpha}_{L^2(M)}.$$

$$\gtrsim \sqrt{\Lambda}$$

$$\| \quad \|_{L^2(\ )}.$$

$$a^\alpha b^{1-\alpha} \lesssim C_\varepsilon a + \varepsilon b.$$

$\partial M = \emptyset$ tō $\partial M \neq \emptyset$