The block moment method in action

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Outline

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The system we are interested in

- Let $T > 0$, $d > 0$ be given.
- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\gamma \subset \partial\Omega$.

Is the following system null-controllable at time T ?

$$\begin{cases} \partial_t y + \begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T)\times\Omega, \\ y = 1_\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

Difficulties

- **Two components** in the state y but only **one control** v.
- This is a **boundary control** system with a control **localised** on γ.
- Geometry :
  - In 1D : the problem is now well understood
  - In higher dimension : very few results in general
- Diffusion ratio d :
  - Case $d = 1$: The analysis is now quite "standard"
  - Case $d \neq 1$: The analysis is much more intricate

Aim of the talk : state a result concerning the case $d \neq 1$ in a 2D cartesian geometry

Remarks

$$\begin{cases} \partial_t y + \begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T)\times\Omega, \\ y = 1_\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

Which tool can we use ?

- ✖ Lebeau-Robbiano type strategy
- ✖ Global Carleman inequalities
- ✔ Moment method

Generalizations

- $-\Delta$ can be replaced by another elliptic operator A (with some care ...)
- The coupling matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ can be replaced by any coupling matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ as soon as the Kalman condition holds

$$m_{21} \neq 0.$$

Outline

2. The 1D case

2.1. Diffusion ratio $d = 1$

The 1D case - $\Omega = (0,1)$

Diffusion ratio $d = 1$

(Fernández–Cara - González–Burgos - de Teresa, '10) (Ammar–Khodja - Benabdallah - González–Burgos - de Teresa, '11)

Theorem

The following system

$$(S) \quad \begin{cases} \partial_t y + \begin{pmatrix} -\partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T)\times(0,1), \\ y = 1_{x=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T)\times\{0,1\}. \end{cases}$$

is null-controllable at any time $T > 0$.

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is null-controllable at any time $T > 0$.

Generalization : for any coupling matrix M, the same result holds if and only if we have **both**

- Kalman condition :

$$m_{21} \neq 0$$

- Non resonance condition : for any $\theta, \tilde{\theta} \in \sigma(M)$, $\lambda, \tilde{\lambda} \in \sigma(-\partial_x^2)$

$$(\theta + \lambda = \tilde{\theta} + \tilde{\lambda}) \implies (\theta = \tilde{\theta} \text{ and } \lambda = \tilde{\lambda}).$$

The 1D case - $\Omega = (0,1)$

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Theorem

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \textit{and} \quad \mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B, \quad \textit{with } B^\star = -(\partial_x)_{|x=0}.$$

$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

System (S) is null-controllable at any time $T > 0$.

Slight change of formalism : We identify $L^2(0,1;\mathbb{C}^2)$ with the tensor product $\mathbb{C}^2 \otimes L^2(0,1)$.

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Sketch of proof : Let $\Lambda := \sigma(-\partial_x^2)$ and $(\phi_\lambda)_{\lambda \in \Lambda}$ the associated eigenfunctions

Set $\Phi_\lambda^0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_\lambda$ and $\Phi_\lambda^1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \phi_\lambda$.

$$\mathcal{A}^\star \Phi_\lambda^0 = \lambda \Phi_\lambda^0, \qquad \mathcal{B}^\star \Phi_\lambda^0 = B^\star \phi_\lambda = -\phi_\lambda'(0) \sim C\sqrt{\lambda},$$

$$\mathcal{A}^\star \Phi_\lambda^1 = \lambda \Phi_\lambda^1 + \Phi_\lambda^0, \qquad \mathcal{B}^\star \Phi_\lambda^1 = 0.$$

Equivalent moment problem

$$\begin{cases} (\mathcal{B}^\star \Phi_\lambda^0) \displaystyle\int_0^T e^{-\lambda(T-t)} v(t)\, dt = -e^{-\lambda T} \langle y_0, \Phi_\lambda^0 \rangle, & \forall \lambda \in \Lambda, \\ (\mathcal{B}^\star \Phi_\lambda^0) \displaystyle\int_0^T (T-t) e^{-\lambda(T-t)} v(t)\, dt = -e^{-\lambda T} \langle y_0, \Phi_\lambda^1 - T\Phi_\lambda^0 \rangle, & \forall \lambda \in \Lambda. \end{cases}$$

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Equivalent moment problem

we set $u(t) := v(T-t)$

$$\begin{cases} (\mathcal{B}^\star \Phi_\lambda^0) \displaystyle\int_0^T e^{-\lambda t} u(t)\,dt = -e^{-\lambda T}\langle y_0, \Phi_\lambda^0\rangle, & \forall \lambda \in \Lambda, \\ (\mathcal{B}^\star \Phi_\lambda^0) \displaystyle\int_0^T t e^{-\lambda t} u(t)\,dt = -e^{-\lambda T}\langle y_0, \Phi_\lambda^1 - T\Phi_\lambda^0\rangle, & \forall \lambda \in \Lambda. \end{cases}$$

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Sketch of proof :

Equivalent moment problem

we set $u(t) := v(T-t)$

$$\begin{cases} \int_0^T e^{-\lambda t} u(t)\,dt = -e^{-\lambda T} \langle y_0, \Psi_\lambda^0 \rangle, & \forall \lambda \in \Lambda, \\ \int_0^T t e^{-\lambda t} u(t)\,dt = -e^{-\lambda T} \langle y_0, \Psi_\lambda^1 - T\Psi_\lambda^0 \rangle, & \forall \lambda \in \Lambda, \end{cases}$$

with

$$\Psi_\lambda^j := \frac{\Phi_\lambda^j}{\mathcal{B}^\star \Phi_\lambda^0}.$$

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$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

System (S) is null-controllable at any time $T > 0$.

Sketch of proof :

Equivalent moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} u(t)\, dt = \omega_\lambda^0, & \forall \lambda \in \Lambda, \\ \displaystyle\int_0^T t e^{-\lambda t} u(t)\, dt = \omega_\lambda^1, & \forall \lambda \in \Lambda \end{cases}$$

with

$$|\omega_\lambda^j| \leqslant C e^{-\lambda T}, \quad \forall \lambda \in \Lambda, \forall j \in \{0,1\}.$$

Solution : $\quad u(t) = \sum_{\lambda \in \Lambda} (\omega_\lambda^0 q_\lambda^0(t) + \omega_\lambda^1 q_\lambda^1(t)),$

where $(q_\lambda^j)_{\lambda,j} \subset L^2(0,T)$ satisfies :

$$\int_0^T t^i e^{-\mu t} q_\lambda^j(t)\, dt = \delta_{\lambda,\mu}\delta_{i,j}.$$

Such a family exists and satisfies

$$\|q_\lambda^j\|_{L^2(0,T)} \leqslant C_T e^{\varepsilon(\lambda)\lambda}, \quad \lim_{\lambda \to +\infty} \varepsilon(\lambda) = 0$$

The 1D case - $\Omega = (0,1)$

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$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

System (S) is null-controllable at any time $T > 0$.

Sketch of proof :

Equivalent moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} u(t)\,dt = \omega_\lambda^0, & \forall \lambda \in \Lambda, \\ \displaystyle\int_0^T t e^{-\lambda t} u(t)\,dt = \omega_\lambda^1, & \forall \lambda \in \Lambda \end{cases}$$

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Such a family exists and satisfies

$$\|q_\lambda^j\|_{L^2(0,T)} \leqslant C_T e^{\varepsilon(\lambda)\lambda}, \quad \lim_{\lambda \to +\infty} \varepsilon(\lambda) = 0$$

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty$$

$$\inf_{\lambda \neq \mu \in \Lambda} |\lambda - \mu| > 0$$

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$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

System (S) is null-controllable at any time $T > 0$. Moreover, $\|v\|_{L^2(0,T)} \leqslant e^{\frac{C}{T}} \|y_0\|$.

Sketch of proof :

Equivalent moment problem

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda t} u(t)\, dt = \omega_\lambda^0, & \forall \lambda \in \Lambda, \\ \displaystyle\int_0^T t e^{-\lambda t} u(t)\, dt = \omega_\lambda^1, & \forall \lambda \in \Lambda \end{cases}$$

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$$|\omega_\lambda^j| \leqslant C e^{-\lambda T}, \quad \forall \lambda \in \Lambda, \forall j \in \{0,1\}.$$

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$$\int_0^T t^i e^{-\mu t} q_\lambda^j(t)\, dt = \delta_{\lambda,\mu} \delta_{i,j}.$$

Such a family exists and satisfies

$$\|q_\lambda^j\|_{L^2(0,T)} \leqslant C e^{\frac{C}{T} + C\sqrt{\lambda}}.$$

2. The 1D case

2.2. Diffusion ratio $d \neq 1$

The 1D case - $\Omega = (0,1)$

Diffusion ratio $d \neq 1$

(Ammar–Khodja - Benabdallah - González–Burgos - de Teresa, '14)

Theorem

$$\mathcal{A} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \textit{and} \quad \mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B, \quad \textit{with } B^\star = -(\partial_x)_{|x=0}.$$

$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

There exists a **minimal null-control time** $T_0(d) \in [0, +\infty]$ *:*

System (S) is NC at any time $T > T_0(d)$ *and is not NC at any time* $T < T_0(d)$*.*

Remarks :

- There is an **infinite countable** set of values of d for which the system is even not approximately controllable.
- For any $\tau \in [0, +\infty]$ there exist (many) values of d such that

$$T_0(d) = \tau.$$

- The same system with distributed control is null-controllable at any time $T > 0$.

The 1D case - $\Omega = (0,1)$

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Spectral analysis

- For any $\lambda \in \Lambda$ we set

$$\Phi_{d\lambda} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_\lambda \quad \Rightarrow \quad \mathcal{A}^\star \Phi_{d\lambda} = d\lambda \Phi_{d\lambda} \quad \text{and} \quad \mathcal{B}^\star \Phi_{d\lambda} = -\phi'_\lambda(0) \neq 0.$$

- For any $\mu \in \Lambda$ we set

$$\widetilde{\Phi}_\mu := \begin{pmatrix} \frac{1}{\mu(1-d)} \\ 1 \end{pmatrix} \otimes \phi_\mu \quad \Rightarrow \quad \mathcal{A}^\star \widetilde{\Phi}_\mu = \mu \widetilde{\Phi}_\mu \quad \text{and} \quad \mathcal{B}^\star \widetilde{\Phi}_\mu = -\frac{\phi'_\mu(0)}{\mu(1-d)} \neq 0.$$

Approximate controllability $\iff \Lambda \cap d\Lambda = \emptyset$.

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Necessary null-controllability condition

We assume $\Lambda \cap d\Lambda = \varnothing$

$$\Phi_{d\lambda} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_\lambda, \qquad \Phi_\mu = \begin{pmatrix} \frac{1}{\mu(1-d)} \\ 1 \end{pmatrix} \otimes \phi_\mu, \qquad \Psi_\bullet = \frac{\Phi_\bullet}{\mathcal{B}^\star \Phi_\bullet}$$

If a control v exists with $\|v\| \leqslant C_T \|y_0\|$, we have

Recall that $u(\cdot) = v(T - \cdot)$

$$\int_0^T e^{-d\lambda t} u(t)\, dt = -e^{-d\lambda T} \langle y_0, \Psi_{d\lambda} \rangle,$$

$$\int_0^T e^{-\mu t} u(t)\, dt = -e^{-\mu T} \langle y_0, \Psi_\mu \rangle.$$

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If a control v exists with $\|v\| \leqslant C_T \|y_0\|$, we have

Recall that $u(\cdot) = v(T - \cdot)$

$$\int_0^T \left(e^{d\lambda T} \frac{e^{-d\lambda t} - e^{-\mu t}}{d\lambda - \mu} + \frac{e^{d\lambda T} - e^{\mu T}}{d\lambda - \mu} e^{-\mu t} \right) u(t)\, dt = -\left\langle y_0, \frac{\Psi_{d\lambda} - \Psi_\mu}{d\lambda - \mu} \right\rangle.$$

$$\implies \left\| \frac{\Psi_{d\lambda} - \Psi_\mu}{d\lambda - \mu} \right\| \leqslant C_T e^{\mu T} \implies \frac{e^{-\mu T}}{|d\lambda - \mu|} \leqslant \tilde{C}_T \implies T \geqslant T_0(d) := \limsup_{\substack{\lambda,\mu \to \infty \\ |d\lambda - \mu| \leqslant 1}} \frac{-\ln |d\lambda - \mu|}{\mu}.$$

The 1D case - $\Omega = (0,1)$

Diffusion ratio $d \neq 1$

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Sufficient null-controllability condition

The moment problem is solved as follows

$$u(t) = \sum_{\lambda \in \Lambda} \omega_{d\lambda} q_{d\lambda}(t) + \sum_{\mu \in \Lambda} \omega_\mu q_\mu(t),$$

where $(q_\bullet)$ is a biorthogonal family

$$\int_0^T e^{-\tilde{\sigma} t} q_\sigma(t)\, dt = \delta_{\sigma,\tilde{\sigma}}, \qquad \forall \sigma, \tilde{\sigma} \in \Lambda \cup d\Lambda.$$

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(Ammar–Khodja - Benabdallah - González–Burgos - de Teresa, '14)

Theorem

$$\mathcal{A} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \textit{and} \quad \mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B, \quad \textit{with } B^\star = -(\partial_x)_{|x=0}.$$

$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

There exists a **minimal null-control time** $T_0(d) \in [0, +\infty]$ *:*

System (S) is NC at any time $T > T_0(d)$ and is not NC at any time $T < T_0(d)$.

Sufficient null-controllability condition

The moment problem is solved as follows

$$u(t) = \sum_{\lambda \in \Lambda} \omega_{d\lambda} q_{d\lambda}(t) + \sum_{\mu \in \Lambda} \omega_\mu q_\mu(t),$$

where $(q_\bullet)$ is a biorthogonal family

$$\int_0^T e^{-\tilde{\sigma} t} q_\sigma(t)\, dt = \delta_{\sigma,\tilde{\sigma}}, \qquad \forall \sigma, \tilde{\sigma} \in \Lambda \cup d\Lambda.$$

Such a biorthogonal family exists and satisfies

$$\|q_\sigma\|_{L^2(0,T)} \leqslant C_T e^{\left(c(\Lambda \cup d\Lambda) + \varepsilon(\sigma)\right)\sigma}, \forall \sigma \in \Lambda \cup d\Lambda.$$

$c(\bullet) :=$ condensation index of a family in $\mathbb{C}$.

If $T > c(\Lambda \cup d\Lambda)$ the series converges ✔

In our case : $c(\Lambda \cup d\Lambda) = T_0(d)$.

2. The 1D case

2.3. Interlude - the block moment method

The 1D case - $\Omega = (0, 1)$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

Question

Is it always true that

$$\text{Eigenvalues condensate} \Longrightarrow \text{No short-time null-controllabity} \quad ?$$

The 1D case - $\Omega = (0,1)$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

Question

Is it always true that

Eigenvalues condensate $\Longrightarrow$ No short-time null-controllabity ?

Example

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes a(x) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B, \quad \text{with } B^\star = -(\partial_x)_{|x=0}.$$

$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

We set $\Lambda_a := \sigma(-\partial_x^2 + a)$ and $(\phi_{a,\lambda})_{\lambda \in \Lambda_a}$ are the associated eigenfunctions.

Theorem (The answer is NO)

There are (plenty) of coefficients $a \in L^\infty(0,1)$ such that the condensation index satisfies

$$c(\Lambda \cup \Lambda_a) > 0,$$

but, however, system (S) is null-controllable at any time $T > 0$.

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes a(x) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B, \quad \text{with } B^\star = -(\partial_x)_{|x=0}.$$

$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

Spectral analysis (with some cheating ...)

We assume $\Lambda \cap \Lambda_a = \emptyset$

- For any $\mu \in \Lambda_a$, we set

$$\Phi_\mu := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_{a,\mu} \quad \Rightarrow \quad \mathcal{A}^\star \Phi_\mu = \mu \Phi_\mu, \quad \text{and} \quad \mathcal{B}^\star \Phi_\mu = -\phi'_{a,\mu}(0).$$

- For any $\lambda \in \Lambda$, we set

$$\Phi_\lambda := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \phi_\lambda + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_\lambda, \quad \Rightarrow \quad \mathcal{A}^\star \Phi_\lambda = \lambda \Phi_\lambda, \quad \text{and} \quad \mathcal{B}^\star \Phi_\lambda = -\psi'_\lambda(0) \neq 0,$$

where $\psi_\lambda \in H_0^1(0,1)$ is the unique solution to

$$(-\partial_x^2 + a)\psi_\lambda = \lambda \psi_\lambda - \phi_\lambda.$$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

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We assume $\Lambda \cap \Lambda_a = \varnothing$

$$\Phi_\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_{a,\mu}, \text{ for } \mu \in \Lambda_a, \quad \Phi_\lambda = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \phi_\lambda + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \psi_\lambda, \text{ for } \lambda \in \Lambda, \quad \Psi_\bullet = \frac{\Phi_\bullet}{\mathcal{B}^\star \Phi_\bullet}.$$

Necessary null-controllability condition

$$T \geqslant \limsup_{\substack{\lambda \in \Lambda, \mu \in \Lambda_a \\ \lambda, \mu \to \infty}} \frac{\ln \left\| \frac{\Psi_\lambda - \Psi_\mu}{\lambda - \mu} \right\|}{\lambda}.$$

Let $\lambda \in \Lambda$ and $\mu \in \Lambda_a$, $|\lambda - \mu| \ll 1$.

We can prove that $\quad \psi_\lambda \approx \dfrac{\phi_{a,\mu}}{\lambda - \mu} + O(1), \quad \Rightarrow \quad \mathcal{B}^\star \Phi_\lambda \approx \dfrac{\sqrt{\mu}}{\lambda - \mu},$

$$\Psi_\mu \approx \frac{1}{\sqrt{\mu}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_{a,\mu}, \qquad \Psi_\lambda \approx \frac{\lambda - \mu}{\sqrt{\mu}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \phi_\lambda + \frac{1}{\sqrt{\mu}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \phi_{a,\mu} + O(|\lambda - \mu|)$$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

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$$\frac{\Psi_\lambda - \Psi_\mu}{\lambda - \mu} = O(1) \quad \Longrightarrow$$ No obstruction to small time null-controllability .

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

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$$(S) \quad \partial_t y + \mathcal{A} y = \mathcal{B} v,$$

Solving the moment equations

$$\int_0^T e^{-\sigma t} u(t)\, dt = -e^{-\sigma T} \langle y_0, \Psi_\sigma \rangle, \quad \forall \sigma \in \Lambda \cup \Lambda_a,$$

The usual way

$$u(t) = \sum_{\sigma \in \Lambda \cup \Lambda_a} \omega_\sigma q_\sigma(t)$$

✖ Only works for $T > c(\Lambda \cup \Lambda_a)$ since

$$\|q_\sigma\|_{L^2(0,T)} \leqslant C_T e^{(c(\Lambda \cup \Lambda_a) + \varepsilon(\sigma))\sigma}.$$

Interlude - the block moment method

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The block moment way

Let $G = \{\lambda, \mu\}$, $\lambda \in \Lambda$, $\mu \in \Lambda_a$ with $|\lambda - \mu| \ll 1$.
We can find a $u_G \in L^2(0,T)$ such that

$$\begin{cases} \displaystyle\int_0^T e^{-\sigma t} u_G(t)\, dt = -e^{-\sigma T} \langle y_0, \Psi_\sigma \rangle, & \forall \sigma \in G, \\ \displaystyle\int_0^T e^{-\sigma t} u_G(t)\, dt = 0, & \forall \sigma \in (\Lambda \cup \Lambda_a) \backslash G, \end{cases}$$

with $\quad \|u_G\|_{L^2(0,T)} \leqslant C_T e^{-\lambda T} \left(\|\Psi_\lambda\| + \left\| \dfrac{\Psi_\lambda - \Psi_\mu}{\lambda - \mu} \right\| \right).$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

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with $\quad \|u_G\|_{L^2(0,T)} \leqslant \tilde{C} C_T e^{-\lambda T}.$

Interlude - the block moment method

(Benabdallah - B. - Morancey, '20)

$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (-\partial_x^2) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes a(x) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes B, \quad \text{with } B^\star = -(\partial_x)_{|x=0}.$$

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$$\|q_\sigma\|_{L^2(0,T)} \leqslant C_T e^{(c(\Lambda \cup \Lambda_a) + \varepsilon(\sigma))\sigma}.$$

The block moment way

$$\Lambda \cup \Lambda_a = \bigsqcup_{k \geqslant 1} G_k,$$

with $\quad \mathrm{diam}(G_k) \leqslant \rho, \quad \#G_k \leqslant 2.$

Then, without any condition on T, the function

$$u = \sum_{k \geqslant 1} u_{G_k},$$

is well-defined in $L^2(0,T)$ and gives a suitable control.

The block moment method

Statement precised

(Benabdallah - B. - Morancey, '20) (B. - Morancey, '21)

Definition

Let $\kappa > 0$, $\rho > 0$, $n \in \mathbb{N}^*$. *We define* $\mathcal{L}_w(n,\rho,\kappa)$ *as the set of all the* $S \subset [0,+\infty[$ *satisfying*

$$\textbf{Asymptotic behavior :} \quad N_S(r) := \#\big(S \cap [0,r)\big) \leqslant \kappa\sqrt{r}, \quad \forall r > 0,$$

$$\textbf{Weak gap condition :} \quad \#\big(S \cap [\mu,\mu+\rho]\big) \leqslant n, \quad \forall \mu \geqslant 0.$$

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Let $\kappa > 0$, $\rho > 0$, $n \in \mathbb{N}^*$. *We define* $\mathcal{L}_w(n, \rho, \kappa)$ *as the set of all the* $S \subset [0, +\infty[$ *satisfying*

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Weak gap condition : $$\#\big(S \cap [\mu, \mu + \rho]\big) \leqslant n, \quad \forall \mu \geqslant 0.$$

Typical situation

For any $i = 1, \ldots, n$, consider a $\Lambda_i \subset [0, +\infty[$ such that

$$N_{\Lambda_i}(r) \leqslant \kappa_i\sqrt{r},$$

$$|\lambda - \mu| > \rho_i, \quad \forall \lambda, \mu \in \Lambda_i, \lambda \neq \mu.$$

Then, we have

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}_w(n, \rho, \kappa),$$

with $\rho = \min \rho_i$, $\kappa = \max \kappa_i$.

The block moment method

Statement precised (Benabdallah - B. - Morancey, '20) (B. - Morancey, '21)

Definition

Let $\kappa > 0$, $\rho > 0$, $n \in \mathbb{N}^*$. *We define* $\mathcal{L}_w(n,\rho,\kappa)$ *as the set of all the* $S \subset [0,+\infty[$ *satisfying*

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$$\textbf{Weak gap condition :} \quad \#\big(S \cap [\mu, \mu+\rho]\big) \leqslant n, \quad \forall \mu \geqslant 0.$$

Any such set can be written $S = \bigsqcup_{k\geqslant 1} G_k$ with

$$\#G_k \leqslant n, \quad \operatorname{diam}(G_k) \leqslant \rho, \quad d(G_k, G_{k+1}) \geqslant \frac{\rho}{n}.$$

Theorem (in the case $n = 2$)

There exists $C > 0$ **depending only on ρ and κ** *such that, for any* $(\omega_\sigma)_{\sigma\in S} \subset \mathbb{R}$, *for any* $T > 0$, $k \geqslant 1$, *there exists a* $u_k \in L^2(0,T)$ *satisfying the*

$$\textbf{Partial moment problem} \quad \int_0^T e^{-\sigma t} u_k(t)\,dt = \begin{cases} \omega_\sigma, & \textit{if } \sigma \in G_k, \\ 0, & \textit{if } \sigma \notin G_k, \end{cases}$$

Uniform estimate $\|u_k\|_{L^2(0,T)} \leqslant C e^{C\sqrt{\sigma_k}\log(\sigma_k) + \frac{C}{T}} \left(|\omega_{\sigma_k}| + \left| \dfrac{\omega_{\sigma_k} - \omega_{\tilde{\sigma}_k}}{\sigma_k - \tilde{\sigma}_k} \right| \right)$, *where* $G_k = \{\sigma_k, \tilde{\sigma}_k\}$.

Outline

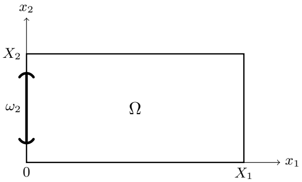
3. The 2D case

3.1. Diffusion ratio $d = 1$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$

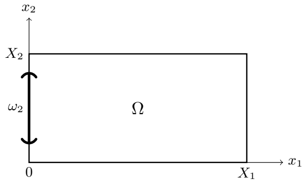
(Benabdallah - B. - González–Burgos - Olive, '14)



$$\begin{cases} \partial_t y + \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T) \times \Omega, \\ y = 1_{\{0\} \times \omega_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$ **(Benabdallah - B. - González–Burgos - Olive, '14)**



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Theorem (with $L^2(\Omega;\mathbb{C}^2) \approx \mathbb{C}^2 \otimes L^2(\Omega_1) \otimes L^2(\Omega_2)$)

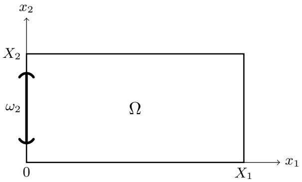
For any non empty $\omega_2 \subset \Omega_2$, the following system

$$(S) \quad \partial_t y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[(-\partial^2_{x_1}) \otimes \mathrm{Id} + \mathrm{Id} \otimes (-\partial^2_{x_2}) \Big] y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v,$$

is null-controllable at any time $T > 0$.

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$ (Benabdallah - B. - González–Burgos - Olive, '14)



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For any non empty $\omega_2 \subset \Omega_2$, the following system

$$(S) \quad \partial_t y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \Big] y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v,$$

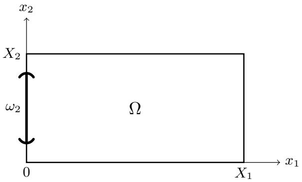
is null-controllable at any time $T > 0$.

Generalizations

- Same result if A_1 and A_2 are any two 1D elliptic operators.

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$ **(Benabdallah - B. - González–Burgos - Olive, '14)**



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$$(S) \quad \partial_t y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \right] y + M \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v,$$

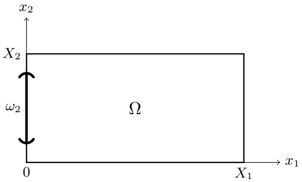
is null-controllable at any time $T > 0$.

Generalizations

- Same result if A_1 and A_2 are any two 1D elliptic operators.
- Same result with a general coupling matrix M satisfying **Kalman + non resonance condition**

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$ (Benabdallah - B. - González–Burgos - Olive, '14)



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For any non empty $\omega_2 \subset \Omega_2$, the following system

$$(S) \quad \partial_t y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \Big] y + M \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v,$$

is null-controllable at any time $T > 0$.

Straighforward case : $\omega_2 = \Omega_2$ $\qquad$ $\Lambda_2 = \sigma(A_2)$, eigenf. $(\phi_{2,\lambda_2})_{\lambda_2 \in \Lambda_2}$

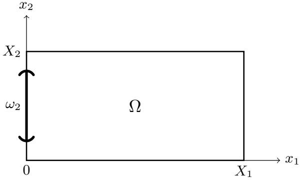
Decomposition of the solution and the control $\quad y = \sum_{\lambda_2 \in \Lambda_2} y_{\lambda_2} \otimes \phi_{2,\lambda_2}, \qquad v = \sum_{\lambda_2 \in \Lambda_2} v_{\lambda_2} \phi_{2,\lambda_2}.$

For each $\lambda_2 \in \Lambda_2$, we are led to a **1D control problem** for $y_{\lambda_2}(t) \in \mathbb{C}^2 \otimes L^2(\Omega_1)$ with $v_{\lambda_2}(t) \in \mathbb{C}$

$$\partial_t y_{\lambda_2} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes (A_1 + \lambda_2) y_{\lambda_2} + M \otimes \mathrm{Id}\, y_{\lambda_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B\, v_{\lambda_2},$$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Diffusion ratio $d = 1$ **(Benabdallah - B. - González–Burgos - Olive, '14)**



$$\begin{cases} \partial_t y + \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T) \times \Omega, \\ y = 1_{\{0\} \times \omega_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

Theorem (with $L^2(\Omega;\mathbb{C}^2) \approx \mathbb{C}^2 \otimes L^2(\Omega_1) \otimes L^2(\Omega_2)$)

For any non empty $\omega_2 \subset \Omega_2$, the following system

$$(S) \quad \partial_t y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \right] y + M \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v,$$

is null-controllable at any time $T > 0$.

Less straightforward case : $\omega_2 \subsetneq \Omega_2$

Use a Lebeau-Robbiano approach to combine the following ingredients **(Miller, '10)**

1. The system with $\omega_2 = \Omega_2$ is null-controllable **at any time** $\tau > 0$ with cost $e^{\frac{C}{\tau}}$.
2. Spectral LR inequality : $\|\psi\| \leqslant C e^{C\sqrt{\mu}} \|\mathbf{1}_{\omega_2} \psi\|, \quad \forall \psi \in E_{2,\mu} := \mathrm{Span}(\phi_{2,\lambda_2}, \lambda_2 \leqslant \mu)$.
3. Dissipation

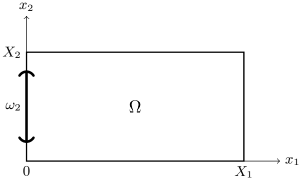
$$\|e^{-t\mathcal{A}^\star} \psi\| \leqslant C e^{-\mu t} \|\Psi\|, \quad \forall \Psi \in L^2(\Omega_1) \otimes \left(E_{2,\mu}\right)^\perp.$$

3. The 2D case

3.2. Diffusion ratio $d \neq 1$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$



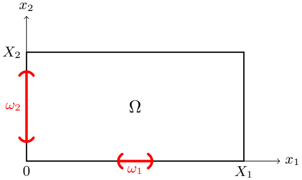
$$\begin{cases} \partial_t y + \begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T) \times \Omega, \\ y = 1_{\{0\} \times \omega_2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

A few comments

- Even if $\omega_2 = \Omega_2$, null-controllability **does not hold** for $T < T_0(d)$.
- If $T_0(d) = 0$: null-controllability holds at any time $T > 0$.
- If $T_0(d) > 0$ and $\omega_2 \subsetneq \Omega_2$: We don't know if NC holds for $T > T_0(d)$!!

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$



$$\begin{cases} \partial_t y + \begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T) \times \Omega, \\ y = 1_{\left(\omega_1 \times \{0\}\right) \cup \left(\{0\} \times \omega_2\right)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

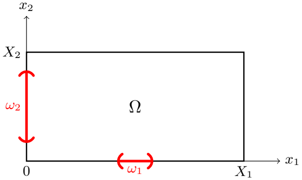
What about controlling also on another side of the rectangle ?

- Approximate controllability holds **for any value of** d.
- For $d = 1$ and any coupling matrix M that satisfies Kalman condition ($m_{21} \neq 0$) : approximate controllability holds. **(Olive, '14)**

The "geometry" of the control support matters !

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$



$$\begin{cases} \partial_t y + \begin{pmatrix} -d\Delta & 0 \\ 0 & -\Delta \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, & \text{in } (0,T) \times \Omega, \\ y = 1_{(\omega_1 \times \{0\}) \cup (\{0\} \times \omega_2)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v, & \text{on } (0,T) \times \partial\Omega. \end{cases}$$

What about controlling also on another side of the rectangle ?

Question

What can be said concerning null-controllability ?

(B. - Olive, '21)

Theorem

For any $d > 0$, and any M satisfying the Kalman condition $m_{21} \neq 0$, the following system

$$(S) \quad \partial_t y + \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \Big] y + M \otimes \mathrm{Id} \otimes \mathrm{Id}\, y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \mathbf{1}_{\omega_1} \otimes B\, v_1 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes B \otimes \mathbf{1}_{\omega_2}\, v_2,$$

is null-controllable at any time $T > 0$ and we have the estimate $\|v\|_{L^2((0,T)\times\partial\Omega)} \leqslant e^{\frac{C}{T}} \|y_0\|$.

Even for $d = 1$, this is a new result since there is no "non resonance" condition !

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

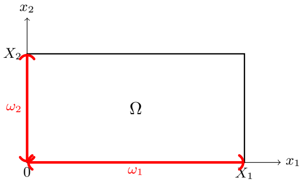
Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

$$\partial_t y + \mathcal{A} y = \mathcal{B} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

with

$$\mathcal{A}^\star = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \Big] y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B}^\star = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes B^\star \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B^\star \otimes \mathrm{Id} \end{pmatrix}.$$



The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2, \quad \Omega_1 = (0, X_1), \quad \Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

$$\partial_t y + \mathcal{A}y = \mathcal{B}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

with

$$\mathcal{A}^\star = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2\Big] y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B}^\star = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes B^\star \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B^\star \otimes \mathrm{Id} \end{pmatrix}.$$

Spectral analysis

We assume $d > 1$

- We set $\Gamma := \{+, -\} \times \Lambda_1 \times \Lambda_2$.
- For each triple $\gamma = (s, \lambda_1, \lambda_2) \in \Gamma$ we set

$$\sigma(\gamma) := \frac{(1+d) + s(d-1)}{2}(\lambda_1 + \lambda_2) = \begin{cases} d(\lambda_1 + \lambda_2) & \text{if } s = +, \\ \lambda_1 + \lambda_2 & \text{if } s = - \end{cases}$$

$$\Phi_\gamma := \begin{pmatrix} 1 \\ \sigma(\gamma) - d(\lambda_1 + \lambda_2) \end{pmatrix} \otimes \psi_{1,\lambda_1} \otimes \psi_{2,\lambda_2} \quad \Rightarrow \quad \mathcal{A}^\star \Phi_\gamma = \sigma(\gamma)\Phi_\gamma, \quad \text{and} \quad \mathcal{B}^\star \Phi_\gamma = \begin{pmatrix} \psi_{1,\lambda_1} \\ \psi_{2,\lambda_2} \end{pmatrix}.$$

  Here we have used again the notation $\psi_{\bullet,\bullet} = \frac{\phi_{\bullet,\bullet}}{B^\star \phi_{\bullet,\bullet}}$.
- The family $(\Phi_\gamma)_{\gamma \in \Gamma}$ is complete in $\mathbb{C}^2 \otimes L^2(\Omega_1) \otimes L^2(\Omega_2)$.

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2, \quad \Omega_1 = (0, X_1), \quad \Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

$$\partial_t y + \mathcal{A} y = \mathcal{B} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

with

$$\mathcal{A}^\star = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \otimes \Big[A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2 \Big] y + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes \mathrm{Id}, \quad \text{and} \quad \mathcal{B}^\star = \begin{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \mathrm{Id} \otimes B^\star \\ \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes B^\star \otimes \mathrm{Id} \end{pmatrix}.$$

Equivalent equations

with $u_1(t) := v_1(T - t)$, $u_2(t) := v_2(T - t)$

$$\int_0^T e^{-\sigma(\gamma)t} \big\langle u_1(t), \psi_{1,\lambda_1(\gamma)} \big\rangle_{L^2(\Omega_1)} \, dt + \int_0^T e^{-\sigma(\gamma)t} \big\langle u_2(t), \psi_{2,\lambda_2(\gamma)} \big\rangle_{L^2(\Omega_2)} \, dt$$
$$= e^{-\sigma(\gamma)T} \big\langle \Phi_\gamma, y_0 \big\rangle_{L^2(\Omega)}, \quad \forall \gamma \in \Gamma.$$

Writing $u_1(t) = \sum_{\lambda_1 \in \Lambda_1} u_{1,\lambda_1}(t) \frac{\psi_{1,\lambda_1}}{\|\psi_{1,\lambda_1}\|^2}$ and $u_2(t) = \sum_{\lambda_1 \in \Lambda_1} u_{2,\lambda_2}(t) \frac{\psi_{2,\lambda_2}}{\|\psi_{2,\lambda_2}\|^2}$, we arrive to

$$\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1(\gamma)}(t) \, dt + \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2(\gamma)} \, dt = e^{-\sigma(\gamma)T} \big\langle \Phi_\gamma, y_0 \big\rangle_{L^2(\Omega)}, \quad \forall \gamma \in \Gamma.$$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

Summary

We look for two countable families of functions in $L^2(0,T)$: $(u_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ and $(u_{2,\lambda_2})_{\lambda_2 \in \Lambda_2}$ s.t.

$$\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1(\gamma)}(t)\, dt + \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2(\gamma)}\, dt = e^{-\sigma(\gamma)T} \left\langle \Phi_\gamma, y_0 \right\rangle_{L^2(\Omega)}, \quad \forall \gamma \in \Gamma.$$

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$$\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1(\gamma)}(t)\,dt + \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2(\gamma)}\,dt = e^{-\sigma(\gamma)T} \langle \Phi_\gamma, y_0 \rangle_{L^2(\Omega)}, \quad \forall \gamma \in \Gamma.$$

This looks like a moment problem but ...

- Each function u_{1,λ_1} appears in **infinitely many equations**, namely all those corresponding to

$$\gamma \in \Gamma_{1,\lambda_1} := \{+,-\} \times \{\lambda_1\} \times \Lambda_2,$$

  and the same for each function u_{2,λ_2}.
- The same integral can appear in two different equations. For instance, for a given $\lambda_1 \in \Lambda_1$

$$\gamma = (+, \lambda_1, \lambda_2) \implies \int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1}(t)\,dt + \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2}\,dt = e^{-\sigma(\gamma)T} \langle \Phi_\gamma, y_0 \rangle_{L^2(\Omega)},$$

$$\tilde{\gamma} = (-, \lambda_1, \tilde{\lambda}_2) \implies \int_0^T e^{-\sigma(\tilde{\gamma})t} u_{1,\lambda_1}(t)\,dt + \int_0^T e^{-\sigma(\tilde{\gamma})t} u_{2,\tilde{\lambda}_2}\,dt = e^{-\sigma(\tilde{\gamma})T} \langle \Phi_{\tilde{\gamma}}, y_0 \rangle_{L^2(\Omega)},$$

  It may happen that $\sigma(\gamma) = d(\lambda_1 + \lambda_2) = (\lambda_1 + \tilde{\lambda}_2) = \sigma(\tilde{\gamma})$ or even that $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \ll 1$.

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

Summary

We look for two countable families of functions in $L^2(0,T)$: $(u_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ and $(u_{2,\lambda_2})_{\lambda_2 \in \Lambda_2}$ s.t.

$$\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1(\gamma)}(t)\, dt + \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2(\gamma)}\, dt = e^{-\sigma(\gamma)T} \left\langle \Phi_\gamma, y_0 \right\rangle_{L^2(\Omega)}, \quad \forall \gamma \in \Gamma.$$

Resolution strategy

- Assume that, for any $\gamma \in \Gamma$, we are given two numbers $\omega_{\gamma,1}$ and $\omega_{\gamma,2}$ such that

$$e^{-\sigma(\gamma)T} \left\langle \Phi_\gamma, y_0 \right\rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}.$$

- The problem is reduced to find $(u_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ and $(u_{2,\lambda_2})_{\lambda_1 \in \Lambda_2}$ such that

$$\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1(\gamma)}(t)\, dt = \omega_{\gamma,1}, \quad \text{and} \quad \int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2(\gamma)}(t)\, dt = \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma.$$

- Putting all in order $\Rightarrow$ infinite countable family of **independent** “standard” moment problems

For each $\lambda_1 \in \Lambda_1$: $\left[\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1}(t)\, dt = \omega_{\gamma,1}, \quad \forall \gamma \in \Gamma_{1,\lambda_1} \right] \qquad (P_{1,\lambda_1}),$

For each $\lambda_2 \in \Lambda_2$: $\left[\int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2}(t)\, dt = \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma_{2,\lambda_2} \right] \qquad (P_{2,\lambda_2}).$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

Where are we ?

We look for $(u_{1,\lambda_1})_{\lambda_1 \in \Lambda_1}$ and $(u_{2,\lambda_2})_{\lambda_1 \in \Lambda_2}$ such that

$$\text{For each } \lambda_1 \in \Lambda_1 : \quad \left[\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1}(t)\, dt = \omega_{\gamma,1}, \quad \forall \gamma \in \Gamma_{1,\lambda_1} \right] \qquad (P_{1,\lambda_1}),$$

$$\text{For each } \lambda_2 \in \Lambda_2 : \quad \left[\int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2}(t)\, dt = \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma_{2,\lambda_2} \right] \qquad (P_{2,\lambda_2}).$$

where

$$e^{-\sigma(\gamma)T} \langle \Phi_\gamma, y_0 \rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}.$$

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$$\text{For each } \lambda_2 \in \Lambda_2 : \quad \left[\int_0^T e^{-\sigma(\gamma)t} u_{2,\lambda_2}(t)\, dt = \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma_{2,\lambda_2} \right] \qquad (P_{2,\lambda_2}).$$

where

$$e^{-\sigma(\gamma)T} \langle \Phi_\gamma, y_0 \rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}.$$

Issues

- For a given $\lambda_1 \in \Lambda_1$, it may exist two **different** $\gamma, \tilde{\gamma} \in \Gamma_{1,\lambda_1}$ with $\sigma(\gamma) = \sigma(\tilde{\gamma})$.

  This imposes $\omega_{\gamma,1} = \omega_{\tilde{\gamma},1}$.

- For a given $\lambda_1 \in \Lambda_1$, it may exist two **different** $\gamma, \tilde{\gamma} \in \Gamma_{1,\lambda_1}$ with $|\sigma(\gamma) - \sigma(\tilde{\gamma})| \ll 1$.

  This requires a control on $\left| \dfrac{\omega_{\gamma,1} - \omega_{\tilde{\gamma},1}}{\sigma(\gamma) - \sigma(\tilde{\gamma})} \right|$

- Similar constraints on $\omega_{\bullet,2}$ ...
- Can we find such values $(\omega_{\gamma,1})_{\gamma \in \Gamma}$ and $(\omega_{\gamma,2})_{\gamma \in \Gamma}$?

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

We assume $\omega_1 = \Omega_1$ and $\omega_2 = \Omega_2$

Proposition (Key result)

There exists $\rho > 0$, and two families $(\omega_{\gamma,1})_{\gamma\in\Gamma}$ and $(\omega_{\gamma,2})_{\gamma\in\Gamma}$ of numbers such that

$$e^{-\sigma(\gamma)T}\left\langle \Phi_\gamma, y_0 \right\rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma$$

$$\Big(\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,1} = \omega_{\tilde{\gamma},1}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma,$$

$$\Big(\lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,2} = \omega_{\tilde{\gamma},2}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma.$$

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Proposition (Key result)

There exists $\rho > 0$, and two families $(\omega_{\gamma,1})_{\gamma\in\Gamma}$ and $(\omega_{\gamma,2})_{\gamma\in\Gamma}$ of numbers such that

$$\boxed{R_\gamma} =: e^{-\sigma(\gamma)T}\left\langle \Phi_\gamma, y_0 \right\rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma$$

$$\boxed{\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma}} =: \begin{cases} \boxed{\gamma \underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma}} =: \Big(\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,1} = \omega_{\tilde{\gamma},1}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma, \\ \boxed{\gamma \underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma}} =: \Big(\lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,2} = \omega_{\tilde{\gamma},2}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma. \end{cases}$$

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There exists $\rho > 0$, and two families $(\omega_{\gamma,1})_{\gamma\in\Gamma}$ and $(\omega_{\gamma,2})_{\gamma\in\Gamma}$ of numbers such that

$$\boxed{R_\gamma} =: e^{-\sigma(\gamma)T}\langle \Phi_\gamma, y_0\rangle_{L^2(\Omega)} = \omega_{\gamma,1} + \omega_{\gamma,2}, \quad \forall \gamma \in \Gamma$$

$$\boxed{\gamma \overset{\rho}{\Longleftrightarrow} \tilde{\gamma}} =: \begin{cases} \boxed{\gamma \underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma}} =: \Big(\lambda_1(\gamma) = \lambda_1(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,1} = \omega_{\tilde{\gamma},1}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma, \\ \boxed{\gamma \underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}} \tilde{\gamma}} =: \Big(\lambda_2(\gamma) = \lambda_2(\tilde{\gamma}) \text{ and } |\sigma(\gamma) - \sigma(\tilde{\gamma})| \leqslant \rho\Big) \Longrightarrow \omega_{\gamma,2} = \omega_{\tilde{\gamma},2}, \quad \forall \gamma, \tilde{\gamma} \in \Gamma. \end{cases}$$

Basic construction: Take any $\gamma_0 \in \Gamma$ and choose, for instance, $\omega_{\gamma_0,1} := R_{\gamma_0}$ and $\omega_{\gamma_0,2} := 0$.

Let $\gamma_1 \in \Gamma$ such that $\gamma_0 \overset{\rho}{\Longleftrightarrow} \gamma_1$.

Case $\gamma_0 \underset{\lambda_1}{\overset{\rho}{\longleftrightarrow}} \gamma_1$

No choice : $\begin{cases} \omega_{\gamma_1,1} := \omega_{\gamma_0,1} \\ \omega_{\gamma_1,2} := R_{\gamma_1} - \omega_{\gamma_1,1}. \end{cases}$

Case $\gamma_0 \underset{\lambda_2}{\overset{\rho}{\longleftrightarrow}} \gamma_1$

No choice : $\begin{cases} \omega_{\gamma_1,2} := \omega_{\gamma_0,2} \\ \omega_{\gamma_1,1} := R_{\gamma_1} - \omega_{\gamma_1,2}. \end{cases}$

The 2D cartesian case - $\Omega = \Omega_1 \times \Omega_2$, $\Omega_1 = (0, X_1)$, $\Omega_2 = (0, X_2)$

Any diffusion ratio $d > 0$

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There exists $\rho > 0$, and two families $(\omega_{\gamma,1})_{\gamma\in\Gamma}$ and $(\omega_{\gamma,2})_{\gamma\in\Gamma}$ of numbers such that

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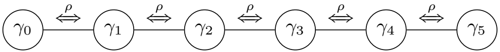
Basic construction: Take any $\gamma_0 \in \Gamma$ and choose, for instance, $\omega_{\gamma_0,1} := R_{\gamma_0}$ and $\omega_{\gamma_0,2} := 0$.



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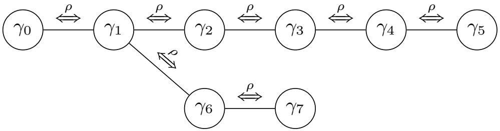

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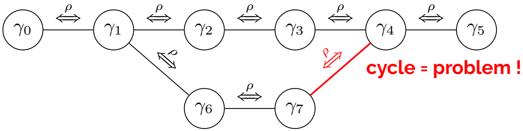

How to avoid this phenomenon ?

Study of the graph structure of $(\Gamma, \overset{\rho}{\Longleftrightarrow})$ and choice of ρ

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The graph is a forest

We can find a $\rho > 0$, small enough, such that there is **no cycle** in the graph $(\Gamma, \overset{\rho}{\Longleftrightarrow})$.

Moreover, every path P in the graph is finite and its length n satisfies

$$n \leq C|\sigma(\gamma)|, \quad \gamma \in P.$$

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Any diffusion ratio $d > 0$

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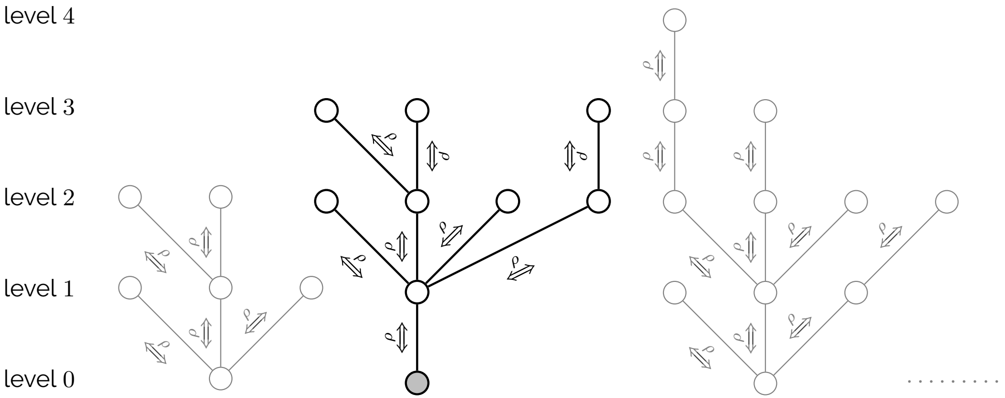
Here is the forest $(\Gamma, \overset{\rho}{\Longleftrightarrow})$ ...



With this value of ρ : We can construct all the $\omega_{\bullet,\bullet}$ and prove the required estimates

$$|\omega_{\gamma,\bullet}| \leqslant Ce^{-\sigma(\gamma)T}.$$

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Now we have to solve infinitely many moment problems

For any $\lambda_1 \in \Lambda_1$, we look for $u_{1,\lambda_1} \in L^2(0,T)$ s.t. $\left[\int_0^T e^{-\sigma(\gamma)t} u_{1,\lambda_1}(t)\,dt = \omega_{\gamma,1}, \quad \forall \gamma \in \Gamma_{1,\lambda_1}\right]$.

- Thanks to property (C_1), we can write this problem a follows

$$\int_0^T e^{-\sigma t} u_{1,\lambda_1}(t)\,dt = \omega_{\sigma,1}, \quad \forall \sigma \in \Sigma_{1,\lambda_1} := \sigma(\Gamma_{1,\lambda_1}).$$

- The family $\Sigma_{1,\lambda_1} = (\lambda_1 + \Lambda_2) \bigcup (d\lambda_1 + d\Lambda_2)$ satisfies:
  - **Uniform** weak gap condition with $n = 2$.
  - **Uniform** estimate of the counting function $N_{\Sigma_{1,\lambda_1}}(r) \leqslant C\sqrt{r}, \quad \forall r > 0$.

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- Block moment approach $\rightsquigarrow \exists u_{1,\lambda_1}$ with $\|u_{1,\lambda_1}\| \leqslant e^{-C\lambda_1 T + \frac{C}{T}} \|y_0\|$.

  Indeed, for any group $G_k = \{\sigma, \tilde\sigma\}$ we have $\boxed{\dfrac{\omega_{\sigma,1} - \omega_{\tilde\sigma,1}}{\sigma - \tilde\sigma} = 0}$, even if $|\sigma - \tilde\sigma|$ is very small.

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Conclusion :

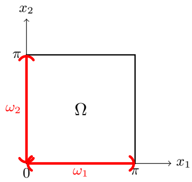
- The following series converge

$$u_1(t) = \sum_{\lambda_1\in\Lambda_1} u_{1,\lambda_1}(t)\frac{\psi_{1,\lambda_1}}{\|\psi_{1,\lambda_1}\|^2}.$$

- The same for u_2 ...
- We are done ! ✔

Examples - Extensions

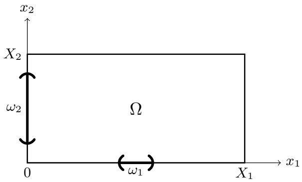
Example

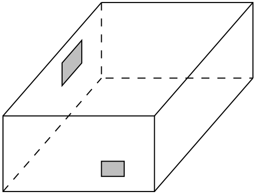


$$\begin{cases} \partial_t y_1 - \Delta y_1 = 0, \\ \partial_t y_2 - \Delta y_2 - y_1 - 3y_2 = 0 \end{cases}$$

- If we control only on $(0,\pi) \times \{0\}$ or on $\{0\} \times (0,\pi)$, this system is not approximately controllable.
- If we control on the two sides $\big((0,\pi) \times \{0\}\big) \bigcup \big(\{0\} \times (0,\pi)\big)$, the system is null-controllable at any time $T > 0$.

Extensions





Outline

Conclusion / Perspectives

Summary

- Boundary controllability of coupled parabolic systems can be intricate:
  - Geometric conditions on the support of the control.
  - Positive minimal null-control time may appear.
- In the 1D case
  - The moment method approach is powerful (yet limited to time invariant systems ...)
  - Block moment approach let us handle spectral condensation phenomena
  - Uniform estimates with respect to the family of eigenvalues (in a certain class)
- In the multi-D case
  - Cartesian setting : reduce the problem to the study of a coupled set of infinitely many “1D” systems.
    The block moment approach helps
  - General geometries : Almost no result as far as I know ...

Extension of the block moment approach to non scalar control problems

(B. - Morancey, ’21)

Theorem (Two boundary controls)

The following system is null-controllable at any time $T > 0$

$$\partial_t y + \begin{pmatrix} -d\partial_x^2 & 0 \\ 0 & -\partial_x^2 \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y = 0, \quad y(t,0) = \begin{pmatrix} u_1(t) \\ 0 \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} u_2(t) \\ u_2(t) \end{pmatrix}$$

Thanks for your attention !

Questions ?

- A. Benabdallah, F.B., M. Morancey, *A block moment method to handle spectral condensation phenomenon in parabolic control problems*, Annales Henri Lebesgue, Vol. 3, pp. 717-793 (2020)

  `https://doi.org/10.5802/ahl.45`

- F.B., G. Olive, *Boundary null-controllability of some multi-dimensional linear parabolic systems by the moment method*, preprint, (2021)

  `https://hal.archives-ouvertes.fr/hal-03175706`

- F.B., M. Morancey, *Analysis of non scalar control problems for parabolic systems by the block moment method*, preprint, (2021)

  `https://hal.archives-ouvertes.fr/hal-02397706`